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INEQUALITIES FOR THE C^*–VALUED NORM ON A HILBERT C^*–MODULE

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Dedicated to Professor Josip Pečarić on the occasion of his 60th birthday

Abstract. The C^*-valued norm is defined on a Hilbert C^*-module by its standard inner product. In this paper we give generalizations of a number of classical inequalities known for either complex numbers or Hilbert space operators. In particular, we study Bohr's inequality for the C^*-valued norm on a Hilbert C^*-module.

1. Preliminaries and introduction

Let A be a C^*-algebra i. e. a Banach linear space with an involution $*$ and the C^*-norm property for a norm $\|\cdot\|$ on A: $\|a^*a\| = \|a\|^2, a \in A$. A left pre-Hilbert C^*-module V over a C^*-algebra A is a complex vector space and a left A-module equipped with an A-valued inner product $(\cdot,\cdot) : V \times V \to A$ with the following properties: for $x, y \in V, a \in A$; $\alpha, \beta \in \mathbf{C}$

1. $(\alpha x_1 + \beta x_2, y) = \alpha (x_1, y) + \beta (x_2, y)$,
2. $(ax, y) = a (x, y)$,
3. $(x, y)^* = (y, x)$,
4. $(x, x) \geqslant 0$ and $(x, x) = 0$ if and only if $x = 0$.

If V is complete with respect to the norm $\|x\| = \|(x,x)\|^{1/2}$, then it is called a (left) Hilbert C^*-module. The scalar multiplication on V is compatible with the module action (see [8],15.A) i. e. $\lambda(ax) = (\lambda a)x = a(\lambda x)$ for all $\lambda \in \mathbf{C}, a \in A, x \in V$.

Recall that an element $a \in A$ is positive if a is selfadjoint with a positive real spectrum, or equivalently, a is of the form b^*b for some $b \in A$. We write $a \geqslant 0$ if a is positive. For selfadjoint $a, b \in A$ we write $a \leqslant b$ if $b - a \geqslant 0$. Every positive element of a C^*-algebra has a unique positive square root.

Besides the standard $\mathbf{C}$-valued norm, there is the C^*-valued norm on V defined by

$$|x| = (x,x)^{\frac{1}{2}}, \ x \in V.$$

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For every $x \in V$, $|x|$ is positive and for every $\lambda \in \mathbf{C}$ we have $|\lambda x| = |\lambda||x|$. However, the triangle inequality $|x+y| \leqslant |x| + |y|$, $x, y \in V$ need not hold (see [4] or [6]).

The aim of this paper is to generalize some inequalities, valid for absolute values of complex numbers or Hilbert space operators, to the C^*-valued norm on a Hilbert C^*-module.

The first question is whether the triangle inequality holds for the C^*-valued norm on a Hilbert C^*-module. A partial answer to this question is given in section 2. We also give an analogue of the classical aritmetic-geometric mean inequality (cf. [2]) in a Hilbert C^*-module setting.

Section 3 is entirely devoted to the study of Bohr's inequality for the C^*-valued norm on a Hilbert C^*-module. We obtain various forms of Bohr inequalities. We also generalize Euler-Lagrange type identity to Hilbert C^*-module setting (Theorem 6) in order to get another Bohr type inequality (Theorem 7).

2. The triangle inequality

We determine the conditions for the triangle inequality to hold for the C^*-valued norm on a Hilbert C^*-module. First we quote Theorem 2.8 from [1].

THEOREM 1. *Let V be a Hilbert C^*-module over a C^*-algebra A with unit e. Let $\varepsilon > 0$. For every $x, y \in V$ there are $u, v \in A$ such that $\|u\| \leqslant 1, \|v\| \leqslant 1$ and*

$$|x+y| \leqslant u|x|u^* + v|y|v^* + \varepsilon e.$$

THEOREM 2. *Let V be a Hilbert C^*-module over a C^*-algebra A. For $x, y \in V$ such that $|x|, |y| \in Z(A)$, we have*

$$|x+y| \leqslant |x| + |y|.$$

Proof. Without loss of generality we may assume that A possesses a unit e. (If it is not a case, we adjoin a unit to A.) Let $x, y \in V$ with $|x|, |y| \in Z(A)$. By Theorem 1,

$$u|x|u^* = |x|^{\frac{1}{2}} uu^* |x|^{\frac{1}{2}} \leqslant \|uu^*\| |x|^{\frac{1}{2}} |x|^{\frac{1}{2}} \leqslant |x|,$$

and similarly $v|y|v^* \leqslant |y|$. Now we have

$$|x+y| \leqslant u|x|u^* + v|y|v^* + \varepsilon e \leqslant |x| + |y| + \varepsilon e.$$

