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COMPLEMENTARINESS WITH RESPECT TO THE LOGARITHMIC MEAN

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Dedicated to Professor Josip Pečarić on the occasion of his 60th birthday

Abstract. We want to determine the complementary of a weighted Gini mean with respect to the logarithmic mean which is again a weighted Gini mean.

1. Introduction

A **mean** is a function $M:\mathbb{R}_+^2 \to \mathbb{R}_+$, with the property

$$\min(a,b) \leqslant M(a,b) \leqslant \max(a,b),\ \forall a,b>0.$$

Each mean is **reflexive**, that is

$$M(a,a)=a,\ \forall a>0.$$

This is also used as the definition of $M(a,a)$, if necessary.

In what follows we use the logarithmic mean $\mathscr{L}$ defined by

$$\mathscr{L}(a,b)=\frac{a-b}{\ln a-\ln b}$$

and the **weighted Gini means** $\mathscr{B}_{r,s;\lambda}$ defined by

$$\mathscr{B}_{r,s;\lambda}(a,b)=\left[\frac{\lambda\cdot a^r+(1-\lambda)\cdot b^r}{\lambda\cdot a^s+(1-\lambda)\cdot b^s}\right]^{\frac{1}{r-s}},\ r\neq s$$

with $\lambda\in[0,1]$ fixed. Of course, $\mathscr{B}_{r,s;0}=\Pi_2$ and $\mathscr{B}_{r,s;1}=\Pi_1$, where Π_1 and Π_2 are the first and the second **projections** defined respectively by

$$\Pi_1(a,b)=a,\quad \Pi_2(a,b)=b,\ \forall a,b\geqslant 0.$$

The special cases of **Lehmer means** $\mathscr{C}_{r;\lambda}$ given by $\mathscr{B}_{r,r-1;\lambda}$ and of **weighted power means** $\mathscr{P}_{r;\lambda}$ given by $\mathscr{B}_{r,0;\lambda}$, will also be mentioned. Note that $\mathscr{A}=\mathscr{P}_{1;1/2}$ and $\mathscr{G}=\mathscr{B}_{1/2,-1/2;1/2}$ are the common arithmetic and geometric mean respectively.

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Given two means M and N, the double sequence of Gauss type defined by:

$$a_{n+1} = M(a_n, b_n), \quad b_{n+1} = N(a_n, b_n),\ n \geqslant 0,$$

is usually convergent to the common limit $P(a_0, b_0)$. Moreover, P defines a mean which is called the **Gaussian product** of the means M and N and is denoted by $P = M \otimes N$. The mean P is $(M,N)-$**invariant** as proven in [1]. This means that P verifies

$$P(M,N) = P,$$

where

$$P(M,N)(a,b) = P(M(a,b),N(a,b)),\ \forall a,b > 0.$$

In fact, using this property of invariance, Gauss showed that $\mathscr{A} \otimes \mathscr{G} = \mathscr{M}$, where

$$\mathscr{M}(a,b) = \frac{\pi}{2} \cdot \left[\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}} \right]^{-1}.$$

As it can be seen in this case, given two means M and N, it is very difficult to find a mean P which is $(M,N)-$invariant. To overcome this difficulty, another method is considered.

DEFINITION 1. A mean N is called **complementary to** M **with respect to** P (or $P-$**complementary** to M) if it verifies

$$P(M,N) = P.$$

If a given mean M has a unique $P-$complementary mean N, denote it by $N = M^P$. Of course, N is the complementary to M with respect to P if and only if P is $(M,N)-$invariant. However, there are easier ways to determine the complementary. For instance, we have the following **trivial cases** of complementariness: for every mean M we have

$$M^M = M, \quad \Pi_1^M = \Pi_2, \quad M^{\Pi_2} = \Pi_2$$

and if M is a symmetric mean then

$$\Pi_2^M = \Pi_1.$$

For the determination of complementaries, three methods have been used: by direct calculation, by using the methods of functional equations, and by the series expansion of means.

