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HEPTAGONAL TRIANGLE AS THE EXTREME TRIANGLE OF DIXMIER–KAHANE–NICOLAS INEQUALITY

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. Let T be a triangle in the Euclidean plane. Let $g(T)$ be the orthic triangle of the triangle T, and let $g^{n+1}(T)$ be the orthic triangle of the triangle $g^n(T)$. In [2] it is proved that for $n \to \infty$ the triangle $g^n(T)$ tends to the point L. It has also been shown that $|OL| \leqslant \frac{4}{3}R$ for all triangles T and that $|OL| = \frac{4}{3}R$ if T is a heptagonal triangle, where (O,R) is the circumscribed circle of the triangle T.

In this paper it will be geometrically proved that the equality in Dixmier–Kahane–Nicolas inequality $|OL| \leqslant \frac{4}{3}R$ is valid in the case of a heptagonal triangle. The relationship between the initial heptagonal triangle T and the obtained point L will also be investigated.

Let T be a triangle in the Euclidean plane. Let $g(T)$ denote the triangle whose vertices are the feet of the altitudes of the triangle T, i.e. the orthic triangle of the triangle T, and let $g^2(T)$ be the orthic triangle of the triangle $g(T)$; generally, let $g^{n+1}(T)$ be the orthic triangle of the triangle $g^n(T)$.

In [2] Dixmier, Kahane and Nicolas have proved, by means of trigonometric series, that for $n \to \infty$ the triangle $g^n(T)$ tends to the point L, a new characteristic point of the triangle T. If (O,R) is the circle circumscribed to the triangle T, then it has also been shown that the inequality $|OL| \leqslant \frac{4}{3}R$ is valid for all triangles T and that the equality $|OL| = \frac{4}{3}R$ is valid if and only if the angles of T are $\frac{4}{7}\pi$, $\frac{2}{7}\pi$, $\frac{1}{7}\pi$. This special triangle is called *heptagonal triangle* according to [1]. It is a very interesting and rare occurrence that a heptagonal triangle is the extreme triangle, because the extreme triangle in most of the different extreme problems concerning the triangles is an equilateral triangle.

Here we shall give the elementary geometric proof that in the case of heptagonal triangle ABC, the Dixmier-Kahane-Nicolas point L is the center of indirect similarity of the triangle ABC and its orthic triangle $B'C'A'$, where A', B', C' are the feet of the altitudes through the vertices A, B, C of the triangle ABC.

For the vertices A, B, C of the heptagonal triangle ABC we can get successively the vertices P_{10}, P_{12}, P_6 of the regular 14–gon $P_0P_1P_2\ldots P_{13}$ inscribed in the unit circle $\mathscr{K}$ with the center O, i.e. let $R = 1$ (*Figure 1*).

The triangles $ABC = P_{10}P_{12}P_6$ and $A_0B_0C_0 = P_{11}P_9P_1$ are symmetrical with respect to the common perpendicular bisector of the parallel chords $\overline{P_{10}P_{11}}$, $\overline{P_{12}P_9}$, $\overline{P_6P_1}$

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of the circle $\mathscr{K}$, so the orthocenters H, H_0 of these triangles are symmetrical with respect to the perpendicular bisector OH', where H' is the midpoint of the segment $\overline{HH_0}$. As $\overline{P_{13}P_6}$ is the diameter of the circle $\mathscr{K}$, then the chords $\overline{P_{12}P_{13}}$, $\overline{P_{12}P_6}$ are perpendicular, and the chords $\overline{P_{10}P_1}$, $\overline{P_{12}P_6}$ are also perpendicular i.e. the line $P_{10}P_1$ is the altitude through the vertex A of the triangle ABC. In the same manner it can be shown that the lines $P_{12}P_{11}$ and P_6P_9 are the altitudes through the vertices B and C. Because of that the feet of these three altitudes are the points $A' = P_{12}P_6 \cap P_{10}P_1$, $B' = P_6P_{10} \cap P_{12}P_{11}$, $C' = P_{10}P_{12} \cap P_6P_9$.

