Mathematical
Inequalities
& Applications
Volume 12, Number 4 (2009), 753–762

EXISTENCE AND BEHAVIOR OF SOLUTIONS FOR VARIATIONAL INEQUALITIES OVER PRODUCTS OF SETS

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. In this paper we study a variational inequality over a product of sets, governed by a multi-valued mapping, in pseudomonotonicity conditions. We are interested in the existence of the solution and, when the inequality depends on a parameter, also in the behavior of the solution at perturbations of the parameter.

1. Introduction

Several problems arising in economics, physics, engineering and other fields can be modeled using systems of variational inequalities or variational inequalities over Cartesian products of sets. Such problems have been much studied and existence results were proved in the presence of different monotonicity or pseudomonotonicity concepts (see [9], [20], [3], [17]).

We consider here a variational inequality over a product of sets, governed by a multi-valued mapping. Assuming that the mapping satisfies a certain pseudomonotonicity condition (used also in [10], [12], [16]) we prove the existence of a solution for the variational inequality.

If the variational inequality depends on a parameter, a problem of interest is the behavior of the solution under perturbations of the parameter. In this paper we are interested in two aspects of this problem: the closedness of the solution mapping and the Hölder continuity of the solution.

2. An existence result

For each $i \in I = \{1,2,\dots,n\}$, let X_i be real Hausdorff topological vector spaces, X_i^* their duals and $X = \Pi_{i\in I} X_i$; $X^* = \Pi_{i\in I} X_i^*$. We denote by $\langle \cdot,\cdot \rangle$ the pairing between X_i^* and X_i and consider the weak topology defined from the duality of the pair of spaces.

Let $F_i : X \to 2^{X_i^*}$, $F = (F_1,\dots,F_n)$. Let $K_i \subset X_i$ be nonempty, convex, closed sets and $K = \Pi_{i\in I} K_i$. Then K is also a nonempty closed and convex set in X.

Mathematics subject classification (2000): 47J20, 58E35, 49N60.
Keywords and phrases: parametric variational inequality over a product of sets; pseudomonotone set-valued mapping.

We formulate the following variational inequality:

$$\begin{aligned}&\text{Find} \quad x=(x_1,\dots,x_n)\in K \text{ such that}\\ &\sum_{i=1}^{n}\sup_{f_i\in F_i(x)}\langle f_i, y_i-x_i\rangle \geqslant 0,\ \forall\, y=(y_1,\dots,y_n)\in K\end{aligned} \tag{VI}$$

To obtain an existence result for the problem (VI) we will use a generalization of the Ky-Fan intersection lemma and a result about the marginal function:

LEMMA 1. ([8]; [15], Theorem V.2.23) *Let V be a topological vector space, $H\subset V$ and $T: H\to 2^V$ such that:*

(i) $\operatorname{cl} T(x_0)$ *is compact for some* $x_0\in H$*,*

(ii) for every $x_1,x_2,\dots,x_n\in H$*,* $co\{x_1,x_2,\dots,x_n\}\subset \bigcup_{i=1}^{n} T(x_i)$*,*

(iii) for each $x\in H$, the intersection of $T(x)$ with any finite dimensional subspace of V is closed,

(iv) for every line segment D of V,

$$\operatorname{cl}\Big(\bigcap_{x\in H\cap D} T(x)\Big)\cap D=\Big(\bigcap_{x\in H\cap D} T(x)\Big)\cap D,$$

Then $\bigcap_{x\in E} T(x)\neq\emptyset$.

If H is convex, closed and $T(x)\subset H$ for every $x\in H$, then the hypothesis (iv) can be replaced with:
(iv') for every line segment D of H, $\operatorname{cl}\Big(\bigcap_{x\in D} T(x)\Big)\cap D=\Big(\bigcap_{x\in D} T(x)\Big)\cap D$.