The first method was used in [10] for the determination of complementaries of an arbitrary mean with respect to the Greek means.

The problem of invariance in a family of means was solved in some cases using the second method. That is the triples (P,M,N) of means were determined in a given family of means, with the property that N is the complementary of M with respect to P. This is also called the Matkowski-Sutô problem and it was solved in [4] for arithmetic means , in [6] for weighted arithmetic means and in [8] for the Beckenbach-Gini means.

The last method was used for the determination of complementaries of some mean with respect to the weighted geometric mean in [3], with respect to the weighted power mean in [11], with respect to the weighted Gini mean in [12], and with respect to the Stolarsky mean in [9]. In all of these cases, the following Euler's recurrence formula was used (see [5]):

THEOREM 2. *If the function f has the Taylor series*

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot x^n,$$

p is a real number and

$$[f(x)]^p = \sum_{n=0}^{\infty} b_n \cdot x^n,$$

then we have the recurrence relation

$$\sum_{k=0}^{n} [k(p+1)-n] \cdot a_k \cdot b_{n-k} = 0,\ n \geqslant 0.$$

In the current paper we consider the complementariness with respect to the logarithmic mean. Although it is a difficult problem to take on, some results do exist. For instance, in [1] it is shown that the complementary of $\mathscr{G}(\mathscr{P}_{1/2;1/2},\Pi_1)$ with respect to $\mathscr{L}$ is $\mathscr{G}(\mathscr{P}_{1/2;1/2},\Pi_2)$. Using the method of series expansion of means, we want to determine the complementaries of Gini means with respect to the logarithmic mean. We study the case when these complementaries are themselves Gini means. Unfortunately the use of Euler's formula is not possible in this case, which makes the calculations more difficult. We therefore develop a method for obtaining a few coefficients of the series "by hand" and then we use the computer algebra program Maple 8, to obtain more coefficients. It should be noted, however, that we cannot use enough coefficients to obtain a final answer to the above-mentioned problem. There is hope that at some point in the future, when various computer-algebra programs will become available, running on more powerful computers, solutions will be obtained to these type of problems.

2. Series expansion of means

For the study of some problems related to means, in [7] the power series expansion of the normalized function $M(1,1-t)$ is used. In some cases it is impossible to determine all the coefficients. A recurrence relation for the coefficients is therefore very useful. To obtain such a formula we can apply Euler's formula. In this way, the first coefficients of the series expansion of the weighted Gini mean $\mathscr{B}_{q,q-r;\nu}$ (with $r \neq 0$) are given in [2]:

$$\mathscr{B}_{q,q-r;\nu}(1,1-x) = 1-(1-\nu)\cdot x+\nu(1-\nu)(2q-r-1)\cdot\frac{x^2}{2!}$$
$$-\nu(1-\nu)\cdot\{\nu[6q^2-6q(r+1)+(r+1)(2r+1)]$$

$$-3q(q-r)-(r-1)(r+1)\}\cdot\frac{x^3}{3!}$$
$$-\nu(1-\nu)\cdot\{\nu^2[-24q^3+36q^2(r+1)-12q(r+1)(2r+1)$$
$$+(r+1)(2r+1)(3r+1)]+\nu[24q^3-12q^2(3r+1)$$
$$+12q(r+1)(2r-1)-3(r+1)(2r+1)$$
$$(r-1)]-4q^3+6q^2(r-1)-2q\left(2r^2-3r-1\right)$$
$$+(r-2)(r-1)(r+1)\}\cdot\frac{x^4}{4!}+\cdots.$$

We also need the series expansion of the logarithmic mean. Denote

$$\mathscr{L}(1,1-x)=\frac{x}{-\ln(1-x)}=\sum_{n=0}^{\infty}l_nx^n.$$

This gives

$$x=\sum_{n=0}^{\infty}l_nx^n\cdot\sum_{n=1}^{\infty}\frac{x^n}{n}=\sum_{n=1}^{\infty}\left(\frac{l_0}{n}+\frac{l_1}{n-1}+...+l_{n-1}\right)x^n,$$

which allows the step by step determination of the coefficients l_n, thus

$$\mathscr{L}(1,1-x)=1-\frac{1}{2}x-\frac{1}{12}x^2-\frac{1}{24}x^3-\frac{19}{720}x^4-\frac{3}{160}x^5-\frac{863}{60480}x^6+...$$

