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REVERSE CAUCHY—SCHWARZ INEQUALITIES FOR POSITIVE C^*–VALUED SESQUILINEAR FORMS

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. We prove two new reverse Cauchy–Schwarz inequalities of additive and multiplicative types in a space equipped with a positive sesquilinear form with values in a C^*-algebra. We apply our results to get some norm and integral inequalities. As a consequence, we improve a celebrated reverse Cauchy–Schwarz inequality due to G. Pólya and G. Szegö.

1. Introduction

The probably first reverse Cauchy–Schwarz inequality for positive real numbers $a_1,\cdots,a_n$ is the following one (see [15, p. 57 and 213–214] and [16, p. 71–72 and 253–255]):

THEOREM. [G. Pólya and G. Szegö (1925)] *Let* $a_1,\cdots,a_n$ *and* $b_1,\cdots,b_n$ *be positive real numbers. If*

$$0 < a \leqslant a_i \leqslant A < \infty, \quad 0 < b \leqslant b_i \leqslant B < \infty$$

for some constants a,b,A,B *and all* $1 \leqslant i \leqslant n$, *then*

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \leqslant \frac{(ab+AB)^2}{4abAB}\left(\sum_{i=1}^n a_i b_i\right)^2. \tag{1.1}$$

The inequality is sharp in the sense that $1/4$ *is the best possible constant.*

We remark that (1.1) can be obviously rewritten in the following equivalent form

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2 \leqslant \frac{(AB-ab)^2}{4abAB}\left(\sum_{i=1}^n a_i b_i\right)^2. \tag{1.2}$$

We say that (1.1) is the multiplicative form of the Pólya–Szegö inequality and that (1.2) is the additive form.

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There exist a lot of generalizations of this classical inequality. For example, Chapter 5 in [1] (36 pages) is devoted only to such reversed discrete Cauchy–Schwarz inequalities. Also similar results for integrals, isotone functionals as well as generalizations in the setting of inner product spaces are today well-studied and understood; see the books [3] and [4]. Moreover, C.P. Niculescu [13] and M. Joita [9] have proved some reverse Cauchy–Schwarz inequalities in the framework of C^*-algebras. We also refer to another interesting even newer paper by D. Ilisević and S. Varosanec [8] of this type; see also the book of T. Furuta, J.M. Hot, J.E. Pečarić and Y. Seo [6] and references therein.

In this paper we continue and complement this research by proving some new generalizations of both (1.1) and (1.2) in a similar framework (see Theorem 3.1 and Theorem 3.3). We also apply our results to get some norm and integral inequalities. As a consequence, we improve inequality (1.2).

2. Preliminaries

A C^**-algebra* is a Banach $*$-algebra $(\mathfrak{A}, \|\cdot\|)$ such that $\|a^*a\| = \|a\|^2$ for each $a \in \mathfrak{A}$. Recall that $a \in \mathfrak{A}$ is called *positive* (we write $a \geqslant 0$) if $a = b^*b$ for some $b \in \mathfrak{A}$. If $a \in \mathfrak{A}$ is positive, then there is a unique positive $b \in \mathfrak{A}$ such that $a = b^2$; such an element b is called the positive square root of a and denoted by $a^{1/2}$. For every $a \in \mathfrak{A}$, the positive square root of a^*a is denoted by $|a|$. For two self-adjoint elements a, b one can define a partial order $\leqslant$ by

$$a \leqslant b \Leftrightarrow b - a \geqslant 0.$$

For $a \in A$, by $\mathrm{Re}\, a$ we denote $\frac{a+a^*}{2}$.

Let $\mathfrak{A}$ be a C^*-algebra and let $\mathfrak{X}$ be an algebraic right $\mathfrak{A}$-module which is a complex linear space with $(\lambda x)a = x(\lambda a) = \lambda(xa)$ for all $x \in \mathfrak{X}$, $a \in \mathfrak{A}$, $\lambda \in \mathbb{C}$. The space $\mathfrak{X}$ is called a *(right) semi-inner product* $\mathfrak{A}$*-module* (or *semi-inner product* C^**-module over the* C^**-algebra* $\mathfrak{A}$) if there exists an $\mathfrak{A}$-valued inner product, i.e., a mapping $\langle\cdot,\cdot\rangle : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{A}$ satisfying

