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CONDITIONS CHARACTERIZING THE HARDY AND REVERSE HARDY INEQUALITIES

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Dedicated to Professor Josip Pečarić on the occasion of his 60th birthday

Abstract. We summarize the conditions characterizing the Hardy and reverse Hardy inequalities for the case $p,q \in \mathbb{R}\setminus\{0\}$, and extend in some cases the number of equivalent conditions.

1. Introduction and preliminaries

We will deal with scales of (equivalent) necessary and sufficient conditions for the validity of the *Hardy inequality*

$$\left(\int_a^b \left(\int_a^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_a^b f(x)^p v(x)dx\right)^{\frac{1}{p}} \tag{1.1}$$

and the *reverse Hardy inequality*

$$\left(\int_a^b f(x)^p v(x)dx\right)^{\frac{1}{p}} \leqslant C\left(\int_a^b \left(\int_a^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \tag{1.2}$$

for different values of the parameters p and q ($p,q \in \mathbb{R}\setminus\{0\}$). Here, it is $-\infty \leqslant a < b \leqslant \infty$, u and v are *weight functions* on (a,b), i.e. measurable and positive a.e..

We denote

$$U(x) := \int_x^b u(t)dt, \quad V(x) := \int_a^x v^{1-p'}(t)dt, \quad p' = \frac{p}{p-1}, p \neq 1,$$

and

$$\tilde{U}(x) := \int_a^x u(t)dt, \quad \tilde{V}(x) := \int_x^b v^{1-p'}(t)dt,$$

and assume that in any particular situation, the functions $U, V, \tilde{U}$ and $\tilde{V}$ are finite. Furthermore, we denote, for $\alpha,\beta \in \mathbb{R}$,

$$\begin{aligned} A_0(x;\alpha,\beta) &:= U^\alpha(x)V^\beta(x), \\ B_0(x;\alpha,\beta) &:= \tilde{U}^\alpha(x)V^\beta(x). \end{aligned} \tag{1.3}$$

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It is well-known that in the triangular domain

$$1 < p \leqslant q < \infty,$$

the Hardy inequality (1.1) holds for all $f \geqslant 0$ if and only if the function

$$A_M(x) := A_0\left(x; \frac{1}{q}, \frac{1}{p'}\right) \tag{1.4}$$

(the so-called *Muckenhoupt* function) is bounded, i.e.

$$A_0 := \sup_{a<x<b} A_M(x) < \infty, \tag{1.5}$$

and for the best constant C in (1.1), we have $C \approx A_0$.

Recently, it was shown (see [3, Theorem 3.1] and also [4]) that there is a number of equivalent criteria of the validity of inequality (1.1): If we denote

$$\begin{aligned}
A_1(s) &:= \sup_{a<x<b} \left(\int_x^b u(t) V^{q(\frac{1}{p'}-s)}(t)\,dt\right)^{\frac{1}{q}} V^s(x), \quad s>0;\\
A_2(s) &:= \sup_{a<x<b} \left(\int_a^x v^{1-p'}(t) U^{p'(\frac{1}{q}-s)}(t)\,dt\right)^{\frac{1}{p'}} U^s(x), \quad s>0;\\
A_3(s) &:= \sup_{a<x<b} \left(\int_a^x u(t) V^{q(\frac{1}{p'}+s)}(t)\,dt\right)^{\frac{1}{q}} V^{-s}(x), \quad s>0;\\
A_4(s) &:= \sup_{a<x<b} \left(\int_x^b v^{1-p'}(t) U^{p'(\frac{1}{q}+s)}(t)\,dt\right)^{\frac{1}{p'}} U^{-s}(x), \quad s>0;\\
A_5(s) &:= \sup_{a<x<b} \left(\int_x^b u(t) V^{\frac{q}{p'(1+sq)}}(t)\,dt\right)^{\frac{1+sq}{q}} U^{-s}(x), \quad s>0;\\
A_6(s) &:= \sup_{a<x<b} \left(\int_x^b v^{1-p'}(t) U^{\frac{p'}{q(1+sp')}}(t)\,dt\right)^{\frac{1+sp'}{p'}} V^{-s}(x), \quad s>0;\\
A_7(s) &:= \sup_{a<x<b} \left(\int_a^x u(t) V^{\frac{q}{p'(1-sq)}}(t)\,dt\right)^{\frac{1-sq}{q}} U^{s}(x), \quad 0<s<\frac{1}{q};\\
A_8(s) &:= \sup_{a<x<b} \left(\int_x^b u(t) V^{\frac{q}{p'(1-sq)}}(t)\,dt\right)^{\frac{1-sq}{q}} U^{s}(x), \quad s>\frac{1}{q};\\
A_9(s) &:= \sup_{a<x<b} \left(\int_x^b v^{1-p'}(t) U^{\frac{p'}{q(1-sp')}}(t)\,dt\right)^{\frac{1-sp'}{p'}} V^{s}(x), \quad 0<s<\frac{1}{p'};\\
A_{10}(s) &:= \sup_{a<x<b} \left(\int_a^x v^{1-p'}(t) U^{\frac{p'}{q(1-sp')}}(t)\,dt\right)^{\frac{1-sp'}{p'}} V^{s}(x), \quad s>\frac{1}{p'};\\
A_{11}(s) &:= \inf_{h>0} \sup_{a<x<b} \left(\int_x^b u(t) h(t)^{q(\frac{1}{p'}-s)}(t)\,dt\right)^{\frac{1}{q}} (h(x)+V(x))^s, \quad s>\frac{1}{p'};
\end{aligned} \tag{1.6}$$

