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ON SPECIAL DIFFERENTIAL SUBORDINATIONS USING SĂLĂGEAN AND RUSCHEWEYH OPERATORS

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. In the present paper we define a new operator using the Sălăgean and Ruscheweyh operators. By L_α^n we denote the operator given by $L_\alpha^n : A \to A$, $L_\alpha^n f(z) = (1-\alpha)R^n f(z) + \alpha S^n f(z)$, for $z \in U$, where $R^n f(z)$ denotes the Ruscheweyh derivative, $S^n f(z)$ is the Sălăgean operator and $A_n = \{f \in \mathscr{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ is the class of normalized analytic functions with $A_1 = A$. A certain subclass, denoted by $S_n(\delta,\alpha)$, of analytic functions in the open unit disc is introduced by means of the new operator. By making use of the concept of differential subordination we derive various properties and characteristics of the class $S_n(\delta,\alpha)$. Also, several differential subordinations are established regarding the operator L_α^n.

1. Introduction

Denote by U the unit disc in the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$, and by $\mathscr{H}(U)$ the space of holomorphic functions in U.

Let $A_n = \{f \in \mathscr{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$, with $A_1 = A$, and $\mathscr{H}[a,n] = \{f \in \mathscr{H}(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$, for $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

Denote by $K = \left\{ f \in A : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U \right\}$ the class of normalized convex functions in U.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \qquad \text{for } z \in U, \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

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DEFINITION 1.1. (Sălăgean [6]) For $f \in A$, $n \in \mathbb{N}$, the operator S^n is defined by $S^n : A \to A$,

$$S^0 f(z) = f(z)$$
$$S^1 f(z) = zf'(z)$$
$$\ldots$$
$$S^{n+1} f(z) = z(S^n f(z))', \quad \text{for } z \in U.$$

REMARK 1.1. If $f \in A$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, for $z \in U$.

DEFINITION 1.2. (Ruscheweyh [5]) For $f \in A$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : A \to A$,

$$R^0 f(z) = f(z)$$
$$R^1 f(z) = zf'(z)$$
$$\ldots$$
$$(n+1) R^{n+1} f(z) = z(R^n f(z))' + nR^n f(z), \quad \text{for } z \in U.$$

REMARK 1.2. If $f \in A$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j$, for $z \in U$.

LEMMA 1.1. (Hallenbeck and Ruscheweyh [4, Th. 3.1.6, p. 71]) *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C} \backslash \{0\}$ be a complex number with Re $\gamma \geqslant 0$. If $p \in \mathscr{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z), \quad \textit{for } z \in U,$$

then

$$p(z) \prec g(z) \prec h(z), \quad \textit{for } z \in U,$$

where $g(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt$, for $z \in U$.

LEMMA 1.2. (Miller and Mocanu [4]) *Let g be a convex function in U and let $h(z) = g(z) + n\alpha zg'(z)$, for $z \in U$, where $\alpha > 0$ and n is a positive integer.*

If $p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \ldots$, for $z \in U$ is holomorphic in U and

$$p(z) + \alpha zp'(z) \prec h(z), \quad \textit{for } z \in U,$$

then

$$p(z) \prec g(z), \quad \textit{for } z \in U,$$

and this result is sharp.

2. Main results

DEFINITION 2.1. [1] Let $\alpha \geqslant 0$, $n \in \mathbb{N}$. Denote by L_{α}^{n} the operator given by $L_{\alpha}^{n} : A \to A$,

$$L_{\alpha}^{n} f(z) = (1-\alpha) R^{n} f(z) + \alpha S^{n} f(z), \quad \text{for } z \in U.$$

REMARK 2.1. L_{α}^{n} is a linear operator and if $f \in A$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $L_{\alpha}^{n} f(z) = z + \sum_{j=2}^{\infty} \left(\alpha j^n + (1-\alpha) C_{n+j-1}^{n}\right) a_j z^j$, for $z \in U$.