(i) $\langle x, x\rangle \geqslant 0$,

(ii) $\langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, z\rangle$,

(iii) $\langle x, ya\rangle = \langle x, y\rangle a$,

(iv) $\langle y, x\rangle = \langle x, y\rangle^*$,

for all $x, y, z \in \mathfrak{X}$, $a \in \mathfrak{A}$, $\lambda \in \mathbb{C}$. Moreover, if

(v) $x = 0$ whenever $\langle x, x\rangle = 0$,

then $\mathfrak{X}$ is called an *inner product* $\mathfrak{A}$*-module* (*inner product* C^**-module over the* C^**-algebra* $\mathfrak{A}$). In this case $\|x\| := \sqrt{\|\langle x, x\rangle\|}$ gives a norm on $\mathfrak{X}$ making it into a normed space, where the latter norm denotes that in the C^*-algebra $\mathfrak{A}$. If this normed space is complete, then $\mathfrak{X}$ is called a *Hilbert* $\mathfrak{A}$*-module* (*Hilbert* C^**-module over the* C^**-algebra* $\mathfrak{A}$). A left inner product $\mathfrak{A}$-module can be defined analogously. Any inner product (resp. Hilbert) space is a left inner product (resp. Hilbert) $\mathbb{C}$-module and any C^*-algebra $\mathfrak{A}$ is a right Hilbert C^*-module over itself via $\langle a,b\rangle = a^*b$, for all $a,b\in\mathfrak{A}$. For more details on inner product C^*-modules see [10] and [11].

The Cauchy–Schwarz inequality asserts that $\langle x,y\rangle\langle y,x\rangle \leqslant \|y\|^2\langle x,x\rangle$ in a semi-inner product module $\mathfrak{X}$ over a C^*-algebra $\mathfrak{A}$; see [10, Proposition 1.1] as well as [5]. This is a generalization of the classical Cauchy–Schwarz inequality stating that if x and y are elements of a semi-inner product space, then $|\langle x,y\rangle|^2 \leqslant \langle x,x\rangle\langle y,y\rangle$. There are mainly two types of reverse Cauchy–Schwarz inequality. In the additive approach (initiated by N. Ozeki [14]) we look for an inequality of the form $k+|\langle x,y\rangle|^2 \geqslant \langle x,x\rangle\langle y,y\rangle$ for some suitable positive constant k (see also (1.2)). In the multiplicative approach (initiated by G. Polya and G. Szegö [15]) we seek for an appropriate positive constant k such that $|\langle x,y\rangle|^2 \geqslant k\langle x,x\rangle\langle y,y\rangle$ (see also (1.1)).

In the next section we prove and discuss some reverse Cauchy–Schwarz inequalities of additive and multiplicative types in a linear space equipped with a positive sesquilinear form with values in a C^*-algebra. For a comprehensive account on Cauchy–Schwarz inequality and its various inverses we refer the reader to books [3] and [4].

3. The main results

Let $\mathfrak{A}$ be a C^*-algebra and let $\mathfrak{X}$ be a linear space. By an $\mathfrak{A}$-valued positive sesquilinear form we mean a mapping $\langle\cdot,\cdot\rangle : \mathfrak{X}\times\mathfrak{X}\to\mathfrak{A}$ which is linear in the first variable and conjugate linear in the second and fulfills $\langle x,x\rangle \geqslant 0 \quad (x\in\mathfrak{X})$.

Our first result of this section is the following additive reverse Cauchy–Schwarz inequality:

THEOREM 3.1. *Let* $\mathfrak{A}$ *be a* C^**-algebra and let* $\mathfrak{X}$ *be a linear space equipped with an* $\mathfrak{A}$*-valued positive sesquilinear form* $\langle\cdot,\cdot\rangle : \mathfrak{X}\times\mathfrak{X}\to\mathfrak{A}$. *Suppose that* $x,y\in\mathfrak{X}$ *are such that*

$$\langle x,y\rangle^* = \langle y,x\rangle \tag{3.1}$$

$$\langle y,y\rangle^{1/2}\langle x,y\rangle = \langle x,y\rangle\langle y,y\rangle^{1/2} \tag{3.2}$$

$$Re\,\langle \Omega y - x, x-\omega y\rangle \geqslant 0 \tag{3.3}$$

for some $\omega,\Omega\in\mathbb{C}$. *Then*

$$\left|\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2}\right|^2 - |\langle y,x\rangle|^2 \leqslant \frac{1}{4}|\Omega-\omega|^2\langle y,y\rangle^2. \tag{3.4}$$

REMARK 3.2. The constant $1/4$ in (3.4) can not in general be replaced by some smaller numbers (see the proof for the special case considered in Proposition 3.4).