$$A_{12}(s) := \inf_{h>0} \sup_{a<x<b} \left(\int_0^x v^{1-p'}(t) h(t)^{p'(\frac{1}{q}-s)}(t)dt \right)^{\frac{1}{p'}} (h(x)+U(x))^s, \quad s > \frac{1}{q};$$

$$A_{13}(s) := \inf_{h>0} \sup_{a<x<b} \left(\int_0^x u(t)(h(t)+V(t))^{q(\frac{1}{p'}+s)}(t)dt \right)^{\frac{1}{q}} h(x)^{-s}, \quad s > 0;$$

$$A_{14}(s) := \inf_{h>0} \sup_{a<x<b} \left(\int_x^b v^{1-p'}(t)(h(t)+U(t))^{p'(\frac{1}{q}+s)}(t)dt \right)^{\frac{1}{p'}} h(x)^{-s}, \quad s > 0,$$

then $A_i(s) \approx A_0$ for $i = 1,2,...,14$ and consequently, (1.1) holds (with $1 < p \leqslant q < \infty$) if and only if any of the numbers $A_i(s)$ is finite.

The aim of this paper is to use some recent results to obtain similar scales of conditions to characterize (1.1) or (1.2) for some other parameters and also to give an overview of this problem for all parameters $p,\ q,\ p \neq 0,\ q \neq 0$. We note that it is natural to study (1.1) for $p \geqslant 1$ and (1.2) for $p < 1$.

Basic information about problems connected with the Hardy inequality can be found in the books [KP] and [KMP].

2. The second quadrant

Now, we assume that $p < 0,\ q > 0$. Recently, it was shown (see [6]) that the *reverse* Hardy inequality (1.2) holds if and only if the function $A_M(x)$ from (1.4) is bounded, i.e. (1.5) holds. Consequently, using Theorem 2.1 in [3] (where we put $\alpha = \frac{1}{q}$, $\beta = \frac{1}{p'}$ and $f = u$, $g = v^{1-p'}$) we have immediately the following result:

THEOREM 2.1. *The reverse Hardy inequality (*1.2*) with $p < 0$ and $q > 0$ holds for all functions $f > 0$ if and only if any of the numbers $A_i = A_i(s)$ from (*1.6*) is finite. It is $A_i \approx A_j$ for $i,j = 0,1,2,...,14$ and the best constant C in (*1.2*) satisfies $C \approx A_i$.*

