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REMARKS ON t–QUASICONVEX FUNCTIONS

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Dedicated to Professor Josip Pečarić
on his 60th birthday

Abstract. Given a convex subset D of a vector space and a constant $t \in (0,1)$, a function $f : D \to \mathbb{R}$ is called t*-quasiconvex* if, for all $x,y \in D$,

$$f(tx+(1-t)y) \leqslant \max\{f(x),f(y)\};$$

f is called *strictly* t*-quasiconvex* if, for all $x,y \in D$ such that $f(x) \neq f(y)$,

$$f(tx+(1-t)y) < \max\{f(x),f(y)\}.$$

The following Kuhn–type theorem is proved: If f is t-quasiconvex and strictly t-quasiconvex then it is $(1/2)$-quasiconvex and strictly $(1/2)$-quasiconvex. It is also shown that lower semicontinuous strictly t-quasiconvex functions are quasiconvex, which generalizes the well-known Karamardian's theorem.

1. Introduction

Let D be a convex subset of a vector space X and $t \in (0,1)$ be a fixed constant. A function $f : D \to \mathbb{R}$ is called t*-quasiconvex* if, for all $x,y \in D$,

$$f(tx+(1-t)y) \leqslant \max\{f(x),f(y)\};$$

f is *midpoint–quasiconvex* if it is $(1/2)$-quasiconvex; f is *quasiconvex* if it is t-quasiconvex for all $t \in (0,1)$ (cf. [8], [3], [4], [13]).

A function $f : D \to \mathbb{R}$ is called *strictly* t*-quasiconvex* if, for all $x,y \in D$ such that $f(x) \neq f(y)$,

$$f(tx+(1-t)y) < \max\{f(x),f(y)\};$$

f is *strictly midpoint–quasiconvex* if it is strictly $(1/2)$-quasiconvex; f is *strictly quasiconvex* if it is strictly t-quasiconvex for all $t \in (0,1)$ (cf. [9], [3], [4], [8]).

Many properties and interrelations between the functions defined above can be found, among others, in [3], [4], [6], [7], [8], [9], [12], [13].

Recall also that a function $f : D \to \mathbb{R}$ is called t*-convex* if, for all $x,y \in D$,

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y);$$

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f is *midpoint–convex* (or *Jensen–convex*) if it is $(1/2)$-convex; f is *convex* if it is t-convex for all $t \in (0,1)$ (cf. [1], [10], [11], [13]).

Clearly, every convex (t-convex) function is both quasiconvex (t-quasiconvex) and strictly quasiconvex (strictly t-quasiconvex), but not conversely. Note also that quasiconvexity does not imply strict quasiconvexity as well as strict quasiconvexity does not imply quasiconvexity (cf. [9], [8]).

It is known by the theorem of Kuhn [11] that every t-convex function is midpoint–convex. Analogous result does not hold for quasiconvexity: Neither t-quasiconvexity implies midpoint–quasiconvexity, nor strict t-quasiconvexity implies strict midpoint–quasiconvexity. However, if a function is both t-quasiconvex and strictly t-quasiconvex then it is midpoint quasiconvex and strictly midpoint–quasiconvex. This is our main result in Section 2. In Section 3 we generalize the well–known Karamardian's theorem [9] stating that lower semicontinuous strictly quasiconvex functions are quasiconvex. Behringer [3] proved that also lower semicontinuous and strictly midpoint–quasiconvex functions are quasiconvex. We will show that the same holds for t-quasiconvexity: Every lower semicontinuous strictly t-quasiconvex function is quasiconvex.

2. Kuhn–type results for quasiconvex functions

In 1984 N. Kuhn [11] proved that every t-convex function (with a fixed $t \in (0,1)$) is midpoint–convex. Consequently, t-convex functions are $\mathbb{Q}$-convex (i.e. t-convex with any rational $t \in (0,1)$), because midpoint–convex functions are $\mathbb{Q}$-convex (cf. [1], [10], [13]). In this section we consider analogous problems for quasiconvexity. We give examples showing that t-quasiconvex functions need not be midpoint–quasiconvex as well as strict t-quasiconvex functions need not be strict midpoint–quasiconvex. We will prove, however, that if a function is both t-quasiconvex and strictly t-quasiconvex then it is midpoint quasiconvex and strictly midpoint–quasiconvex.