Since $\varepsilon > 0$ is arbitrary, by taking $\varepsilon \to 0$ we conclude that $|x+y| \leqslant |x| + |y|$. $\square$

LEMMA 1. *Let A be a C^*-algebra and let $a, b \in A$ be positive and such that $a \in Z(A)$ or $b \in Z(A)$. Then*

$$ab \leqslant \frac{a^2+b^2}{2}.$$

Proof. Since $a \in Z(A)$ or $b \in Z(A)$, we have

$$a^2 - 2ab + b^2 = (a-b)^2 = (a-b)^*(a-b) \geqslant 0,$$

and the claim follows. □

Using this fact we have the following version of a classical arithmetic-geometric mean inequality for a norm $|\cdot|$ on V.

COROLLARY 1. *Let V be a Hilbert C^*-module over a C^*-algebra A. For $x, y \in V$, $|x| \in Z(A)$ or $|y| \in Z(A)$ imply*

$$|x|^k |y|^k \leqslant \frac{|x|^{2k} + |y|^{2k}}{2}, \; k \in \mathbf{N}.$$

Proof. The result follows from Lemma 1 inserting $a := |x|^k, b := |y|^k$, $k \in \mathbf{N}$. □

3. The Bohr's inequality

Let us discuss Bohr's inequality for the C^*-valued "norm" on a Hilbert C^*-module V. It was shown in [5] that for operators $A, B \in \mathbf{B}(\mathbf{H})$ (where $\mathbf{B}(\mathbf{H})$ denotes the C^*-algebra of all bounded operators on a complex separable Hilbert space $\mathbf{H}$) and for real numbers $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we have $|A - B|^2 \leqslant p|A|^2 + q|B|^2$.

THEOREM 3. *Let V be a Hilbert C^*-module and let $x, y \in V$. Let $p, q \in \mathbf{R}$ be such that $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. A C^*-valued norm on V satisfies the Bohr's inequality:*

$$|x + y|^2 \leqslant p|x|^2 + q|y|^2.$$

Proof. Similarly to [5], we first prove that for $p \leqslant q$ the inequality $|x+y|^2 + |(1-p)x + y|^2 \leqslant p|x|^2 + q|y|^2$ holds. We have

$$\begin{aligned}
&|x+y|^2 + |(1-p)x+y|^2 - p|x|^2 - q|y|^2 \\
&\quad = [1 + (1-p)^2 - p]|x|^2 + (2-q)|y|^2 + (2-p)[(x,y) + (y,x)] \\
&\quad = (p-2)(p-1)|x|^2 + (2-q)|y|^2 - (p-2)[(x,y) + (y,x)] \\
&\quad = (p-2)(p-1)|x|^2 + \frac{p-2}{p-1}|y|^2 - (p-2)[(x,y) + (y,x)] \\
&\quad = (p-2)\left|\sqrt{p-1}\,x - \frac{1}{\sqrt{p-1}}y\right|^2 \leqslant 0,
\end{aligned}$$

since $p - 2 \leqslant 0$. Similarly, if $p \geqslant q$, we have

$$|x+y|^2 + |x + (1-q)y|^2 \leqslant p|x|^2 + q|y|^2.$$

It follows that the Bohr's inequality $|x+y|^2 \leqslant p|x|^2 + q|y|^2$ holds for real numbers $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. □

Notice that this theorem generalizes the result of [5] if we consider $\mathbf{B}(H)$ as a Hilbert C^*-module over itself. From the proof of Theorem 3, the next result follows.