REMARK 2.2. For $\alpha = 0$, $L_{0}^{n} f(z) = R^n f(z)$, where $z \in U$ and for $\alpha = 1$, $L_{1}^{n} f(z) = S^n f(z)$, where $z \in U$.

For $n = 0$, $L_{\alpha}^{0} f(z) = (1-\alpha) R^0 f(z) + \alpha S^0 f(z) = f(z) = R^0 f(z) = S^0 f(z)$, where $z \in U$ and for $n = 1$, $L_{\alpha}^{1} f(z) = (1-\alpha) R^1 f(z) + \alpha S^1 f(z) = z f'(z) = R^1 f(z) = S^1 f(z)$, where $z \in U$.

THEOREM 2.1. *Let g be a convex function, $g(0) = 1$, and let h be the function $h(z) = g(z) + z g'(z)$, for $z \in U$.*

If $\alpha \geqslant 0$, $n \in \mathbb{N}$, $f \in A$ satisfies the differential subordination

$$(L_{\alpha}^{n} f(z))' \prec h(z), \quad \text{for } z \in U, \tag{2.1}$$

then

$$\frac{L_{\alpha}^{n} f(z)}{z} \prec g(z), \quad \text{for } z \in U,$$

and this result is sharp.

Proof. By using the properties of operator L_{α}^{n}, we have

$$L_{\alpha}^{n} f(z) = z + \sum_{j=2}^{\infty} \left(\alpha j^n + (1-\alpha) C_{n+j-1}^{n}\right) a_j z^j, \quad \text{for } z \in U.$$

Consider $p(z) = \frac{L_{\alpha}^{n} f(z)}{z} = \frac{z + \sum_{j=2}^{\infty} \left(\alpha j^n + (1-\alpha) C_{n+j-1}^{n}\right) a_j z^j}{z} = 1 + p_1 z + p_2 z^2 + \ldots$, for $z \in U$.

We deduce that $p \in \mathscr{H}[1,1]$.

Let $L_{\alpha}^{n} f(z) = z p(z)$, for $z \in U$. Differentiating we obtain $(L_{\alpha}^{n} f(z))' = p(z) + z p'(z)$, for $z \in U$.

Then (2.1) becomes

$$p(z) + z p'(z) \prec h(z) = g(z) + z g'(z), \quad \text{for } z \in U.$$

By using Lemma 1.2, we have

$$p(z) \prec g(z), \quad \text{for } z \in U, \quad \text{i.e.} \quad \frac{L_{\alpha}^{n} f(z)}{z} \prec g(z), \quad \text{for } z \in U. \qquad \square$$

THEOREM 2.2. *Let h be an holomorphic function which satisfies the inequality* $\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, *for* $z \in U$, *and* $h(0) = 1$.

If $\alpha \geqslant 0,\ n \in \mathbb{N},\ f \in A$ *satisfies the differential subordination*

$$\left(L_\alpha^n f(z)\right)' \prec h(z), \quad \text{for } z \in U, \tag{2.2}$$

then

$$\frac{L_\alpha^n f(z)}{z} \prec q(z), \quad \text{for } z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. *The function q is convex and it is the best dominant.*

Proof. Let

$$p(z) = \frac{L_\alpha^n f(z)}{z} = \frac{z + \sum_{j=2}^{\infty}\left(\alpha j^n + (1-\alpha)C_{n+j-1}^n\right)a_j z^j}{z}$$
$$= 1 + \sum_{j=2}^{\infty}\left(\alpha j^n + (1-\alpha)C_{n+j-1}^n\right)a_j z^{j-1} = 1 + z + \sum_{j=2}^{\infty} p_j z^{j-1},$$
$$\text{for } z \in U,\ p \in \mathscr{H}[1,1].$$