LEMMA 2. ([15]) *Let U and V be topological spaces, $G: U\to 2^V$ a set-valued mapping and $g: U\times V\to\mathbb{R}$. Denote by $h: U\to\mathbb{R}$, $h(u)=\sup_{v\in G(u)} g(u,v)$ the marginal function. If the conditions:*

(i) g is upper semi-continuous on $U\times V$,

(ii) $G(u_0)$ is compact for some $u_0\in U$,

(iii) G is upper semi-continuous at u_0,

are satisfied, then h is upper semi-continuous at u_0.

For each $z\in X$, we denote $T(z)=\{x\in K \mid \sum_{i=1}^{n}\sup_{f_i\in F_i(x)}\langle f_i, z_i-x_i\rangle\geqslant 0\}$. Obviously, if $x\in T(z)$ for each $z\in K$, then x is a solution of the problem (VI).

LEMMA 3. *For each* $z^1, z^2, \ldots, z^p \in K$, $co\{z^1, \ldots, z^p\} \subset T(z^1) \cup \ldots \cup T(z^p)$.

Proof. Suppose that there exist $\lambda_1, \ldots, \lambda_p \geqslant 0$ with $\lambda_1 + \ldots + \lambda_p = 1$ such that $\bar{z} = \sum_{j=1}^{p} \lambda_j z^j \notin T(z^l)$, that is $\sum_{i=1}^{n} \sup_{f_i \in F_i(\bar{z})} \langle f_i, z_i^l - \bar{z}_i \rangle < 0$, for each $l = 1, \ldots, p$. We fix $f = (f_1, \ldots, f_n) \in F(\bar{z})$. Obviously,

$$\sum_{i=1}^{n} \langle f_i, z_i^l - \bar{z}_i \rangle \leqslant \sum_{i=1}^{n} \sup_{f_i \in F_i(\bar{z})} \langle f_i, z_i^l - \bar{z}_i \rangle < 0,$$

for each $l = 1, \ldots, p$. We have

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \langle f_i, \bar{z}_i - \bar{z}_i \rangle = \sum_{i=1}^{n} \langle f_i, \sum_{j=1}^{p} \lambda_j z_i^j - (\sum_{j=1}^{p} \lambda_j) \bar{z}_i \rangle = \sum_{i=1}^{n} \sum_{j=1}^{p} \lambda_j \langle f_i, z_i^j - \bar{z}_i \rangle \\ &= \sum_{j=1}^{p} \sum_{i=1}^{n} \lambda_j \langle f_i, z_i^j - \bar{z}_i \rangle = \sum_{j=1}^{p} \lambda_j \sum_{i=1}^{n} \langle f_i, z_i^j - \bar{z}_i \rangle < 0, \end{aligned}$$

which is a contradiction. □

Many notions of pseudomonotonicity have been defined (see [14], [19]) mainly starting from the algebraic one introduced by Karamardian in 1976 and the topological one introduced by Brézis in 1968. The following condition appears in several papers. We mention here [11], [16] (where it was studied in the single-valued and set-valued case and compared to some of the classical concepts of pseudomonotonicity) and [12] (where it appears also in the more general context of equilibrium problems and it is called 0-segmentary closedness).

In the case of a multi-function defined on a product of sets, we have:

DEFINITION 4. $F = (F_1, \ldots, F_n)$, $F_i : X \to 2^{X_i^*}$ is said to be a *C-pseudomonotone* (0-*segmentary closed*) mapping if, for each $x, y \in X$ and each net $\{x^\alpha\} \subset X$, with $x^\alpha \rightharpoonup x$,

$$\sum_{i=1}^{n} \sup_{f_i \in F_i(x^\alpha)} \langle f_i, (1-t)x_i + t y_i - x_i^\alpha \rangle \geqslant 0, \ \forall\, t \in [0,1], \forall \alpha$$

implies $\sum_{i=1}^{n} \sup_{f_i \in F_i(x)} \langle f_i, y_i - x_i \rangle \geqslant 0$.