COROLLARY 2. *Let V be a Hilbert C^*-module and let $x,y \in V$. Let $p,q \in \mathbf{R}$ be such that $p,q > 1$, $p \leqslant q$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$|x+y|^2 + |(1-p)x+y|^2 \leqslant p|x|^2 + q|y|^2$$

and equality holds if and only if $p(=q)=2$ or $(p-1)x=y$.

The next form of Bohr's inequality (given for complex numbers in [7]) is a consequence of Theorem 3.

COROLLARY 3. *Let V be a Hilbert C^*-module and let $x,y \in V$. For $\alpha,\beta \in \mathbf{R}$ such that $\alpha \neq 0, \beta \neq 0$ and $\alpha+\beta \neq 0$ we have that*

$$(1)\quad \frac{|x+y|^2}{\alpha+\beta} \leqslant \frac{|x|^2}{\alpha} + \frac{|y|^2}{\beta} \quad if\ \alpha,\beta > 0,$$
$$(2)\quad \frac{|x+y|^2}{\alpha+\beta} \geqslant \frac{|x|^2}{\alpha} + \frac{|y|^2}{\beta} \quad if\ \alpha,\beta < 0.$$

Proof. Put $p=\frac{\alpha+\beta}{\alpha}$, $q=\frac{\alpha+\beta}{\beta}$. Notice that $\frac{1}{p}+\frac{1}{q}=1$. For $\alpha,\beta>0$ or $\alpha,\beta<0$, we have $p,q>1$. If $\alpha,\beta>0$, we have from Theorem 3

$$|x+y|^2 \leqslant \frac{\alpha+\beta}{\alpha}|x|^2 + \frac{\alpha+\beta}{\beta}|y|^2,$$

which is equivalent to (1). Similarly, for $\alpha,\beta<0$ we get (2). □

We also have the following refinement of Bohr's inequality.

THEOREM 4. *Let V be a Hilbert C^*-module over a C^*-algebra A and let $x,y \in V$ be such that $|x|,|y| \in Z(A)$. For $p,q \in \mathbf{R}$ such that $p,q>1, p \leqslant q$ and $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$|x+y|^4 + |(1-p)x+y|^4 \leqslant (p|x|^2+q|y|^2)^2.$$

Proof. Notice that $a=|x+y|^2 \geqslant 0$ and $b=|(1-p)x+y|^2 \geqslant 0$. If $|x|$ and $|y|$ are elements in $Z(A)$, then a and b commute and therefore ([8], p. 308) their product is positive. Consequently, $a^2+b^2 \leqslant (a+b)^2$ and the inequality follows. □

We have the following extension of Theorem 3.

THEOREM 5. *Let V be a Hilbert C^*-module, $x,y \in V$ and let $p,q \in \mathbf{R}$ be such that $p \neq 0$, $q \neq 0$ and $\frac{1}{p}+\frac{1}{q}=1$. We have*

$$|x+y|^2 \leqslant p|x|^2+q|y|^2 \quad if \quad pq>0,$$
$$|x+y|^2 \geqslant p|x|^2+q|y|^2 \quad if \quad pq<0.$$

Proof. Notice that

$$\begin{aligned}
&\frac{1}{q}|x|^2+\frac{1}{p}|y|^2-\frac{1}{pq}|x+y|^2\\
&\quad=\frac{1}{q}|x|^2+\frac{1}{p}|y|^2-\frac{1}{pq}[|x|^2+(x,y)+(y,x)+|y|^2]\\
&\quad=\left[\frac{1}{q}-\frac{1}{pq}\right]|x|^2+\left[\frac{1}{p}-\frac{1}{pq}\right]|y|^2-\frac{1}{pq}[(x,y)+(y,x)]\\
&\quad=\frac{1}{q^2}|x|^2-\frac{1}{pq}[(x,y)+(y,x)]+\frac{1}{p^2}|y|^2=\left|\frac{1}{q}x-\frac{1}{p}y\right|^2.
\end{aligned}$$

Hence,

$$p|x|^2+q|y|^2=|x+y|^2+\frac{|px-qy|^2}{pq}.$$

Depending on sign of the product pq, the claims follow. □

From Theorem 5 we can get generalizations of the results of Cheung and Pečarić stated in [3] for bounded operators on a Hilbert space.