Differentiating, we obtain $\left(L_\alpha^n f(z)\right)' = p(z) + zp'(z)$, for $z \in U$ and (2.2) becomes

$$p(z) + zp'(z) \prec h(z), \quad \text{for } z \in U.$$

Using Lemma 1.1, we have

$$p(z) \prec q(z), \text{ for } z \in U, \quad i.e. \quad \frac{L_\alpha^n f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt, \text{ for } z \in U,$$

and q is the best dominant. □

THEOREM 2.3. *Let g be a convex function such that $g(0) = 1$ and let h be the function defined by $h(z) = g(z) + zg'(z)$, for $z \in U$.*

If $\alpha \geqslant 0,\ n \in \mathbb{N},\ f \in A$ *and the differential subordination*

$$\left(\frac{zL_\alpha^{n+1} f(z)}{L_\alpha^n f(z)}\right)' \prec h(z), \quad \text{for } z \in U \tag{2.3}$$

holds, then

$$\frac{L_\alpha^{n+1} f(z)}{L_\alpha^n f(z)} \prec g(z), \quad \text{for } z \in U,$$

and this result is sharp.

Proof. For $f \in A$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ we have

$$L_\alpha^n f(z) = z + \sum_{j=2}^{\infty}\left(\alpha j^n + (1-\alpha)C_{n+j-1}^n\right)a_j z^j, \quad \text{for } z \in U.$$

Consider

$$p(z) = \frac{L_\alpha^{n+1} f(z)}{L_\alpha^n f(z)} = \frac{z + \sum_{j=2}^{\infty} \left(\alpha j^{n+1} + (1-\alpha) C_{n+j}^{n+1}\right) a_j z^j}{z + \sum_{j=2}^{\infty} \left(\alpha j^n + (1-\alpha) C_{n+j-1}^{n}\right) a_j z^j}.$$

We have $p'(z) = \frac{\left(L_\alpha^{n+1} f(z)\right)'}{L_\alpha^n f(z)} - p(z) \cdot \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)}$ and we obtain $p(z) + z \cdot p'(z) = \left(\frac{z L_\alpha^{n+1} f(z)}{L_\alpha^n f(z)}\right)'$.

Relation (2.3) becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z), \quad \text{for } z \in U.$$

By using Lemma 1.2, we have

$$p(z) \prec g(z), \quad \text{for } z \in U, \quad i.e. \quad \frac{L_\alpha^{n+1} f(z)}{L_\alpha^n f(z)} \prec g(z), \quad \text{for } z \in U. \qquad \square$$

Following the work done by A. Cătaş [2] (see also [3]) we introduce

DEFINITION 2.2. Let $\delta \in [0,1)$, $\alpha \geqslant 0$ and $n \in \mathbb{N}$. A function $f \in A$ is said to be in the class $S_n(\delta, \alpha)$ if it satisfies the inequality

$$\operatorname{Re} \left(L_\alpha^n f(z)\right)' > \delta, \quad \text{for } z \in U. \tag{2.4}$$

THEOREM 2.4. *The set $S_n(\delta, \alpha)$ is convex.*

Proof. Let the functions

$$f_j(z) = z + \sum_{j=2}^{\infty} a_{jk} z^j, \quad \text{for } k = 1,2, \quad z \in U,$$

be in the class $S_n(\delta, \alpha)$. It is sufficient to show that the function

$$h(z) = \eta_1 f_1(z) + \eta_2 f_2(z)$$

is in the class $S_n(\delta, \alpha)$ with η_1 and η_2 nonnegative such that $\eta_1 + \eta_2 = 1$.

Since $h(z) = z + \sum_{j=2}^{\infty} \left(\eta_1 a_{j1} + \eta_2 a_{j2}\right) z^j$, for $z \in U$, then

$$L_\alpha^n h(z) = z + \sum_{j=2}^{\infty} \left[\alpha j^n + (1-\alpha) C_{n+j-1}^{n}\right] \left(\eta_1 a_{j1} + \eta_2 a_{j2}\right) z^j, \quad \text{for } z \in U. \tag{2.5}$$

Differentiating (2.5) we obtain

$$\left(L_\alpha^n h(z)\right)' = 1 + \sum_{j=2}^{\infty} \left[\alpha j^n + (1-\alpha) C_{n+j-1}^{n}\right] \left(\eta_1 a_{j1} + \eta_2 a_{j2}\right) j z^{j-1}, \quad \text{for } z \in U.$$