The common perpendicular bisector of the chords $\overline{P_{12}P_{13}}$, $\overline{P_{11}P_0}$, $\overline{P_{10}P_1}$, $\overline{P_6P_5}$ of the circle $\mathscr{K}$ is parallel to its chord $\overline{P_{12}P_6}$, and the chords $\overline{P_{12}P_{11}}$, $\overline{P_{13}P_0}$ are symmetrical with respect to this bisector, while the chords $\overline{P_{13}P_0}$, $\overline{P_{12}P_1}$ are mutually parallel. Because of that the lines on which the chords $\overline{P_{12}P_{11}}$, $\overline{P_{12}P_1}$ lie, are symmetrical with respect to the chord $\overline{P_{12}P_6}$, so it means that the triangle $P_{12}P_1H$ is isosceles and the point A' is the midpoint of the side $\overline{P_1H}$ of that triangle. Similarly it may be shown that the points B' and C' are the midpoints of the segments $\overline{P_{11}H}$ and $\overline{P_9H}$. Thus, the points A', B', C' are successively the midpoints of the segments $\overline{C_0H}$, $\overline{A_0H}$, $\overline{B_0H}$. It means that the homothecy with the center H and coefficient $\frac{1}{2}$ maps the triangle $A_0B_0C_0$ to the triangle $B'C'A'$. The composition of the symmetry with respect to the line OH' and the mentioned homothecy is then an indirect similarity σ with the coefficient $\frac{1}{2}$, which maps the given triangle ABC to its orthic triangle $B'C'A'$.

Indirect similarity σ maps the circumcenter O, the orthocenter H and the centroid G of the triangle ABC to the circumcenter O', the orthocenter H' and the centroid G' of the triangle $B'C'A'$. The point O' is in fact the Euler center of the triangle ABC and it is the midpoint of the segment $\overline{OH}$, and the points G and G' lie on the thirds of the segments $\overline{OH}$ and $\overline{O'H'}$ starting from the points O and O'. Because of that the line GG' is parallel to the line OH' and it intersects the segment $\overline{H'H}$ at the point L, which is on the third of that segment starting from the point H'. The point G' is the centroid of the rectangular triangle OHH', and the point G' is the midpoint of the segment $\overline{GL}$.

Let us consider the indirect similarity σ_0, which is the composition of the homothecy with the center L and the coefficient $\frac{1}{2}$ and the symmetry with respect to the line GG'. This similarity obviously maps the points G and H to the points G' and H', however the indirect similarity σ has also this property. As the (indirect) similarity is uniquely determined with two pairs of associated points, it follows $\sigma = \sigma_0$. So, we have proved the following theorem.

THEOREM 1. *If the heptagonal triangle ABC and its orthic triangle $A'B'C'$ have the centroids G and G' and the orthocenters H and H', then the triangles ABC and $B'C'A'$ are indirect similar. The indirect similarity $\sigma : ABC \to B'C'A'$ has mutually perpendicular axes GG' and HH', the center $L = GG' \cap HH'$ and coefficient $\frac{1}{2}$. The restrictions of this similarity on its axes GG' and HH' are homothecies with the center L and the coefficients $\frac{1}{2}$ and $-\frac{1}{2}$ (Figure 1).*

Let $A''B''C''$ be the orthic triangle of the triangle $A'B'C'$ and G'' the centroid and H'' the orthocenter of the triangle $A''B''C''$, then the triangle $B'C'A'$ is also heptagonal, thus the triangles $B'C'A'$ and $C''A''B''$ are indirect similar, according to Theorem 1, and

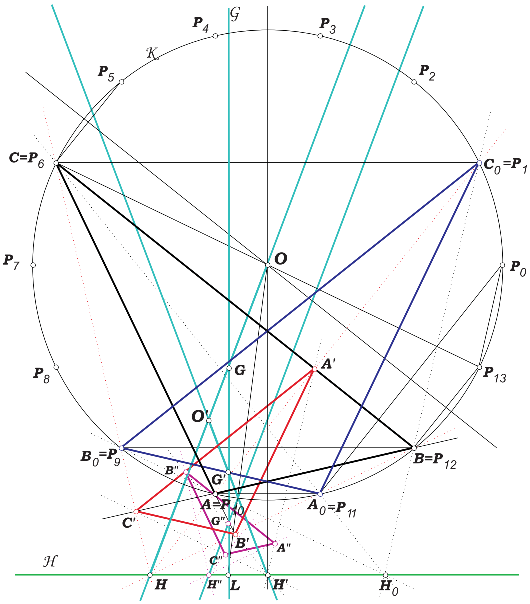

Figure 1.