LEMMA 5. *If the function F is C-pseudomonotone, then for each $x, y \in K$*

$$\mathrm{w-cl}\Big(\bigcap_{z \in [x,y]} T(z) \Big) \cap [x,y] = \Big(\bigcap_{z \in [x,y]} T(z) \Big) \cap [x,y].$$

Proof. It is enough to prove (see [16]) that, for each $y \in K$, $x \in \text{w-cl}\Big(\bigcap_{z \in [x,y]} T(z) \Big)$ implies $x \in \bigcap_{z \in [x,y]} T(z)$. Let $\{x^\alpha\}$ be a net such that $x^\alpha \in \bigcap_{z \in [x,y]} T(z)$, with $x^\alpha \rightharpoonup x$.

This means that $x^\alpha \in K$ and

$$\sum_{i=1}^{n} \sup_{f_i \in F_i(x^\alpha)} \langle f_i, (1-t)x_i + ty_i - x_i^\alpha \rangle \geqslant 0, \ \forall\, t \in [0,1],\ \alpha \in \mathbf{N}.$$

Since F is C-pseudomonotone, this implies $\sum_{i=1}^{n} \sup_{f_i \in F_i(x)} \langle f_i, y_i - x_i \rangle \geqslant 0$. For each $t \in [0,1]$, we have

$$\sum_{i=1}^{n} \sup_{f_i \in F_i(x)} \langle f_i, (1-t)x_i + ty_i - x_i \rangle = t \sum_{i=1}^{n} \sup_{f_i \in F_i(x)} \langle f_i, y_i - x_i \rangle \geqslant 0$$

that is $x \in T(z)$, for each $z \in [x,y]$. □

THEOREM 6. *Let $K_i \subset X_i$ be convex, closed, nonempty, for each $i \in I = \{1,\ldots,n\}$. Let $F = (F_1,\ldots,F_n)$, $F_i : K \to 2^{X_i^*}$. Assume that the following conditions hold:*

(a) F is C-pseudomonotone,

(b) there exist $B \subset X$ weakly compact and $z^0 \in K$ such that

$$\sum_{i=1}^{n} \sup_{f_i \in F_i(x)} \langle f_i, z_i^0 - x_i \rangle < 0, \textit{ for each } x \in K \setminus B,$$

(c) For each finite dimensional subspace Z of X, F_i is upper semi-continuous on $K \cap Z$, with the weak topology in X_i^*,*

(d) $F_i(x)$ is weak compact for each $x \in K$, for each $i \in I$.*

Then the problem (VI) has at least a solution.

Proof. We check the hypotheses of Lemma 1 for the sets

$$T(z) = \{x \in K \mid \sum_{i=1}^{n} \sup_{f_i \in F_i(x)} \langle f_i, z_i - x_i \rangle \geqslant 0\}, \text{ for } z \in K.$$

(i) From (b) we have that $T(z_0) \subset B$, which gives w-cl$\,T(z_0) \subset$ w-cl$\,B = B$. Since B is weakly compact, it follows that w-cl$\,S(z_0)$ is also weakly compact.
(ii) Follows directly from Lemma 3.
(iii) Let Z be a finite-dimensional subspace of X. We have to prove that $Z \cap T(z)$ is closed, for each $z \in K$.

Consider a net $\{x^\alpha\} \subset Z \cap T(z)$, with $x^\alpha \to x$. This means that $x \in Z \cap K$ and $\sum_{i=1}^{n} \sup_{f_i \in F_i(x^\alpha)} \langle f_i, z_i - x_i^\alpha \rangle \geqslant 0$. The mapping $\langle \cdot, \cdot \rangle$ is continuous for the first component with the weak* topology and the second one with the strong topology. Defining the function $g : X_i^* \times X \to \mathbb{R}$, by $g(f,x) = \langle f, z_i - x_i \rangle$ we apply Lemma 2 and obtain the upper semi-continuity of the function $\sup_{f_i \in F_i(\cdot)} \langle f_i, z_i - \cdot \rangle$, for each $i \in I$. It follows:

$$\limsup_{\alpha} \sup_{f_i \in F_i(x^\alpha)} \langle f_i, z_i - x_i^\alpha \rangle \leqslant \sup_{f_i \in F_i(x)} \langle f_i, z_i - x_i \rangle.$$

$$\sum_{i=1}^{n} \sup_{f_i \in F_i(x)} \langle f_i, z_i - x_i \rangle \geqslant \sum_{i=1}^{n} \limsup_{\alpha} \sup_{f_i \in F_i(x^\alpha)} \langle f_i, z_i - x_i^\alpha \rangle$$
$$\geqslant \limsup_{\alpha} \sum_{i=1}^{n} \sup_{f_i \in F_i(x^\alpha)} \langle f_i, z_i - x_i^\alpha \rangle \geqslant 0.$$

This gives $x \in T(z)$, so $x \in T(z) \cap Z$.
(iv) Follows directly from Lemma 5. □

The pseudomonotonicity introduced by Karamardian for single-valued functions was generalized in several ways for set-valued functions (see [14], [19]). In the case of a multi-function defined on a product of sets, a possible generalization is the following:

DEFINITION 7. $F = (F_1, \ldots, F_n)$ is said to be *generalized algebraic-pseudomonotone* if from

$$\sum_{i=1}^{n} \sup_{f_i \in F_i(x)} \langle f_i, y_i - x_i \rangle \geqslant 0 \quad \text{it follows that} \quad \sum_{i=1}^{n} \inf_{g_i \in F_i(y)} \langle g_i, y_i - x_i \rangle \geqslant 0.$$

REMARK 8. If F is generalized algebraic pseudomonotone and for each $i \in I$, $F_i(x)$ is weakly* compact and the application $t \mapsto F_i((1-t)x + ty)$ is upper semicontinuous at 0 with the weak* topology in X_i^* (for each $x, y \in X$), then F is C-pseudomonotone.

Proof. Let $x, y \in X$, $\{x^\alpha\} \subset X$, a net with $x^\alpha \rightharpoonup x$ and

$$\sum_{i=1}^{n} \sup_{f_i \in F_i(x^\alpha)} \langle f_i, (1-t)x_i + ty_i - x_i^\alpha \rangle \geqslant 0, \ \forall\, t \in [0,1], \quad \forall \alpha.$$

From the generalized-algebraic pseudomonotonicity, it follows that

$$\sum_{i=1}^{n} \langle g_i, (1-t)x_i + ty_i - x_i^\alpha \rangle \geqslant 0, \text{ for each } g_i \in F_i((1-t)x + ty).$$

For each $i \in I$, we can pass to the limit, $x_i^\alpha \rightharpoonup x_i$ getting

$$\sum_{i=1}^{n} \langle g_i, t(y_i - x_i) \rangle \geqslant 0, \ \forall\, g_i \in F_i((1-t)x + ty), \ \forall\, t \in [0,1] \text{ and}$$

$$\sum_{i=1}^{n} \sup_{g_i \in F_i((1-t)x+ty)} \langle g_i, y_i - x_i \rangle \geqslant \sum_{i=1}^{n} \langle g_i, y_i - x_i \rangle \geqslant 0, \ \forall\, t \in (0,1].$$

According to Lemma 2 the function $h(t) = \sup_{g_i \in F_i((1-t)x+ty)} \langle g_i, y_i - x_i \rangle$ is upper semicontinuous at 0, for each $i \in I$. We get from this

$$\sum_{i=1}^{n} \sup_{f_i \in F_i(x)} \langle f_i, y_i - x_i \rangle \geqslant \sum_{i=1}^{n} \limsup_{t \to 0} \sup_{g_i \in F_i((1-t)x+ty)} \langle g_i, y_i - x_i \rangle$$
$$\geqslant \limsup_{t \to 0} \sum_{i=1}^{n} \sup_{g_i \in F_i((1-t)x+ty)} \langle g_i, y_i - x_i \rangle \geqslant 0. \qquad \square$$

REMARK 9. If $F_i : X \to 2^{X_i^*}$ is monotone for each $i \in I$, then $F = (F_1, \dots, F_n)$ is generalized algebraic-pseudomonotone.