Hence

$$\text{Re}\left(L_\alpha^n h(z)\right)' = 1+\text{Re}\left(\eta_1 \sum_{j=2}^{\infty} j\left[\alpha j^n + (1-\alpha) C_{n+j-1}^n\right] a_{j1} z^{j-1}\right) \\ +\text{Re}\left(\eta_2 \sum_{j=2}^{\infty} j\left[\alpha j^n + (1-\alpha) C_{n+j-1}^n\right] a_{j2} z^{j-1}\right). \tag{2.6}$$

Taking into account that $f_1, f_2 \in S_n(\delta,\alpha)$ we deduce

$$\text{Re}\left(\eta_k \sum_{j=2}^{\infty} j\left[\alpha j^n + (1-\alpha) C_{n+j-1}^n\right] a_{jk} z^{j-1}\right) > \eta_k(\delta-1), \quad \text{for } k=1,2. \tag{2.7}$$

Using (2.7) we get from (2.6)

$$\text{Re}\left(L_a^n h(z)\right)' > 1+\eta_1(\delta-1)+\eta_2(\delta-1)=\delta, \quad \text{for } z\in U,$$

which is equivalent that $S_n(\delta,\alpha)$ is convex. □

THEOREM 2.5. *Let g be a convex function in U and let $h(z)=g(z)+\frac{1}{c+2}zg'(z)$, where $z\in U, c>0$.*

If $f\in S_n(\delta,\alpha)$ and $F(z)=I_c(f)(z)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t)\,dt$, for $z\in U$, then

$$\left(L_\alpha^n f(z)\right)' \prec h(z), \quad \textit{for } z\in U, \tag{2.8}$$

implies

$$\left(L_\alpha^n F(z)\right)' \prec g(z), \quad \textit{for } z\in U,$$

and this result is sharp.

Proof. We obtain that

$$z^{c+1}F(z)=(c+2)\int_0^z t^c f(t)\,dt. \tag{2.9}$$

Differentiating (2.9), with respect to z, we have $(c+1)F(z)+zF'(z)=(c+2)f(z)$ and

$$(c+1)L_\alpha^n F(z)+z\left(L_\alpha^n F(z)\right)'=(c+2)L_\alpha^n f(z), \quad \text{for } z\in U. \tag{2.10}$$

Differentiating (2.10) we have

$$\left(L_\alpha^n F(z)\right)'+\frac{1}{c+2}z\left(L_\alpha^n F(z)\right)''=\left(L_\alpha^n f(z)\right)', \quad \text{for } z\in U. \tag{2.11}$$

Using (2.11), the differential subordination (2.8) becomes

$$\left(L_\alpha^n F(z)\right)'+\frac{1}{c+2}z\left(L_\alpha^n F(z)\right)'' \prec g(z)+\frac{1}{c+2}zg'(z). \tag{2.12}$$

If we denote

$$p(z)=\left(L_\alpha^n F(z)\right)', \quad \text{for } z\in U, \tag{2.13}$$

then $p\in\mathscr{H}[1,1]$.

Replacing (2.13) in (2.12) we obtain

$$p(z)+\frac{1}{c+2}zp'(z)\prec g(z)+\frac{1}{c+2}zg'(z), \quad \text{for } z\in U.$$

Using Lemma 1.2 we have

$$p(z)\prec g(z), \quad \text{for } z\in U, \quad i.e. \quad (L_\alpha^n F(z))'\prec g(z), \quad \text{for } z\in U,$$

and g is the best dominant. □

EXAMPLE 2.1. If $f\in S_1\left(1,\frac{1}{2}\right)$, then $f'(z)+zf''(z)\prec\frac{3-2z}{3(1-z)^2}$ implies $F'(z)+zF''(z)\prec\frac{1}{1-z}$, where $F(z)=\frac{3}{z^2}\int_0^z tf(t)\,dt$.