EXAMPLE 1. (cf. [12]) Consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}0, & \text{if } x=\frac{k}{3^n}, \quad k\in\mathbb{Z},\ n\in\mathbb{N},\\ 1, & \text{otherwise.}\end{cases}$$

Then f is $(1/3)$-quasiconvex, but it is not midpoint–quasiconvex.

EXAMPLE 2. Consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}0, & \text{if } x\notin A\\ 1, & \text{if } x\in A\end{cases}$$

where

$$A=\left\{\frac{k}{2^n},\ k\in\mathbb{Z},\ 3\nmid k,\ n\in\mathbb{N}\right\}$$

(the symbol $3\nmid k$ denotes that k is not divisible by 3). Observe that f is not strictly $(1/2)$-quasiconvex, because $f(0)=0\neq 1=f(1)$ and $f(1/2)=1=\max\{f(0),f(1)\}$.

On the other hand f is strictly $(1/3)$-quasiconvex. To prove this take any $x,y\in\mathbb{R}$. The case when both $x,y\in A$ or $x,y\notin A$ is trivial. Without loss of generality we assume that $x\in A$ (let $x=\frac{k}{2^n}$, $3\nmid k$) and $y\notin A$. It is enough to show that $u=\frac{1}{3}x+\frac{2}{3}y\notin A$ and $v=\frac{2}{3}x+\frac{1}{3}y\notin A$, whence $f(u)=f(v)=0<1=\max\,\{f(x),f(y)\}$. On the contrary, suppose that $u=\frac{1}{3}x+\frac{2}{3}y\in A$, i.e. $u=\frac{p}{2^r}$ for some $p\in\mathbb{Z}$, $3\nmid p$, $r\in\mathbb{N}$. Then

$$y=\frac{2^n 3p-2^r k}{2^{n+r+1}}\,.$$

Since $3\nmid(2^n3p-2^rk)$, this means that $y\in A$, a contradiction. Thus $u\notin A$. Using the same method we prove that $v\notin A$.

THEOREM 1. *Let D be a convex subset of a vector space and $t\in(0,1)$ be a fixed constant. If a function $f:D\to\mathbb{R}$ is t-quasiconvex and strictly t-quasiconvex then it is midpoint quasiconvex and strictly midpoint–quasiconvex.*

Proof. Fix $x,y\in D$, $x\neq y$, and put

$$u:=tx+(1-t)\frac{x+y}{2} \qquad\text{and}\qquad v:=t\frac{x+y}{2}+(1-t)y. \tag{1}$$

Then, we have the following identity (cf. [5]):

$$\frac{x+y}{2}=(1-t)u+tv. \tag{2}$$

We will consider two cases.

Case 1: $f(x)=f(y)$.

We have to prove that

$$f\Big(\frac{x+y}{2}\Big)\leqslant\max\,\{f(x),f(y)\}=f(x).$$

Suppose, on the contrary, that $f(\frac{x+y}{2})>f(x)$. Then also $f(\frac{x+y}{2})>f(y)$. By (1) and the strict t-quasiconvexity

$$f(u)<f\Big(\frac{x+y}{2}\Big) \qquad\text{and}\qquad f(v)<f\Big(\frac{x+y}{2}\Big).$$

Hence, using (2) and t-quasiconvexity, we obtain

$$f\Big(\frac{x+y}{2}\Big)\leqslant\max\,\{f(u),f(v)\}<f\Big(\frac{x+y}{2}\Big),$$

which is an obvious contradiction.

Case 2: $f(x)\neq f(y)$.

Without loss of generality we may assume that $f(x)<f(y)$.