Proof. Set

$$\begin{aligned} D_1 &:= \mathrm{Re}[(\Omega\langle y,y\rangle - \langle x,y\rangle)(\langle y,x\rangle - \bar{\omega}\langle y,y\rangle)] \\ &= -\mathrm{Re}(\Omega\bar{\omega})\langle y,y\rangle^2 - \langle x,y\rangle\langle y,x\rangle + \mathrm{Re}[\Omega\langle y,y\rangle\langle y,x\rangle + \bar{\omega}\langle x,y\rangle\langle y,y\rangle]. \end{aligned}$$

It follows from (3.3) that

$$D_2 := \langle y,y\rangle^{1/2}\mathrm{Re}\,\langle \Omega y - x, x - \omega y\rangle \langle y,y\rangle^{1/2} \geqslant 0\,.$$

Hence, by also using (3.2), we find that

$$\begin{aligned} D_2 &= -\mathrm{Re}(\Omega\bar{\omega})\langle y,y\rangle^2 - \langle y,y\rangle^{1/2}\langle x,x\rangle\langle y,y\rangle^{1/2} \\ &\quad + \langle y,y\rangle^{1/2}\mathrm{Re}[\Omega\langle y,x\rangle + \bar{\omega}\langle x,y\rangle]\langle y,y\rangle^{1/2} \\ &= -\mathrm{Re}(\Omega\bar{\omega})\langle y,y\rangle^2 - \langle y,y\rangle^{1/2}\langle x,x\rangle\langle y,y\rangle^{1/2} \\ &\quad + \mathrm{Re}[\Omega\langle y,y\rangle\langle y,x\rangle + \bar{\omega}\langle x,y\rangle\langle y,y\rangle]\,. \end{aligned}$$

Therefore $D_1 - D_2 \leqslant D_1$ and we conclude that

$$\begin{aligned} \langle y,y\rangle^{1/2}\langle x,x\rangle\langle y,y\rangle^{1/2} - \langle x,y\rangle\langle y,x\rangle &\leqslant \mathrm{Re}[(\Omega\langle y,y\rangle - \langle x,y\rangle)(\langle y,x\rangle - \bar{\omega}\langle y,y\rangle)] \\ &\leqslant \frac{1}{4}\left|(\overline{\Omega} - \overline{\omega})\langle y,y\rangle\right|^2\,. \end{aligned}$$

The last inequality is obtained by applying (3.1) and the elementary inequality $\mathrm{Re}(u^*v) \leqslant \frac{1}{4}|u+v|^2 \quad (u,v \in \mathfrak{A})$ for $u = \overline{\Omega}\langle y,y\rangle - \langle y,x\rangle$ and $v = \langle y,x\rangle - \overline{\omega}\langle y,y\rangle$. The proof is complete. $\square$

Our multiplicative reverse Cauchy–Schwarz inequality reads as follows:

THEOREM 3.3. *Let $\mathfrak{A}$ be a C^*-algebra and let $\mathfrak{X}$ be a linear space equipped with an $\mathfrak{A}$-valued positive sesquilinear form $\langle\cdot,\cdot\rangle : \mathfrak{X}\times\mathfrak{X} \to \mathfrak{A}$. Suppose that $x,y \in \mathfrak{X}$ are such that* (3.1) *holds, $\langle x,y\rangle$ is normal and* (3.3) *holds for some $\omega,\Omega \in \mathbb{C}$ with $Re(\bar{\omega}\Omega) > 0$. Then*

$$\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2} + \langle y,y\rangle^{1/2}\langle x,x\rangle^{1/2} \leqslant \frac{|\Omega| + |\omega|}{[Re(\bar{\omega}\Omega)]^{1/2}}\,|\langle x,y\rangle|\,. \tag{3.5}$$

Proof. It follows from (3.3) that

$$\mathrm{Re}(\Omega\langle x,y\rangle^* + \bar{\omega}\langle x,y\rangle) - \langle x,x\rangle - [\mathrm{Re}(\bar{\omega}\Omega)]\langle y,y\rangle \geqslant 0\,.$$