3. Complementaries with respect to $\mathscr{L}$

Denote the $\mathscr{L}-$complementary of the mean M by $M^{\mathscr{L}}$. We establish the following

THEOREM 3. *If the mean M has the series expansion*

$$M(1,1-x)=1+\sum_{n=1}^{\infty}a_nx^n,$$

then the first coefficients of the series expansion of $M^{\mathscr{L}}$ are

$$\begin{aligned}M^{\mathscr{L}}(1,1-x) = {} & 1-(a_1+1)x+\frac{1}{3}\left(2a_1^2+2a_1-3a_2\right)x^2\\
&+\frac{1}{9}\left(-4a_1^3-3a_1^2+a_1+12a_1a_2-9a_3+6a_2\right)x^3\\
&+\frac{1}{135}\left(44a_1^4+28a_1^3-9a_1^2-180a_1^2a_2-90a_1a_2+7a_1+180a_1a_3\right.\\
&\left.+90a_2^2+15a_2+90a_3-135a_4\right)x^4\\
&+\frac{1}{810}(-208a_1^5-124a_1^4+30a_1^3+1056a_1^3a_2+504a_1^2a_2-29a_1^2\\
&-1080a_1^2a_3-1080a_2^2a_1+25a_1-540a_1a_3-108a_1a_2+1080a_1a_4\\
&-810a_5+42a_2+90a_3+540a_4+1080a_2a_3-270a_2^2)x^5+...\end{aligned}$$

Proof. Denoting $M^{\mathscr{L}} = N$ we have the condition $\mathscr{L}(M,N) = \mathscr{L}$, thus

$$\frac{M-N}{\mathscr{L}} = \ln M - \ln N.$$

Denoting

$$N(1,1-x) = 1 + \sum_{n=1}^{\infty} b_n x^n,$$

we have

$$\frac{\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} b_n x^n}{\sum_{n=0}^{\infty} l_n x^n} = \ln\left(\sum_{n=0}^{\infty} a_n x^n\right) - \ln\left(\sum_{n=0}^{\infty} b_n x^n\right).$$

Substituting

$$\frac{\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} b_n x^n}{\sum_{n=0}^{\infty} l_n x^n} = \sum_{n=1}^{\infty} c_n x^n,$$

we obtain

$$c_n = \sum_{i=1}^{n} \frac{a_{n+1-i} - b_{n+1-i}}{i}.$$

Differentiating the resulting equality

$$\sum_{n=1}^{\infty} c_n x^n = \ln\left(\sum_{n=0}^{\infty} a_n x^n\right) - \ln\left(\sum_{n=0}^{\infty} b_n x^n\right),$$

we deduce

$$\sum_{n=1}^{\infty} n c_n x^{n-1} = \frac{\sum_{n=1}^{\infty} n a_n x^{n-1}}{\sum_{n=0}^{\infty} a_n x^n} - \frac{\sum_{n=1}^{\infty} n b_n x^{n-1}}{\sum_{n=0}^{\infty} b_n x^n},$$

thus

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)\left[\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n\right]$$

$$= \left(\sum_{n=0}^{\infty} b_n x^n\right)\left[\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right] - \left(\sum_{n=0}^{\infty} a_n x^n\right)\left[\sum_{n=0}^{\infty} (n+1) b_{n+1} x^n\right],$$

or

$$\sum_{n=0}^{\infty} \left[d_0 (n+1) c_{n+1} + d_1 n c_n + \ldots + d_n c_1\right] x^n$$

$$= \sum_{n=0}^{\infty} \left[b_0 (n+1) a_{n+1} + b_1 n a_n + \ldots + b_n a_1 - a_0 (n+1) b_{n+1} - a_1 n b_n - \ldots - a_n b_1\right] x^n,$$

where $d_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$. The equality of the coefficients of the same power of x allows the step by step determination of b_n.