We have to prove that

$$f\Big(\frac{x+y}{2}\Big)<f(y).$$

Suppose, on the contrary, that $f(\frac{x+y}{2}) \geqslant f(y)$. We have two possibilities:

(i) $f\left(\frac{x+y}{2}\right) > f(y)$. Then also $f\left(\frac{x+y}{2}\right) > f(x)$. Hence, by (1) and the strict t-quasiconvexity,

$$f(v) < f\left(\frac{x+y}{2}\right) \qquad \text{and} \qquad f(u) < f\left(\frac{x+y}{2}\right).$$

Consequently, using (2) and t-quasiconvexity, we obtain

$$f\left(\frac{x+y}{2}\right) \leqslant \max\{f(u), f(v)\} < f\left(\frac{x+y}{2}\right),$$

a contradiction.

(ii) $f\left(\frac{x+y}{2}\right) = f(y)$. Then $f\left(\frac{x+y}{2}\right) > f(x)$. Hence, by (1), t-quasiconvexity and strict t-quasiconvexity, we get

$$f(v) \leqslant f\left(\frac{x+y}{2}\right) \qquad \text{and} \qquad f(u) < f\left(\frac{x+y}{2}\right).$$

Now, if $f(v) < f\left(\frac{x+y}{2}\right)$, then, using (2) and t-quasiconvexity, we obtain

$$f\left(\frac{x+y}{2}\right) \leqslant \max\{f(u), f(v)\} < f\left(\frac{x+y}{2}\right),$$

a contradiction.
If $f(v) = f\left(\frac{x+y}{2}\right)$, then $f(u) < f(v)$ and, using once more (2) and strict t-quasiconvexity, we obtain

$$f\left(\frac{x+y}{2}\right) < \max\{f(u), f(v)\} = f(v) = f\left(\frac{x+y}{2}\right),$$

a contradiction. This finishes the proof.

Let D be a convex subset of a vector space. A function $f : D \to \mathbb{R}$ is called $\mathbb{Q}$*-quasiconvex* (*strictly* $\mathbb{Q}$*-quasiconvex*) if it is t-quasiconvex (strictly t-quasiconvex) for all rational $t \in (0,1)$. It is known that midpoint–quasiconvexity does not imply $\mathbb{Q}$-quasiconvexity, and strict midpoint–quasiconvexity does not imply strict $\mathbb{Q}$-quasiconvexity (cf. [3], [4]). However, it has been proved by Behringer ([3], Thm. 4.6) that midpoint–quasiconvexity together with strict midpoint–quasiconvexity implies $\mathbb{Q}$-quasiconvexity and strict $\mathbb{Q}$-quasiconvexity. Combining this result with Theorem 1 we obtain the following corollary.

COROLLARY 1. *Let D be a convex subset of a vector space and $t \in (0,1)$. If a function $f : D \to \mathbb{R}$ is t-quasiconvex and strictly t-quasiconvex then it is $\mathbb{Q}$-quasiconvex and strictly $\mathbb{Q}$-quasiconvex.*

In [3] Behringer proved that midpoint–quasiconvexity together with strict quasiconvexity implies quasiconvexity ([3], Thm. 3.2), and strict midpoint–quasiconvexity together with quasiconvexity implies strict quasiconvexity ([3], Thm. 3.1). As a consequence of these results and our Theorem 1 we get their generalizations.

COROLLARY 2. *Let D be a convex subset of a vector space and $t \in (0,1)$. If a function $f : D \to \mathbb{R}$ is t-quasiconvex and strictly quasiconvex then it is quasiconvex. If a function $f : D \to \mathbb{R}$ is strictly t-quasiconvex and quasiconvex then it is strictly quasiconvex.*

3. A generalization of the Karamardian theorem

It is known that strictly quasiconvex functions need not be, in general, quasiconvex. However, under the additional assumption of lower semicontinuity, strict quasiconvexity implies quasiconvexity. This well-known result was proved by Karamardian [9] in 1967 (for real–valued functions on a convex subset of $\mathbb{R}^n$). In 1978 Behringer [3] showed that Karamardian's theorem still holds for strictly midpoint–quasiconvex functions. More precisely, he proved that strictly midpoint–quasiconvex and lower semicontinuous functions (defined on a convex subset of a normed space and having values in a connected quasi-ordered space) are quasiconvex. In this section we present a further generalization of Karamardian's theorem. We prove that every strictly t-quasiconvex and lower semicontinuous function is quasiconvex. We give the proof in the case where f is a real-valued function defined on a convex subset of a topological vector space, but the same proof holds if f has values in a connected quasi-ordered space.