the indirect similarity $\sigma' : B'C'A' \to C''A''B''$ has mutually perpendicular axes $G'G''$ and $H'H''$, the center $L' = G'G'' \cap H'H''$ and the coefficient $\frac{1}{2}$. If φ is that oriented angle between Euler lines GH and $G'H'$ of the triangles ABC and $A'B'C'$, whose angle bisector is parallel to the line GG' then, by the application of the indirect similarity σ' it follows that $-\varphi$ is that oriented angle of Euler lines $G'H'$ and $G''H''$ of the triangles $A'B'C'$ and $A''B''C''$, whose angle bisector is parallel to the line $G'G''$, which implies that the lines GH and $G''H''$ are parallel and therefore the angle bisectors GG' and $G'G''$ of these angles are also parallel, i.e. these axes are coincident. The same fact is also valid for the other axes HH' and $H'H''$ of these two similarities. Thus, these two similarities σ and σ' are coincident. The composition $\sigma \circ \sigma$ is the homothecy with the center L and the coefficient $\frac{1}{4}$. Because of that the following result holds.

THEOREM 2. *If $A'B'C'$ is the orthic triangle of the triangle ABC and $A''B''C''$ the orthic triangle of the triangle $A'B'C'$, then the same indirect similarity maps the triangle ABC to the triangle $B'C'A'$ and the triangle $B'C'A'$ to the triangle $C''A''B''$. The triangle ABC is mapped to the triangle $C''A''B''$ by the homothecy with the coefficient $\frac{1}{4}$ and the center L at the intersection of two perpendicular lines $\mathcal{G}$ and $\mathcal{H}$ out of which the first line passes through the centroids, and the second one passes through the orthocenters of the triangles ABC, $A'B'C'$ and $A''B''C''$ (Figure 1).*

The iterative application of Theorem 2 implies that the point L from Theorem 2 is the Dixmier-Kahane-Nicolas point of the heptagonal triangle ABC.

Let us set the regular 14-gon $P_0P_1P_2\dots P_{13}$ in the Gauss plane such that the points P_0, P_1, ..., P_{13} have the complex coordinates 1, ε, ..., ε^{13}, where $\varepsilon = e^{\frac{1}{14}2\pi i}$. The centroid G of the triangle ABC has the coordinate $g = \frac{1}{3}(\varepsilon^{10} + \varepsilon^{12} + \varepsilon^6)$, and the orthocenter H of that triangle the coordinate $h = 3g = \varepsilon^6 + \varepsilon^{10} + \varepsilon^{12}$. In the same manner $h_0 = \varepsilon + \varepsilon^9 + \varepsilon^{11}$ is the coordinate of the orthocenter H_0 of the triangle $A_0B_0C_0$. Because of $\varepsilon^7 = -1$, $\varepsilon^{14} = 1$ and $1 + \varepsilon^2 + \varepsilon^4 + \varepsilon^6 + \varepsilon^8 + \varepsilon^{10} + \varepsilon^{12} = 0$ we get now

$$h - h_0 = \varepsilon^6 + \varepsilon^{10} + \varepsilon^{12} - \varepsilon - \varepsilon^9 - \varepsilon^{11} = \varepsilon^6 + \varepsilon^{10} + \varepsilon^{12} + \varepsilon^8 + \varepsilon^2 + \varepsilon^4 = -1,$$

$$hh_0 = \varepsilon^7(1+\varepsilon^8+\varepsilon^{10})(1+\varepsilon^4+\varepsilon^6) = -(1+\varepsilon^8+\varepsilon^{10}+\varepsilon^4+\varepsilon^{12}+1+\varepsilon^6+1+\varepsilon^2) = -2,$$

therefore $|HH_0| = 1$ and $|OH|^2 = |OH_0|^2 = 2$, i.e. $|OH| = |OH_0| = \sqrt{2}$. From the rectangular triangle OHH' because of $|HH'| = \frac{1}{2}$ we have $|OH'| = \frac{1}{2}\sqrt{7}$, and then from the rectangular triangle $OH'L$ because of $|H'L| = \frac{1}{6}$ we finally get $|OL| = \frac{4}{3}$. So we have.