REMARK 10. There exist functions that are C-pseudomonotone but are not generalized algebraic-pseudomonotone.

For example, let $n = 1$ and $F : \mathbb{R} \to 2^{\mathbb{R}}$, $F(x) = \begin{cases} \left(-\infty, -\frac{1}{x}\right], \text{ for } x < 0, \\ \mathbb{R}, \text{ for } x = 0, \\ \left[-\frac{1}{x}, \infty\right), \text{ for } x > 0. \end{cases}$

Taking $x = -1$ and $y = 1$, we have that $\sup\limits_{f \in F(x)} \langle f, y - x\rangle = \sup\limits_{f \in (-\infty, 1]} f \cdot 2 = 2 \geqslant 0$, while $\inf\limits_{g \in F(y)} \langle g, y - x\rangle = \inf\limits_{g \in [-1, \infty)} g \cdot 2 = -2 < 0$. On the other hand, is easy to see that F is C-pseudomonotone, since $0 \in F(x)$ for each $x \in \mathbb{R}$.

Another simple example of a function that is not generalized algebraic-pseudomonotone but is C-pseudomonotone and also has compact values is $G : \mathbb{R} \to 2^{\mathbb{R}}$,

$$G(x) = \begin{cases} 1, \text{ for } x < 0, \\ [-1, 1], \text{ for } x = 0, \\ -1, \text{ for } x > 0. \end{cases}$$

3. Behavior of the solution

If the variational inequality depends on some parameter, a much studied problem is what happens to the solution when the parameter is perturbed. We consider a family of variational inequalities of the type:

$$x \in K(\mu) \quad \text{such that} \sum_{i \in I} \sup_{f_i \in F_i(x, \mu)} \langle f_i, y_i - x_i\rangle \geqslant 0, \ \forall\, y_i \in K_i(\mu),\ i \in I. \qquad (\mathrm{VI})_\mu$$

where the parameter μ belongs to a metric space M.

Denote by $S(\mu)$ the set of solutions of the problem $(\mathrm{VI})_\mu$ and suppose that $S(\mu) \neq \emptyset$, for each $\mu \in M$. We study next the following problem: for a fixed parameter μ_0, if $x(\mu) \in S(\mu)$ and $x(\mu) \rightharpoonup x_0$ in X, when $\mu \to \mu_0$, in what conditions $x_0 \in S(\mu_0)$?

This property, called the closure of the graph of the solution function was studied, for instance, in [4] for hemivariational inequalities and in [5] for equilibrium problems.

We recall a characterization of the Mosco convergence for sets: a generalized sequence of sets $E_\alpha \subset X$ converges to a set $E \subset X$ in Mosco sense ($E_\alpha \xrightarrow{M} E$) if and only if:

(1) if $z_{\alpha_j} \in E_{\alpha_j}$ and $z_{\alpha_j} \rightharpoonup z$, then $z \in E$,
(2) for each $z \in E$, there exists $z_\alpha \in E_\alpha$ such that $z_\alpha \to z$.

Another notion of pseudomonotonicity will be used in what follows. It generalizes the pseudomonotonicity in the sense of Brézis (see [7], [6], [13]).