THEOREM 2.6. *Let* $h(z)=\frac{1+(2\delta-1)z}{1+z}$, *where* $\delta\in[0,1)$ *and* $c>0$. *If* $\alpha\geqslant 0$, $n\in\mathbb{N}$ *and* $I_c(f)(z)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t)\,dt$, *for* $z\in U$, *then*

$$I_c\left[S_n(\delta,\alpha)\right]\subset S_n(\delta^*,\alpha), \tag{2.14}$$

where $\delta^*=2\delta-1+(c+2)(2-2\delta)\int_0^1\frac{t^{c+1}}{t+1}dt$.

Proof. The function h is convex and using the same steps as in the proof of Theorem 2.5 we get from the hypothesis of Theorem 2.6 that

$$p(z)+\frac{1}{c+2}zp'(z)\prec h(z),$$

where $p(z)$ is defined in (2.13).

Using Lemma 1.1 we deduce that

$$p(z)\prec g(z)\prec h(z), \quad i.e. \quad (L_\alpha^n F(z))'\prec g(z)\prec h(z),$$

where

$$g(z)=\frac{c+2}{z^{c+2}}\int_0^z t^{c+1}\frac{1+(2\delta-1)t}{1+t}dt=2\delta-1+\frac{(c+2)(2-2\delta)}{z^{c+2}}\int_0^z\frac{t^{c+1}}{t+1}dt.$$

Since g is convex and $g(U)$ is symmetric with respect to the real axis, we deduce

$$\begin{aligned}\operatorname{Re}\,(L_\alpha^n F(z))' &\geqslant \min_{|z|=1}\operatorname{Re} g(z)=\operatorname{Re} g(1)=\delta^*\\ &=2\delta-1+(c+2)(2-2\delta)\int_0^1\frac{t^{c+1}}{t+1}dt.\end{aligned} \tag{2.15}$$

From (2.15) we deduce inclusion (2.14). □

THEOREM 2.7. *Let g be a convex function such that $g(0)=1$ and let h be the function $h(z)=g(z)+zg'(z)$, for $z\in U$.*

If $\alpha\geqslant 0$, $n\in\mathbb{N}$, $f\in A$ and the differential subordination

$$\left(L_{\alpha}^{n+1}f(z)\right)'+\frac{(1-\alpha)nz(R^nf(z))''}{n+1}\prec h(z),\quad \textit{for } z\in U \tag{2.16}$$

holds, then

$$[L_{\alpha}^{n}f(z)]'\prec g(z),\quad \textit{for } z\in U.$$

This result is sharp.

Proof. By using the properties of operator L_{α}^{n}, we obtain

$$L_{\alpha}^{n+1}f(z)=(1-\alpha)R^{n+1}f(z)+\alpha S^{n+1}f(z),\quad \text{for } z\in U. \tag{2.17}$$

Then (2.16) becomes

$$\left((1-\alpha)R^{n+1}f(z)+\alpha S^{n+1}f(z)\right)'+\frac{(1-\alpha)nz(R^nf(z))''}{n+1}\prec h(z),$$

with $z\in U$.

After a short calculation, we obtain

$(1-\alpha)(R^nf(z))'+\alpha(S^nf(z))'+z\left((1-\alpha)(R^nf(z))''+\alpha(S^nf(z))''\right)\prec h(z)$, for $z\in U$.

Let

$$\begin{aligned} p(z) &= (1-\alpha)(R^nf(z))'+\alpha(S^nf(z))'=(L_{\alpha}^{n}f(z))' \\ &= 1+\sum_{j=2}^{\infty}\left(\alpha j^{n+1}+(1-\alpha)jC_{n+j-1}^{n}\right)a_jz^{j-1}=1+p_1z+p_2z^2+\ldots. \end{aligned} \tag{2.18}$$

We deduce that $p\in\mathscr{H}[1,1]$.