3. The fourth quadrant

Now, we assume that

$$p > 0, \quad q < 0. \tag{3.1}$$

(i) First, we consider the strip $0 < p < 1,\ q < 0$. As it was shown in [6], the *reverse* Hardy inequality (1.2) holds if and only if

$$A_* := \inf_{a<x<b} A_M(x) > 0 \tag{3.2}$$

and the best constant in (1.2) satisfies $C \approx A_*$. Here, according to (1.3), $A_M(x) = A_0(x; \frac{1}{q}, \frac{1}{p'})$ where both parameters $\frac{1}{q}, \frac{1}{p'}$ are negative. But if we denote

$$\tilde{A}_M(x) = \frac{1}{A_M(x)} = A_0\left(x; -\frac{1}{q}, -\frac{1}{p'}\right),$$

then we can rewrite condition (3.2) as

$$A_0^* := \sup_{a<x<b} \tilde{A}_M(x) < \infty, \tag{3.3}$$

where the parameters $-\frac{1}{q}$, $-\frac{1}{p'}$ in $\tilde{A}_M(x)$ are positive. Using now, analogously as in Section 2, Theorem 2.1 in [3] where we choose $\alpha = -\frac{1}{q}$, $\beta = -\frac{1}{p'}$ and $f = u$, $g = v^{1-p'}$ we obtain the following result:

THEOREM 3.1. *The reverse Hardy inequality (1.2) with $0 < p < 1$ and $q < 0$ holds for all functions $f > 0$ if and only if any of the following numbers is finite:*

$$
\begin{aligned}
A_1^*(s) &:= \sup_{a<x<b} \left(\int_x^b u(t) V^{q(\frac{1}{p'}+s)}(t)dt \right)^{-\frac{1}{q}} V^s(x), \quad s > 0;\\
A_2^*(s) &:= \sup_{a<x<b} \left(\int_a^x v^{1-p'}(t) U^{p'(\frac{1}{q}+s)}(t)dt \right)^{-\frac{1}{p'}} U^s(x), \quad s > 0;\\
A_3^*(s) &:= \sup_{a<x<b} \left(\int_a^x u(t) V^{q(\frac{1}{p'}-s)}(t)dt \right)^{-\frac{1}{q}} V^{-s}(x), \quad s > 0;\\
A_4^*(s) &:= \sup_{a<x<b} \left(\int_x^b v^{1-p'}(t) U^{p'(\frac{1}{q}-s)}(t)dt \right)^{-\frac{1}{p'}} U^{-s}(x), \quad s > 0;\\
A_5^*(s) &:= \sup_{a<x<b} \left(\int_x^b u(t) V^{\frac{q}{p'(1-sq)}}(t)dt \right)^{\frac{sq-1}{q}} U^{-s}(x), \quad s > 0;\\
A_6^*(s) &:= \sup_{a<x<b} \left(\int_x^b v^{1-p'}(t) U^{\frac{p'}{q(1-sp')}}(t)dt \right)^{\frac{sp'-1}{p'}} V^{-s}(x), \quad s > 0;\\
A_7^*(s) &:= \sup_{a<x<b} \left(\int_a^x u(t) V^{\frac{q}{p'(1+sq)}}(t)dt \right)^{-\frac{1+sq}{q}} U^s(x), \quad 0 < s < -\frac{1}{q};\\
A_8^*(s) &:= \sup_{a<x<b} \left(\int_x^b u(t) V^{\frac{q}{p'(1+sq)}}(t)dt \right)^{-\frac{1+sq}{q}} U^s(x), \quad s > -\frac{1}{q};\\
A_9^*(s) &:= \sup_{a<x<b} \left(\int_x^b v^{1-p'}(t) U^{\frac{p'}{q(1+sp')}}(t)dt \right)^{-\frac{1+sp'}{p'}} V^s(x), \quad 0 < s < -\frac{1}{p'};\\
A_{10}^*(s) &:= \sup_{a<x<b} \left(\int_a^x v^{1-p'}(t) U^{\frac{p'}{q(1+sp')}}(t)dt \right)^{-\frac{1+sp'}{p'}} V^s(x), \quad s > -\frac{1}{p'};\\
A_{11}^*(s) &:= \inf_{h>0} \sup_{a<x<b} \left(\int_x^b u(t) h(t)^{q(\frac{1}{p'}+s)}(t)dt \right)^{-\frac{1}{q}} (h(x)+V(x))^s, \quad s > -\frac{1}{p'};\\
A_{12}^*(s) &:= \inf_{h>0} \sup_{a<x<b} \left(\int_a^x v^{1-p'}(t) h(t)^{p'(\frac{1}{q}+s)}(t)dt \right)^{-\frac{1}{p'}} (h(x)+U(x))^s, \quad s > -\frac{1}{q};\\
A_{13}^*(s) &:= \inf_{h>0} \sup_{a<x<b} \left(\int_a^x u(t)(h(t)+V(t))^{q(\frac{1}{p'}-s)}(t)dt \right)^{-\frac{1}{q}} h(x)^{-s}, \quad s > 0;
\end{aligned}
\tag{3.4}
$$

$$A_{14}^*(s) := \inf_{h>0} \sup_{a<x<b} \left(\int_x^b v^{1-p'}(t)(h(t)+U(t))^{p'(\frac{1}{q}-s)}(t)dt \right)^{-\frac{1}{p'}} h(x)^{-s}, \quad s>0.$$