Moreover, since $\langle x,y\rangle$ is normal, $Re(\langle x,y\rangle) \leqslant |\langle x,y\rangle|$ so that

$$\begin{aligned} \frac{1}{[\mathrm{Re}(\bar{\omega}\Omega)]^{1/2}}\langle x,x\rangle + [\mathrm{Re}(\bar{\omega}\Omega)]^{1/2}\langle y,y\rangle &\leqslant \frac{\mathrm{Re}(\Omega\langle x,y\rangle^* + \bar{\omega}\langle x,y\rangle)}{[\mathrm{Re}(\bar{\omega}\Omega)]^{1/2}} \\ &\leqslant \frac{|\Omega| + |\omega|}{[\mathrm{Re}(\bar{\omega}\Omega)]^{1/2}}\,|\langle x,y\rangle|\,. \end{aligned} \tag{3.6}$$

Furthermore, the trivial estimate

$$\left(\frac{1}{[\mathrm{Re}(\bar{\omega}\Omega)]^{1/4}}\langle x,x\rangle^{1/2} - [\mathrm{Re}(\bar{\omega}\Omega)]^{1/4}\langle y,y\rangle^{1/2}\right)^2 \geqslant 0$$

implies that

$$\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2} + \langle y,y\rangle^{1/2}\langle x,x\rangle^{1/2} \leqslant \frac{1}{[\mathrm{Re}(\bar{\omega}\Omega)]^{1/2}}\langle x,x\rangle + [\mathrm{Re}(\bar{\omega}\Omega)]^{1/2}\langle y,y\rangle. \tag{3.7}$$

By combining (3.6) and (3.7) we obtain (3.5) and the proof is complete. □

PROPOSITION 3.4. *Let φ be a positive linear functional on a C^*-algebra $\mathfrak{A}$ and let $x,y\in\mathfrak{A}$ be such that*

$$Re\,\varphi((x-\omega y)^*(\Omega y - x)) \geqslant 0$$

for some $\omega,\Omega\in\mathbb{C}$.

(a) Then

$$\varphi(x^*x)\varphi(y^*y) - |\varphi(y^*x)|^2 \leqslant \frac{1}{4}|\Omega-\omega|^2\varphi(y^*y)^2. \tag{3.8}$$

(b) Moreover, if for each $y\in\mathfrak{A}$ there exists an element $z\in\mathfrak{A}$ such that $\varphi(z^*y)=0$, then the constant $\frac{1}{4}$ is sharp in (3.8).

(c) Furthermore, if $Re(\bar{\omega}\Omega)>0$, then

$$\varphi(x^*x)^{1/2}\varphi(y^*y)^{1/2} \leqslant \frac{1}{2}\frac{|\Omega|+|\omega|}{[Re(\bar{\omega}\Omega)]^{1/2}}\,|\varphi(y^*x)|. \tag{3.9}$$

REMARK 3.5. Proposition 3.4 (a) and (b) is related to Theorem 1 of [2], where the case with inner product spaces was considered.

Proof. (a) By defining $\langle u,v\rangle := \varphi(v^*u)$ one obtains a positive sesquilinear form from $\mathfrak{A}\times\mathfrak{A}$ into $\mathbb{C}$. Equality (3.1) holds by [12, p. 88] and equality (3.2) is trivially fulfilled. Thus Theorem 3.1 gives the additive type reverse Cauchy–Schwarz inequality (3.8) and Theorem 3.3 yields the multiplicative type reverse Cauchy–Schwarz inequality (3.9). It remains to prove the sharpness assertion in (b). In fact, let $y\in\mathfrak{A}$ with $\varphi(y^*y)=1$. Choose $z\in\mathfrak{A}$ with $\varphi(z^*z)=1$ and $\varphi(z^*y)=0$. Put $x=\frac{\Omega+\omega}{2}y+\frac{\Omega-\omega}{2}z$. Then

$$\begin{aligned}\varphi((x-\omega y)^*(\Omega y - x)) &= \varphi\left(\left(\frac{\Omega-\omega}{2}y+\frac{\Omega-\omega}{2}z\right)^*\left(\frac{\Omega-\omega}{2}y-\frac{\Omega-\omega}{2}z\right)\right)\\ &= \left|\frac{\Omega-\omega}{2}\right|^2\varphi(y^*y-z^*z)\\ &= 0.\end{aligned}$$