Namely, we can extract q from the absolute value of the last term of the equality

$$p|x|^2+q|y|^2=|x+y|^2+\frac{|px-qy|^2}{pq}$$

and get

$$p|x|^2+q|y|^2=|x+y|^2+\frac{|q\frac{p}{q}x-qy|^2}{pq}$$

i. e.

$$p|x|^2+q|y|^2=|x+y|^2+\frac{q^2|\frac{p}{q}x-y|^2}{pq}.$$

Using $\frac{p}{q}=p-1$, we get

$$p|x|^2+q|y|^2=|x+y|^2+\frac{1}{p-1}|(1-p)x+y|^2.$$

Similarly, by extracting p, we get

$$p|x|^2+q|y|^2=|x+y|^2+\frac{1}{q-1}|x+(1-q)y|^2.$$

We have the following cases.

1. For $1<p\leqslant 2$ we have

$$p|x|^2+q|y|^2\geqslant|x+y|^2+|(1-p)x+y|^2$$

and (at the same time $q\geqslant 2$)

$$p|x|^2+q|y|^2\leqslant|x+y|^2+|x+(1-q)y|^2.$$

2. For $p \geqslant 2$ we have
$$p|x|^2 + q|y|^2 \leqslant |x+y|^2 + |(1-p)x+y|^2$$
and
$$p|x|^2 + q|y|^2 \geqslant |x+y|^2 + |x+(1-q)y|^2.$$

3. For $p < 1$ (then $q < 1$ and $pq < 0$), by Theorem 5, we have
$$p|x|^2 + q|y|^2 \leqslant |x+y|^2 \leqslant |x+y|^2 + |(1-p)x+y|^2$$
and
$$p|x|^2 + q|y|^2 \leqslant |x+y|^2 \leqslant |x+y|^2 + |x+(1-q)y|^2.$$

We can get another Bohr's type inequality by using the next Euler-Lagrange type identity on V.

THEOREM 6. *Let V be a Hilbert C^*-module over a unital C^*-algebra A with unit e. Let: $x,y \in V$, α,β,γ non-zero real numbers, $a,b \in Z(A)$ with ab^* selfadjoint and $\alpha aa^* + \beta bb^* = \gamma e$. We have the Euler-Lagrange identity:*
$$\frac{|x|^2}{\alpha} + \frac{|y|^2}{\beta} - \frac{|ax+by|^2}{\gamma} = \frac{|\beta bx - \alpha ay|^2}{\alpha\beta\gamma}.$$

Proof. Using the assumptions, we get:
$$\begin{aligned}
&\beta\gamma|x|^2 + \alpha\gamma|y|^2 - \alpha\beta|ax+by|^2 \\
&\quad = \beta\gamma(x,x) + \alpha\gamma(y,y) - \alpha\beta[a(x,x)a^* + a(x,y)b^* + b(y,x)a^* + b(y,y)b^*] \\
&\quad = \beta(x,x)[\gamma e - \alpha aa^*] + \alpha(y,y)[\gamma e - \beta bb^*] - \alpha\beta ab^*[(x,y)+(y,x)] \\
&\quad = \beta^2 bb^*|x|^2 + \alpha^2 aa^*|y|^2 - \alpha\beta ab^*[(x,y)+(y,x)] \\
&\quad = |\beta bx - \alpha ay|^2.
\end{aligned}$$
□

As a consequence of Theorem 6 we can state the next result.

THEOREM 7. *Let V be a Hilbert C^*-module over a unital C^*-algebra A with unit e. Let: $x,y \in V$, α,β,γ non-zero real numbers, $a,b \in Z(A)$ with ab^* selfadjoint and $\alpha aa^* + \beta bb^* = \gamma e$. Then the following Bohr's type inequalities hold:*
$$\frac{|ax+by|^2}{\gamma} \leqslant \frac{|x|^2}{\alpha} + \frac{|y|^2}{\beta} \quad if \quad \alpha\beta\gamma > 0,$$
$$\frac{|ax+by|^2}{\gamma} \geqslant \frac{|x|^2}{\alpha} + \frac{|y|^2}{\beta} \quad if \quad \alpha\beta\gamma < 0.$$

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