DEFINITION 11. A set-valued function $G: X \to 2^{X_i^*}$ is said to be *generalized topologically pseudomonotone* if for a net $\{x^\alpha\} \subset X$ with $x^\alpha \rightharpoonup x$ in X, with

$$\liminf_{\alpha} \sup_{g \in G(x^\alpha)} \langle g, x_i - x_i^\alpha \rangle \geqslant 0,$$

it follows that $\limsup_{\alpha} \sup_{g \in G(x^\alpha)} \langle g, y_i - x_i^\alpha \rangle \leqslant \sup_{g \in G(x)} \langle g, y_i - x_i \rangle$, for each $y = (y_1, \dots, y_n) \in X$.

We state now

THEOREM 12. *Let $K_i(\mu) \subset X_i$ be convex, closed, nonempty, $K(\mu) = K_1(\mu) \times \ldots \times K_n(\mu)$; $F = (F_1, \dots, F_n)$ with $F_i : X \times M \to 2^{X_i^*}$. Assume that the following conditions hold:*

(a) for each $i \in I$, the function $F_i(\cdot, \mu_0)$ is generalized topologically pseudomonotone,

(b) $K(\mu) \xrightarrow{M} K(\mu_0)$ when $\mu \to \mu_0$,

(c) for any convergent net $\{x(\mu)\}$ when $\mu \to \mu_0$,

$$\limsup_{\mu \to \mu_0} \{ \sum_{i=1}^{n} \sup_{f_i \in F_i(x(\mu),\mu)} \langle f_i, y_i - x_i(\mu) \rangle - \sum_{i=1}^{n} \sup_{g_i \in F_i(x(\mu),\mu_0)} \langle g_i, y_i - x_i(\mu) \rangle \} \leqslant 0,$$

for each $y = (y_1, \dots, y_n) \in X$,

(d) for any nets $x(\mu) \rightharpoonup x^0$ and $\hat{x}(\mu) \to x^0$ in X when $\mu \to \mu_0$,

$$\sup_{f_i \in F_i(x(\mu),\mu)} \langle f_i, \hat{x}_i(\mu) - x_i(\mu) \rangle \geqslant 0 \quad \textit{implies} \quad \liminf_{\mu \to \mu_0} \sup_{f_i \in F_i(x(\mu),\mu)} \langle f_i, x_i^0 - x_i(\mu) \rangle \geqslant 0.$$

Then if $x(\mu) \in S(\mu)$ and $x(\mu) \rightharpoonup x^0$ when $\mu \to \mu_0$, then $x^0 \in S(\mu_0)$.

Proof. Let $x(\mu) \in S(\mu)$, with $x(\mu) \rightharpoonup x^0$, for $\mu \to \mu_0$. Since $K(\mu) \xrightarrow{M} K(\mu_0)$, we have $x^0 \in K(\mu_0)$ and there exists a net, denoted $\hat{x}(\mu)$, such that $\hat{x}(\mu) \in K(\mu)$ and $\hat{x}(\mu) \to x^0$ in X.

For an $i \in I$ fixed, we take $y = (x_1(\mu), \dots, \hat{x}_i(\mu), \dots, x_n(\mu))$ in $(\mathrm{VI})_\mu$ and obtain

$$\sup_{f_i \in F_i(x(\mu),\mu)} \langle f_i, \hat{x}_i(\mu) - x_i(\mu) \rangle \geqslant 0.$$

From hypothesis (d), it follows that $\liminf_{\mu \to \mu_0} \sup_{f_i \in F_i(x(\mu),\mu)} \langle f_i, x_i^0 - x_i(\mu) \rangle \geqslant 0$.