Hence $\mathrm{Re}\varphi((x-\omega y)^*(\Omega y - x))\geqslant 0$ holds. If

$$\varphi(x^*x)\varphi(y^*y) - |\varphi(y^*x)|^2 \leqslant C|\Omega-\omega|^2\varphi(y^*y)^2,$$

for some $C \geqslant 0$, then

$$\begin{aligned}\left|\frac{\Omega-\omega}{2}\right|^2 &= \varphi\left(\left|\frac{\Omega+\omega}{2}\right|^2 y^*y + \left|\frac{\Omega-\omega}{2}\right|^2 z^*z\right) - \left|\frac{\Omega+\omega}{2}\right|^2 \varphi(y^*y)\\ &= \varphi(x^*x) - |\varphi(y^*x)|^2\\ &\leqslant C|\Omega-\omega|^2,\end{aligned}$$

from which we conclude that $1/4 \leqslant C$. The proof is complete. □

REMARK 3.6. Let φ be a positive linear functional on a C^*-algebra $\mathfrak{A}$, x,y be self-adjoint elements of $\mathfrak{A}$ such that $\omega y \leqslant x \leqslant \Omega y$ for some scalars $\omega, \Omega > 0$. Then

$$\varphi((x-\omega y)^*(\Omega y - x)) \geqslant 0$$

if $xy = yx$ (in particular, when $\mathfrak{A}$ is commutative), since

$$\varphi((x-\omega y)(\Omega y - x)) = \varphi((\Omega y - x)^{1/2}(x-\omega y)(\Omega y - x)^{1/2}) \geqslant 0.$$

In particular, if x and y are commuting strictly positive elements of a unital C^*-algebra $\mathfrak{A}$, one may consider $\omega = \frac{\inf\sigma(x)}{\sup\sigma(y)}$ and $\Omega = \frac{\sup\sigma(x)}{\inf\sigma(y)}$, where $\sigma(a)$ denotes the spectrum of $a \in \mathfrak{A}$; cf. [13].

We can apply Proposition 3.4 and Remark 3.6 to derive some new inequalities as well as some well-known ones. We only give the following such results:

COROLLARY 3.7. *Let $\mathscr{H}$ be a Hilbert space and $T,S \in \mathbb{B}(\mathscr{H})$ be strictly positive operators such that $TS = ST$. Then*

$$\|Tx\|^2\|Sx\|^2 - |\langle Tx, Sx\rangle|^2 \leqslant \left(\frac{\sup\sigma(T)\sup\sigma(S) - \inf\sigma(T)\inf\sigma(S)}{2}\right)^2 \min\left\{\frac{\|Sx\|^4}{\sup\sigma(S)^2\inf\sigma(S)^2}, \frac{\|Tx\|^4}{\sup\sigma(T)^2\inf\sigma(T)^2}\right\}$$

and

$$\|Tx\|\|Sx\| \leqslant \frac{1}{2}\left(\sqrt{\frac{\inf\sigma(T)\inf\sigma(S)}{\sup\sigma(T)\sup\sigma(S)}} + \sqrt{\frac{\sup\sigma(T)\sup\sigma(S)}{\inf\sigma(T)\inf\sigma(S)}}\right)|\langle Tx, Sx\rangle|$$

for all $x \in \mathscr{H}$.

Proof. It is sufficient to set $\mathfrak{A} = \mathbb{B}(\mathscr{H})$, to consider $\varphi(R) := \langle Rx, x\rangle \quad (R \in \mathbb{B}(\mathscr{H}))$ and to apply Proposition 3.4 and Remark 3.6. □

COROLLARY 3.8. *Let* (X,Σ,μ) *be a probability space and* $f,g \in L^\infty(\mu)$ *with* $0 < a \leqslant f \leqslant A,\ 0 < b \leqslant g \leqslant B$. *Then*

$$\int_X f^2 d\mu \int_X g^2 d\mu - \left(\int_X fg d\mu\right)^2 \leqslant \frac{(AB-ab)^2}{4} \min\left\{\frac{1}{B^2b^2}\left(\int_X g^2 d\mu\right)^2, \frac{1}{A^2a^2}\left(\int_X f^2 d\mu\right)^2\right\} \tag{3.10}$$

and

$$\left(\int_X f^2 d\mu\right)^{1/2}\left(\int_X g^2 d\mu\right)^{1/2} \leqslant \frac{1}{2}\left(\sqrt{\frac{ab}{AB}}+\sqrt{\frac{AB}{ab}}\right)\int_X fg d\mu. \tag{3.11}$$

Proof. It is enough to assume $\mathfrak{A}$ to be the commutative C^*-algebra $L^\infty(X,\mu)$, to consider $\varphi(h) := \int_X h d\mu \quad (h \in L^\infty(\mu))$, use Proposition 3.4 and Remark 3.6 and an obvious symmetry argument. $\square$

REMARK 3.9. The second inequality of Corollary 3.8 is due to C.N. Niculescu [13].