THEOREM 3. *The Dixmier-Kahane-Nicolas point L of the heptagonal triangle ABC is the center of indirect similarity of this triangle and its orthic triangle. If (O,R) is the circumscribed circle of the triangle ABC, then $|OL| = \frac{4}{3}R$.*

The second statement of Theorem 3 is also proved in [1].

Complex coordinates ε^{10}, ε^{12}, ε^6 of the points A, B, C have the product $\varepsilon^{28} = 1$, whose cube roots are 1, η, η^2, where $\eta = e^{\frac{2}{3}\pi i}$. The points with complex coordinates 1, η, η^2 are the so called *Boutin points* of the triangle ABC, and the lines which join the Boutin points with the circumcenter O of that triangle are the so called *Boutin axes* of the triangle ABC. One of the three Boutin points of the triangle ABC is the point P_0. The diameter P_0P_7 of the circle $\mathscr{K}$ is parallel to the already considered chords of this circle which have the common perpendicular bisector OH'. Because of that the Boutin axis OP_0 of the triangle ABC is perpendicular to the line OH', and then to the line GG' too, and it is parallel to the line HH'. So, we have just proved the following statement.

THEOREM 4. *The line $\mathscr{G}$ from Theorem 2 is perpendicular to one Boutin axis of the heptagonal triangle ABC, and the line $\mathscr{H}$ is parallel to this axis.*

The lines $\mathscr{G}$ and $\mathscr{H}$ from Theorems 2 and 4 will be called *centroidal and orthocentric axes* of the heptagonal triangle ABC. The triangles ABC and $A_0B_0C_0$ have the common orthocentric axis $\mathscr{H} = HH_0$.

The triangle $P_4P_2P_8$ is symmetrical to the triangle ABC with respect to the diameter P_0P_7 of the circle $\mathscr{K}$, thus it is also a heptagonal triangle. The lines CP_4, AP_2, BP_8, which are the chords P_4P_6, P_2P_{10}, P_8P_{12} of the circle $\mathscr{K}$, are successively parallel to the chords P_2P_8, P_8P_4, P_4P_2. Because of that the lines CP_4, AP_2, BP_8 form the triangle, whose midpoints of the sides are the points P_4, P_2, P_8, and the so obtained triangle is also heptagonal.

The points P_2, P_9 are the midpoints of arcs BC of the circle $\mathscr{K}$, and the lines AP_2, AP_9 are the angle bisectors of the angle A of the triangle ABC. Analogously, the lines BP_8, BP_1 are the angle bisectors of the angle B, and the lines CP_{11}, CP_4 are the angle bisectors of the angle C of that triangle. Because of that three by three lines AP_2, BP_8, CP_{11}; AP_2, BP_1, CP_4; AP_9, BP_8, CP_4; AP_9, BP_1, CP_{11} intersect successively at the centers I, I_a, I_b, I_c of inscribed and excribed circles of the triangle ABC. The pairs of bisectors AP_2, AP_9; BP_8, BP_1; CP_{11}, CP_4 are perpendicular, so the triangle ABC is the orthic triangle of the triangle II_bI_a, whose orthocenter is the point I_c. This triangle II_bI_a has for its sides $\overline{I_bI_a}$, $\overline{I_aI}$, $\overline{II_b}$ the lines CP_4, AP_2, BP_8, which, as we have proved, form the triangle where the points P_4, P_2, P_8 are the midpoints of its sides.