Then, using (c) with $y = (x_1(\mu), \dots, x_i^0, \dots, x_n(\mu))$,

$$\begin{aligned} &\liminf_{\mu \to \mu_0} \sup_{g_i \in F_i(x(\mu),\mu_0)} \langle g_i, x_i^0 - x_i(\mu) \rangle \\ &\qquad \geqslant \liminf_{\mu \to \mu_0} \sup_{g_i \in F_i(x(\mu),\mu_0)} \langle g_i, x_i^0 - x_i(\mu) \rangle \\ &\qquad\quad + \limsup_{\mu \to \mu_0} \{ \sup_{f_i \in F_i(x(\mu),\mu)} \langle f_i, x_i^0 - x_i(\mu) \rangle - \sup_{g_i \in F_i(x(\mu),\mu_0)} \langle g_i, x_i^0 - x_i(\mu) \rangle \} \\ &\qquad \geqslant \liminf_{\mu \to \mu_0} \sup_{f_i \in F_i(x(\mu),\mu)} \langle f_i, x_i^0 - x_i(\mu) \rangle \geqslant 0. \end{aligned}$$

The pseudomonotonicity of $F_i(\cdot,\mu_0)$ implies now that for each $y_i \in K_i(\mu_0)$,

$$\sup_{f_i \in F_i(x^0,\mu_0)} \langle f_i, y_i - x_i^0 \rangle \geqslant \limsup_{\mu \to \mu_0} \sup_{f_i \in F_i(x(\mu),\mu_0)} \langle f_i, y_i - x_i(\mu) \rangle, \text{ for each } i \in I.$$

By summing up these relations, using the properties of "limsup" and again hypothesis (c), it follows

$$\begin{aligned}
\sum_{i=1}^{n} \sup_{f_i \in F_i(x^0,\mu_0)} \langle f_i, y_i - x_i^0 \rangle &\geqslant \limsup_{\mu \to \mu_0} \sum_{i=1}^{n} \sup_{f_i \in F_i(x(\mu),\mu_0)} \langle f_i, y_i - x_i(\mu) \rangle \\
&\geqslant \limsup_{\mu \to \mu_0} \sum_{i=1}^{n} \sup_{f_i \in F_i(x(\mu),\mu_0)} \langle f_i, y_i - x_i(\mu) \rangle \\
&\quad + \limsup_{\mu \to \mu_0} \{ \sum_{i=1}^{n} \sup_{f_i \in F_i(x(\mu),\mu)} \langle f_i, y_i - x_i(\mu) \rangle - \sum_{i=1}^{n} \sup_{f_i \in F_i(x(\mu),\mu_0)} \langle f_i, y_i - x_i(\mu) \rangle \} \\
&\geqslant \limsup_{\mu \to \mu_0} \sum_{i=1}^{n} \sup_{f_i \in F_i(x(\mu),\mu)} \langle f_i, y_i - x_i(\mu) \rangle \geqslant 0.
\end{aligned}$$

This last inequality means that $x^0 \in S(\mu_0)$, which concludes the proof. □

REMARK 13. For $n = 1$, if $G : X \to 2^{X^*}$ is generalized-topological pseudomonotone, then it is also C-pseudomonotone.

Proof. Let $x, y \in X$ and $\{x^\alpha\} \subset X$ with $x^\alpha \rightharpoonup x$ and

$$\sup_{f \in G(x^\alpha)} \langle g, (1-t)x + ty - x^\alpha \rangle \geqslant 0, \ \forall\, t \in [0,1], \ \alpha \in \mathbf{N}. \tag{1}$$

For $t = 0$ we have $\liminf_\alpha \sup_{f \in G(x^\alpha)} \langle f, x - x^\alpha \rangle \geqslant 0$. Then, from the topological pseudomonotonicity

$$\sup_{f \in G(x)} \langle f, y - x \rangle \geqslant \limsup_{\alpha} \sup_{f \in G(x^\alpha)} \langle f, y - x^\alpha \rangle \geqslant 0,$$

taking $t = 1$ in (1). □

REMARK 14. A more classic way of defining the pseudomonotonicity in the sense of Brézis for set-valued functions (see also [6]) is the following: a function $G : X \to 2^{X_i^*}$ is called generalized-Brézis-pseudomonotone if, for $\{x^\alpha\} \subset X$ with $x^\alpha \rightharpoonup x$ in X, $g_\alpha \in G(x^\alpha)$ with

$$\liminf_{\alpha} \langle g_\alpha, x_i - x_i^\alpha \rangle \geqslant 0,$$

follows that, for each $y \in X$, there exists $g_y \in G(x)$, such that

$$\limsup_{\alpha} \langle g_\alpha, y_i - x_i^\alpha \rangle \leqslant \langle g_y, y_i - x_i \rangle.$$

If G is generalized-topological pseudomonotone and $G(x)$ is weak* compact for each $x \in X$, then G is also generalized-Brézis-pseudomonotone.