By applying (3.11) with a weighted counting measure $\mu = \sum_{i=1}^n w_i\delta_i$, where w_i's are positive numbers and δ_i's are the Dirac delta functions, we obtain with change of notation that if $a_1,\cdots,a_n$ and $b_1,\cdots,b_n$ satisfy the conditions in the Pólya–Szegö theorem, then

$$\sum_{i=1}^n a_i^2 w_i \sum_{i=1}^n b_i^2 w_i \leqslant \frac{(AB+ab)^2}{4ABab}\left(\sum_{i=1}^n a_i b_i w_i\right)^2,$$

which is the Greub–Rheinboldt inequality [7]. In the same way, from (3.10) it follows that

$$\sum_{i=1}^n a_i^2 w_i \sum_{i=1}^n b_i^2 w_i - \left(\sum_{i=1}^n a_i b_i w_i\right)^2 \leqslant \frac{(AB-ab)^2}{4}\min\left\{\frac{1}{B^2b^2}\left(\sum_{i=1}^n b_i^2 w_i\right)^2, \frac{1}{A^2a^2}\left(\sum_{i=1}^n a_i^2 w_i\right)^2\right\}. \tag{3.12}$$

In particular, by using this inequality with $w_i = 1\ (i = 1,\cdots,n)$, we get the following strict improvement of the Pólya–Szegö inequality (1.2).

COROLLARY 3.10. *Suppose that* $a_1,\cdots,a_n$ *and* $b_1,\cdots,b_n$ *are positive real numbers such that* $0 < a \leqslant a_i \leqslant A < \infty, 0 < b \leqslant b_i \leqslant B < \infty$ *for some constants* a,b,A,B

and all $1 \leqslant i \leqslant n$. *Then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \tag{3.13}$$
$$\leqslant \frac{(AB-ab)^2}{4} \min\left\{ \frac{1}{A^2a^2}\left(\sum_{i=1}^{n} a_i^2\right)^2, \frac{1}{B^2b^2}\left(\sum_{i=1}^{n} b_i^2\right)^2, \frac{1}{abAB}\left(\sum_{i=1}^{n} a_i b_i\right)^2 \right\}.$$

Any of the constants in the bracket above can be the strictly least one.

Proof. The inequality (3.13) follows by just combining (1.2) with (3.12). Moreover, the proof of the final statement is as follows: By choosing $n > 1, a = 1, a_i = A = n, b_i = b = B = 1/n$ $(1 \leqslant i \leqslant n)$ we find that the second constant is strictly less than the first one and the third one. Analogously, by choosing $n > 1, a = a_i = A = 1/n, b = 1, b_i = B = n$ $(1 \leqslant i \leqslant n)$ we find that the first constant is strictly less than the second and the third one. We also note that by the Schwarz inequality,

$$\frac{1}{abAB}\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leqslant \frac{1}{abAB}\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \tag{3.14}$$

and obviously

$$\frac{1}{abAB}\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leqslant \min\left\{ \frac{1}{A^2a^2}\left(\sum_{i=1}^{n} a_i^2\right)^2, \frac{1}{B^2b^2}\left(\sum_{i=1}^{n} b_i^2\right)^2 \right\}$$

if and only if

$$\frac{1}{Aa}\sum_{i=1}^{n} a_i^2 = \frac{1}{Bb}\sum_{i=1}^{n} b_i^2. \tag{3.15}$$

We conclude that the third term is strictly less than the first two whenever (3.15) holds and we have strict inequality in the Schwarz inequality (3.14). Hence our claim is proved. □

REMARK 3.11. By combining (3.12) with the Greub–Rheinboldt inequality we can in a similar way derive also a weighted version of Corollary 3.10 (and, thus, also strictly improve the Greub–Rheinboldt inequality).

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