THEOREM 2. *Let D be a convex subset of a (Hausdorff) topological vector space and $t \in (0,1)$ be a fixed constant. If a function $f : D \to \mathbb{R}$ is lower semicontinuous and strictly t-quasiconvex then it is quasiconvex.*

Proof. Fix a number $c \in \mathbb{R}$ and define the level set

$$L_c = \{x \in D : f(x) \leqslant c\}.$$

Since a function is quasiconvex if and only if all its level sets are convex (cf. e.g. [13], [8]), it suffices to prove that L_c is convex. By the lower semicontinuity of f the set L_c is closed in D. We will show that L_c is weakly convex, that is for every $x, y \in L_c$, $x \neq y$, there exists a $p \in (0,1)$ (dependent on x and y) such that

$$px + (1-p)y \in L_c \,. \tag{3}$$

Then the proof will be completed because closed and weakly convex sets are convex (cf. [2]). Take $x, y \in L_c$, $x \neq y$. By the definition of L_c, $f(x) \leqslant c$ and $f(y) \leqslant c$. We will consider two cases.

Case 1: $f(x) \neq f(y)$.

Then, by the strict t-quasiconvexity of f, we have

$$f(tx + (1-t)y) < \max\{f(x), f(y)\} \leqslant c \,,$$

which means that $tx + (1-t)y \in L_c$. Thus (3) holds with $p = t$.

Case 2: $f(x) = f(y)$.

Suppose, contrary to (3), that, for every $p \in (0,1)$,

$$f(px + (1-p)y) > \max\{f(x), f(y)\} = f(x) = f(y) \,. \tag{4}$$

Define points

$$x_1 = \frac{1}{2-t}x + \frac{1-t}{2-t}y \qquad \text{and} \qquad y_1 = \frac{1-t}{2-t}x + \frac{1}{2-t}y.$$

Then

$$x_1 = tx + (1-t)y_1 \qquad \text{and} \qquad y_1 = (1-t)x_1 + ty, \tag{5}$$

and, by (4),

$$f(x_1) > f(y) \qquad \text{and} \qquad f(y_1) > f(x). \tag{6}$$

Now, by (5), (6) and the strict t-quasiconvexity, we get

$$f(x_1) < \max\{f(x), f(y_1)\} = f(y_1)$$

and

$$f(y_1) < \max\{f(x_1), f(y)\} = f(x_1).$$

This obvious contradiction shows that L_c is weakly convex and finishes the proof.

As an immediate consequence of the above Theorem 2 and Corollary 2 we get the following stronger version of this theorem.

COROLLARY 3. *Let D be a convex subset of a (Hausdorff) topological vector space and $t \in (0,1)$ be a fixed constant. If a function $f : D \to \mathbb{R}$ is lower semicontinuous and strictly t-quasiconvex then it is quasiconvex and strictly quasiconvex.*

REMARK 1. In a similar way as in the above theorem we can prove that if a function $f : D \to \mathbb{R}$ is lower semicontinuous and t-quasiconvex then it is quasiconvex (the level sets of f are convex because they are closed and t-convex). For $t = 1/2$ this result was obtained by Behringer ([3], Thm. 2.2). Also upper semicontinuous t-quasiconvex functions are quasiconvex (cf. ([7], Thm. 2, Cor. 3). However, even continuous t-quasiconvex functions need not be strictly quasiconvex. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \min\{x, 0\}$, $x \in \mathbb{R}$, is continuous and quasiconvex but not strictly t-quasiconvex with any $t \in (0,1)$.

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