Thus, the points P_4, P_2, P_8 are the midpoints of the sides $\overline{I_bI_a}$, $\overline{I_aI}$, $\overline{II_b}$ of the heptagonal triangle II_bI_a. The points $P_6 = C$, $P_{10} = A$, $P_{12} = B$ are the feet of the altitudes of this triangle. Does the remaining vertex P_0 of the regular heptagon $P_0P_2P_4P_6P_8P_{10}P_{12}$ have any geometrical meaning for the triangle II_bI_a? We have already known that the point P_0 is the Boutin point of the triangle ABC, and it has the same meaning for its symmetric triangle $P_4P_2P_8$. Let us now find one more nice characterization of this point for the triangle II_bI_a.

As the line P_6P_{13} is the diameter of the circle $\mathscr{K}$, then the line P_4P_{13} is perpendicular to the line P_4P_6, i.e. the line P_4P_{13} is the perpendicular bisector of the side $\overline{I_bI_a}$ of the triangle II_bI_a, and analogously it can be shown that the lines P_2P_3 and P_8P_5 are perpendicular bisectors of the sides I_aI and II_b of that triangle. Because of that the lines P_4P_{13}, P_2P_3, P_8P_5 intersect at the circumcenter S of the triangle II_bI_a. The circle $\mathscr{K}$ is the Euler circle of that triangle, and the radius of the circumscribed circle of that triangle is 2.

The symmetry with respect to the line OH' maps the triangle ABC with the vertices P_{10}, P_{12}, P_6 to the triangle $A_0B_0C_0$ with the vertices P_{11}, P_9, P_1, and then the altitudes $P_{10}P_1$, $P_{12}P_{11}$, P_6P_9 of the triangle ABC are mapped to the altitudes $P_{11}P_6$, P_9P_{10}, P_1P_{12} of the triangle $A_0B_0C_0$ with the intersection H_0. However, the point $P_9P_{10} \cap P_6P_{11}$ is in fact the point $AP_9 \cap CP_{11} = I_c$. Thus $I_c = H_0$. The point O is the center of Euler circle $\mathscr{K}$ of the triangle II_bI_a , i.e. it is the midpoint of the circumcenter S and the orthocenter H_0. Because of the symmetry of the points S, H_0 with respect to the point O and the symmetry of the points H, H_0 with respect to the line OH' and because of the perpendicularity of the lines OH' and OP_0 it follows that the points S and H are symmetrical with respect to the line OP_0. So the triangle P_0HS is an isosceles triangle with the base $|SH| = 2 \cdot |OH'| = \sqrt{7}$ and the altitude $|OP_0| + |HH'| = \frac{3}{2}$, wherefrom the equalities $|P_0H| = |P_0S| = 2$ easily follow. It means that the point P_0 lies on the circumscribed circle of the triangle II_bI_a, and the following statement holds:

THEOREM 5. *If I, I_a, I_b, I_c are the centers of inscribed and excribed circles of the heptagonal triangle ABC with the vertices $A = P_{10}$, $B = P_{12}$, $C = P_6$ at three vertices of the regular heptagon $P_0P_2P_4P_6P_8P_{10}P_{12}$, then II_bI_a is a heptagonal triangle with the orthocenter I_c, the points P_2, P_4, P_8 are successively the midpoints of the sides $\overline{II_a}$, $\overline{I_aI_b}$, $\overline{I_bI}$, and the point P_0 is one intersection of the circumscribed and Euler circle of that triangle (Figure 2).*

Theorem 5 can be applied on the triangle ABC and its orthic triangle $A'B'C'$ instead of the triangle II_bI_a and its orthic triangle ABC, and then it means that the points A', B', C', the midpoints of the sides of the triangle ABC and the intersection U of the