There is $A_i^*(s) \approx A_0^*$ *for* $i=1,2,\dots,14$ *and* $C \approx \frac{1}{A_i^*}$.

(ii) Now, let us consider the remaining part of the quadrant (3.1), i.e. the domain

$$p \geqslant 1, \quad q<0.$$

• If $p>1$, then it was shown in [6], that the Hardy inequality (1.1) holds if and only if

$$B := \inf_{a<x<b} B_0\left(x; \frac{1}{q}, \frac{1}{p'}\right) > 0 \tag{3.5}$$

where $B_0(x;\cdot,\cdot)$ is defined by (1.3). This condition is equivalent to the condition

$$B_0^* := \sup_{a<x<b} B_0\left(x; -\frac{1}{q}, -\frac{1}{p'}\right) < \infty. \tag{3.6}$$

This type of conditions was investigated in [5], [6] and [9], and Theorem 2.1 in [5] leads to the following result:

THEOREM 3.2. *Let* $p>1$, $q<0$. *Let* s, θ *and* ν *be real parameters such that* $\nu>0$, s, $\theta \geqslant 0$, *and* $s+\theta>0$.

Define

$$\begin{aligned} \overline{B}_1(x;s,\theta,\nu) &:= \tilde{U}^{\frac{s}{q}}(x) \left(\int_a^x u(t)\tilde{U}^{-\frac{s+1}{\nu q}-1}(t)V^{\frac{\theta-1}{\nu p'}}(t)dt \right)^{\nu} V^{-\frac{\theta}{p'}}(x), \\ \overline{B}_2(x;s,\theta,\nu) &:= \tilde{U}^{-\frac{s}{q}}(x) \left(\int_x^b u(t)\tilde{U}^{\frac{s-1}{\nu q}-1}(t)V^{-\frac{\theta+1}{\nu p'}}(t)dt \right)^{\nu} V^{\frac{\theta}{p'}}(x). \end{aligned} \tag{3.7}$$

Then the Hardy inequality (1.1) *holds for all measurable functions* $f \geqslant 0$ *if and only if any of the quantities* $\overline{B}_i(s,\theta,\nu) = \sup_{a<x<b} \overline{B}_i(x;s,\theta,\nu)$ $(i=1,2)$ *is finite. Moreover, for the best constant* C *in* (1.1) *we have* $C \approx \frac{1}{\overline{B}_i}, i=1,2$.

Let us note that for some special choice of the parameters s, θ and ν, we obtain expressions similar to those of Section 1 and Section 2:

$$\begin{aligned} B_1^*(s) &:= \sup_{a<x<b} \overline{B}_1\left(x; -sq, 0, -\frac{s+1}{q}\right) \\ &= \sup_{a<x<b} \tilde{U}^{-s}(x) \left(\int_a^x u(t)V^{\frac{q}{(1+s)p'}}(t)dt \right)^{-\frac{s+1}{q}}, \quad s>0; \\ B_2^*(s) &:= \sup_{a<x<b} \overline{B}_1\left(x; 0, s, -\frac{1}{q}\right) \\ &= \sup_{a<x<b} \left(\int_a^x u(t)V^{\frac{(1-sp')q}{p'}}(t)dt \right)^{-\frac{1}{q}} V^{-s}(x), \quad s>0; \end{aligned}$$

$$
\begin{aligned}
B_3^*(s) &:= \sup_{a<x<b} \overline{B}_2\left(x; s, 0, -\frac{sq+1}{q}\right) \\
&= \sup_{a<x<b} \tilde{U}^s(x)\left(\int_x^b u(t)V^{\frac{q}{(1+sq)p'}}(t)dt\right)^{-\frac{sq+1}{q}}, \quad s>0; \\
B_4^*(s) &:= \sup_{a<x<b} \overline{B}_2\left(x; 0, s, -\frac{sq+1}{q}\right) \\
&= \sup_{a<x<b} \left(\int_x^b u(t)V^{\frac{(1+sp')q}{p'}}(t)dt\right)^{-\frac{1}{q}} V^s(x), \quad s>0.
\end{aligned} \tag{3.8}
$$