We mention also a situation when the variational inequality depends on two parameters $\lambda \in \Lambda$ and $\mu \in M$, and another concept of stability of the solution. In what follows, X_i are real normed vector spaces, X_i^* their duals and the norm on the product space X is defined by $\|x\|_X = (\|x_1\|_{X_1}^2 + \ldots + \|x_n\|_{X_n}^2)^{1/2}$. Consider:

$$(VI)_{\lambda,\mu} \qquad x \in K(\lambda) \quad \text{such that} \sum_{i\in I} \sup_{f_i \in F_i(x,\mu)} \langle f_i, y_i - x_i \rangle \geqslant 0,\ \forall\, y_i \in K_i(\lambda),\ i \in I.$$

We can state:

THEOREM 15. *Let $K_i \subset X_i$ be convex, closed, nonempty, for each $i \in I$. Let $F = (F_1, \ldots, F_n)$, $F_i : K \to 2^{X_i^*}$. Assume that the following hold:*

(a) there exists a neighborhood U of λ_0 such that for each $\lambda_1, \lambda_2 \in U$,

$$K_i(\lambda_1) \subset K_i(\lambda_2) + l_i B_{X_i}(0, d^\alpha(\lambda_1, \lambda_2)),\ \textit{with}\ l_i > 0, \alpha > 0,$$

(b) there exist U_0, V_0 neighborhoods of λ_0 and μ_0 such that for each $\lambda \in U_0$, $\mu_1, \mu_2 \in V_0$, $x, y \in K(\lambda)$, $x \neq y$,

$$\Big| \sup_{f_i \in F_i(x,\mu_1)} \langle f_i, y_i - x_i \rangle - \sup_{g_i \in F_i(x,\mu_2)} \langle g_i, y_i - x_i \rangle \Big| \leqslant m_i \|y_i - x_i\|_{X_i} d^\gamma(\mu_1, \mu_2),$$

$m_i > 0$, $\gamma > 0$,

(c) for each $\lambda \in U_0$, $\mu \in V_0$, $x, y \in K(\lambda)$, $x \neq y$, for each $f_i \in F_i(x,\mu)$, $g_i \in F_i(y,\mu)$, $\langle f_i - g_i, y_i - x_i \rangle \leqslant -c_i \|x_i - y_i\|_{X_i}^2$, $\quad c_i > 0$,

(d) for each $\lambda \in U_0$, $\mu \in V_0$, $x \in K(\lambda)$, $y, z \in K(\lambda)$, for each $f_i \in F_i(x,\mu)$,

$$|\langle f_i, y_i - z_i \rangle| \leqslant n_i \|y_i - z_i\|_{X_i}^2, \quad n_i > 0.$$

Then in a neighborhood of (λ_0, μ_0), the solution $x(\lambda, \mu)$ of $(VI)_{\lambda,\mu}$ is unique and

$$\|x(\lambda_1, \mu_1) - x(\lambda_2, \mu_2)\|_X \leqslant k_1 d^\gamma(\mu_1, \mu_2) + k_2 d^\alpha(\lambda_1, \lambda_2).$$

Proof. The proof uses directly a particular case of Corollary 2.3 of [2], in the settings that appear also in [1], for equilibrium problems. □

Acknowledgements. The author wishes to thank the referee for the useful suggestions.

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(Received September 25, 2008)

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