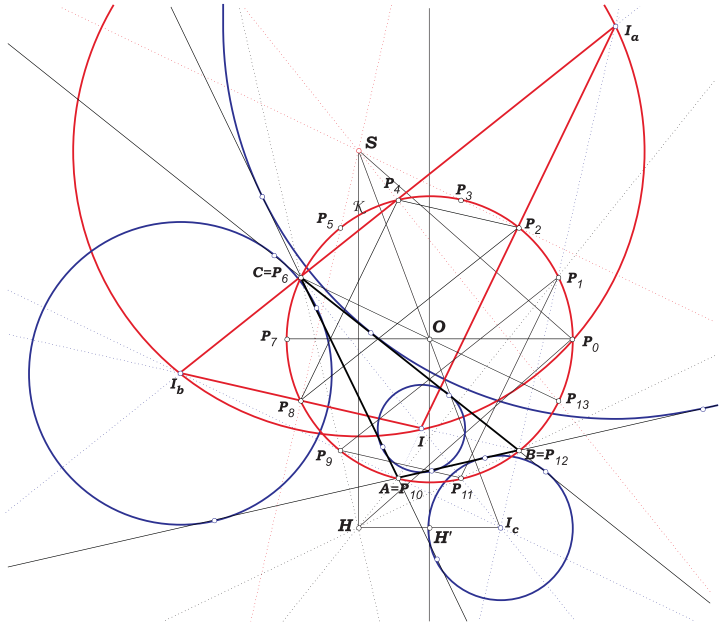

Figure 2.

circumscribed and Euler circle of that triangle are the vertices of one regular heptagon, not in this sequence necessarily (*Figure 1*). As the points A', B', C' are the midpoints of the segments $\overline{HP_1}$, $\overline{HP_{11}}$, $\overline{HP_9}$, then the homothecy with the center H and the coefficient $\frac{1}{2}$ maps the points P_1, P_{11}, P_9 to the points A', B', C'. However, the points P_1, P_{11}, P_9 are the images of the points C, A, B by the symmetry with respect to the line OH'. Indirect similarity σ'', which is the composition of this symmetry and one homothecy, maps the points P_6, P_{10}, P_{12} successively to the images of the points P_1, P_{11}, P_9 with regard to the homothecy $(H, \frac{1}{2})$; thus this similarity maps the remaining vertices P_0, P_2, P_4 and P_8 of the regular heptagon $P_0P_2P_4P_6P_8P_{10}P_{12}$ successively to the images of vertices P_7, P_5, P_3 and P_{13} of the regular heptagon $P_7P_5P_3P_1P_{13}P_{11}P_9$ by this homothecy. The similarity σ'' maps vertices C, A, B of the orthic triangle II_bI_a to the corresponding vertices A', B', C' of the orthic triangle of the triangle ABC, and then it also maps the intersection P_0 of the circumscribed and Euler circle of the triangle II_bI_a to the intersection of circumscribed and Euler circle of the triangle ABC, and here it is the image of the point P_7 by homothecy $(H, \frac{1}{2})$, i.e. the midpoint U of the segment HP_7. What could be said about the second intersection V of the circumscribed and Euler circle of the triangle ABC?

According to the known formula for the lengths of the medians of the triangle by means of the lengths of its sides it follows that the median H_0O' of the triangle OHH_0 with the length of the sides $1, \sqrt{2}, \sqrt{2}$ has the length 1. If U' is the point symmetrical to the point O' with respect to the point U, then the segment $O'U'$ is parallel and equal to the segments OP_7 and H_0H, therefore $O'U'HH_0$ is a rhombus because of

$|O'H_0| = 1 = |HH_0|$ (*Figure 1*). The diagonal $O'H$ of this rhombus is the perpendicular bisector of the segments UV and $U'H_0$, and as the point U is the midpoint of the segment $O'U'$, it follows that the point V is the midpoint of the segment $O'H_0$. As the points O' and $H_0 = I_c$ are the centers of Euler circle and excribed circle of the triangle ABC, and these two circles touch each other outside at the corresponding Feuerbach point Φ_c, then the point V is that Feuerbach point, and besides that it follows that this excribed circle has the radius $\frac{1}{2}$. We have proved the following theorem.

THEOREM 6. *The feet of the altitudes and the midpoints of the sides of a heptagonal triangle are six vertices of one regular heptagon inscribed in Euler circle of that triangle, and the seventh vertex of that heptagon is one intersection of this circle with the circumscribed circle of that triangle. The second intersection of these two circles is the Feuerbach point of the considered triangle, where its Euler circle touches its inscribed circle. These two last circles have the same radii.*

A number of statements of Theorem 6 can be found in [4].

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