Notice that B_0^* from (3.6) is a special case of $B_2^*(s):$ it is

$$
B_0^* = B_2^*\left(\frac{1}{p'}\right).
$$

• Let $p=1$. If we replace in $B_0(x; \frac{1}{q}, \frac{1}{p'})$ the term $V^{\frac{1}{p'}}(x)$ by $\operatorname*{supess}_{a<t<x} v(t)$, then (3.4) (i.e., (3.6)) is this time necessary and sufficient for the Hardy inequality (1.1) to hold. For details see again [6]. Also in this case, we can obtain equivalent conditions as in Theorem 3.2, replacing in (3.7) the term $V^{\frac{1}{p'}}(x)$ by $\operatorname*{supess}_{a<t<x} v(t)$.

4. Concluding remarks

(i) For completeness, let us mention that for the third quadrant, i.e., for

$$
p<0, \quad q<0, \tag{4.1}
$$

the problem was investigated in [1] and [9]. More precisely, for the case

$$
-\infty < q \leqslant p < 0,
$$

the necessary and sufficient condition for the *reverse* Hardy inequality to hold reads

$$
B := \sup_{a<x<b} B_0\left(x; -\frac{1}{q}, -\frac{1}{p'}\right) < \infty \tag{4.2}
$$

while for $-\infty < p < q < \infty$, the corresponding condition reads

$$
B := \left(\int_a^b \tilde{U}^{\frac{r}{p}}(x)V^{\frac{r}{p'}}(x)u(x)dx\right)^{\frac{1}{r}} < \infty \tag{4.3}
$$

where $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$. For details see [9].

Condition (4.2) was extended in [7] to

$$
B(s) := \sup_{a<x<b} \left(\int_a^x u(t)V^{\frac{p-s}{p}}(t)dt\right)^{-\frac{1}{q}} V^{\frac{1-s}{p}}(x) < \infty, \quad p<s<1, \tag{4.4}
$$

and in [5] results similar to that of Theorem 3.2 are presented.

(ii) The case

$$0 < p < 1, \quad 0 < q < 1$$

can be handled via duality, using the results mentioned in **(i)**. For details see again [1] and [9].

(iii) The case

$$1 < q < p < \infty \quad \text{and} \quad 0 < q < 1,\ p > 1$$

is also well known and the corresponding necessary and sufficient condition for the Hardy inequality (1.1) to hold reads

$$B_M := \left(\int_a^b U^{\frac{r}{q}}(x) V^{\frac{r}{q'}}(x) v^{1-p'}(x)dx \right)^{\frac{1}{r}} < \infty \tag{4.5}$$

where $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$.

An equivalent condition was obtained by L.-E. Persson and V. Stepanov; it reads

$$B_{PS} := \left(\int_0^\infty \left[\int_0^x u(t) V^q(t)dt \right]^{\frac{r}{q}} V^{-\frac{r}{q}}(x) v^{1-p'}(x)dx \right)^{\frac{1}{r}} < \infty \tag{4.6}$$

(see [KP, Theorem 1.2]). Both conditions, (4.5) and (4.6), have been extended to scales of conditions; for details see [8].

(iv) As far as concerns the remaining part of the $(p,q)-$plane, namely the strip

$$0 < p < 1,\ q > 1,$$

this case is investigated in [2]. The corresponding condition looks like:

$$A = \left(\int_a^b \tilde{V}^{\frac{r}{p'}}(t) dU^{\frac{r}{q}}(t) \right)^{\frac{1}{r}} + \frac{\tilde{V}^{\frac{1}{p'}}(a)}{U^{\frac{1}{q}}(a)} < \infty, \tag{4.7}$$

where $\frac{1}{r} := \frac{1}{p} - \frac{1}{q}$.

(v) Obviously, all results mentioned can be formulated via duality arguments for the case when the term $\int_a^x f(t)dt$ in (1.1) and (1.2) is replaced by $\int_x^b f(t)dt$.

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