Using the notation in (2.18), the differential subordination becomes

$$p(z)+zp'(z)\prec h(z)=g(z)+zg'(z).$$

By using Lemma 1.2, we have

$$p(z)\prec g(z),\quad \text{for } z\in U,\quad \textit{i.e.}\quad (L_{\alpha}^{n}f(z))'\prec g(z),\quad \text{for } z\in U,$$

and this result is sharp. □

EXAMPLE 2.2. If $n=1$, $\alpha=1$, $f\in A$, we deduce that $f'(z)+3zf''(z)+z^2f'''(z)\prec g(z)+zg'(z)$, which yields that $f'(z)+zf''(z)\prec g(z)$, for $z\in U$.

THEOREM 2.8. *Let* $h(z)=\frac{1+(2\beta-1)z}{1+z}$ *be a convex function in* U*, where* $0\leqslant\beta<1$.

If $\alpha\geqslant 0$*,* $n\in\mathbb{N}$*,* $f\in A$ *and satisfies the differential subordination*

$$[L_{\alpha}^{n+1}f(z)]'+\frac{(1-\alpha)nz(R^{n}f(z))''}{n+1}\prec h(z),\quad for\ z\in U, \tag{2.19}$$

then

$$(L_{\alpha}^{n}f(z))'\prec q(z),\quad for\ z\in U,$$

where q *is given by* $q(z)=2\beta-1+2(1-\beta)\frac{\ln(1+z)}{z}$*, for* $z\in U$*. The function* q *is convex and it is the best dominant.*

Proof. Following the same steps as in the proof of Theorem 2.7 and considering $p(z)=(L_{\alpha}^{n}f(z))'$, the differential subordination (2.19) becomes

$$p(z)+zp'(z)\prec h(z)=\frac{1+(2\beta-1)z}{1+z},\quad \text{for } z\in U.$$

By using Lemma 1.1 for $\gamma=1$ and $n=1$, we have $p(z)\prec q(z)$, i.e.,

$$\begin{aligned}(L_{\alpha}^{n}f(z))'\prec q(z)&=\frac{1}{z}\int_{0}^{z}h(t)dt=\frac{1}{z}\int_{0}^{z}\frac{1+(2\beta-1)t}{1+t}dt\\&=2\beta-1+2(1-\beta)\frac{1}{z}\ln(z+1),\quad \text{for } z\in U.\end{aligned}$$ □

THEOREM 2.9. *Let* h *be an holomorphic function which satisfies the inequality* $\text{Re}\left[1+\frac{zh''(z)}{h'(z)}\right]>-\frac{1}{2}$*, for* $z\in U$*, and* $h(0)=1$.

If $\alpha\geqslant 0$*,* $n\in\mathbb{N}$*,* $f\in A$ *and satisfies the differential subordination*

$$\left(L_{\alpha}^{n+1}f(z)\right)'+\frac{(1-\alpha)nz(R^{n}f(z))''}{n+1}\prec h(z),\quad for\ z\in U, \tag{2.20}$$

then

$$(L_{\alpha}^{n}f(z))'\prec q(z),\quad for\ z\in U,$$

where q *is given by* $q(z)=\frac{1}{z}\int_{0}^{z}h(t)dt$*. The function* q *is convex and it is the best dominant.*

Proof. Using the properties of operator L_{α}^{n} and considering $p(z)=(L_{\alpha}^{n}f(z))'$, we obtain

$$\left(L_{\alpha}^{n+1}f(z)\right)'+\frac{(1-\alpha)nz(R^{n}f(z))''}{n+1}=p(z)+zp'(z),\quad \text{for } z\in U.$$

Then (2.20) becomes

$$p(z)+zp'(z)\prec h(z),\quad \text{for } z\in U.$$

Since $p\in\mathscr{H}[1,1]$, using Lemma 1.1, we deduce

$$p(z)\prec q(z),\ \text{for } z\in U,\quad i.e.\quad (L_{\alpha}^{n}f(z))'\prec q(z)=\frac{1}{z}\int_{0}^{z}h(t)dt,\quad \text{for } z\in U,$$

and q is the best dominant. □

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