4. $\mathscr{L}-$Complementaries of $\mathscr{B}_{q,q-r;\nu}$

Using the previous result, we conclude the following

THEOREM 4. *The first terms of the series expansion of the $\mathscr{L}-$complementary of $\mathscr{B}_{m,m-r;\nu}$ are*

$$\begin{aligned}\mathscr{B}^{\mathscr{L}}_{m,m-r;\nu}(1,1-x) = {} & 1-\nu x+\nu(1-\nu)(3r-6m-1)\cdot\frac{x^2}{6}\\ & +\nu(1-\nu)\big(6\nu r^2-18\nu rm-3r\nu+18\nu m^2+6m\nu-\nu+6r-3r^2\\ & -12m+9rm-9m^2-1\big)\cdot\frac{x^3}{18}+\nu(1-\nu)\big(-26+255r+180m\nu\\ & -90r\nu+630rm-510m+1080r^2m\nu-1620rm^2\nu-270r^3\nu\\ & +1080m^3\nu-630m^2-210r^2+45r^3+270rm^2-180r^2m-53\nu\\ & +1620rm^2\nu^2-1080r^2m\nu^2+270r^3\nu^2-1080m^3\nu^2-7\nu^2\\ & +1620m^2\nu-180m^3-165r^2\nu^2-540m^2\nu^2+180m\nu^2+540mr\nu^2\\ & -1620rm\nu+525r^2\nu\big)\cdot\frac{x^4}{1080}+\cdots.\end{aligned}$$

COROLLARY 5. *If*

$$\mathscr{B}^{\mathscr{L}}_{m,m-r;\nu}=\mathscr{B}_{p,p-q;\mu},\ \textit{for } rq\neq 0$$

then we can get non-trivial cases only if

$$\nu=\mu=\frac{1}{2}.$$

Proof. Equating the coefficients of x we have

$$\mu=1-\nu.$$

Passing to the coefficients of x^2, we get the trivial cases $\nu=0$ and $\nu=1$, or the condition

$$2m-r=\frac{2}{3}-2p+q.$$

This condition, for the coefficients of x^3 , can be true in one of following three cases:

$$1)\ \nu=\mu=\frac{1}{2},\quad 2m-r=\frac{2}{3}-2p+q;$$

$$2)\ r=q,\quad \mu=1-\nu,\quad m=q-p+\frac{1}{3};$$

or

$$3)\ r=-q,\quad \mu=1-\nu,\quad m=\frac{1}{3}-q.$$

Equating the coefficients of x^k, $k=1,2,...,7$, we get: 1) $\nu=\mu=1/2$ and r,m,p as solutions of complicated equations of sixth degree, with coefficients dependent of q; 2) $\nu=\mu=1/2$, $r=q=0.53748...$, $p=0.31544...$, $m=1/3-p$; 3) $\nu=\mu=1/2$, $r=-q=-0.53748...$, $p=0.31544...$, $m=1/3-p$.

REMARK 6. Unfortunately none of the above determined possible solutions can be verified. To find the right solutions of the above problem, we have to add more equations to the system. These equations are quite complex and they are unsolvable by Maple 8 running on a usual computer. We can only obtain some special results.

COROLLARY 7. *If*

$$\mathscr{C}^{\mathscr{L}}_{m;\nu}=\mathscr{B}_{p,p-q;\mu},\ \textit{for}\ q\neq 0$$

then the only non-trivial case can be given by:

$$\nu=\mu=\frac{1}{2},\quad m=1.03284...,\quad p=0.54010...,\quad q=1.47922...$$

COROLLARY 8. *There is no nontrivial case such that*

$$\mathscr{C}^{\mathscr{L}}_{m;\nu}=\mathscr{C}_{p;\mu}.$$

COROLLARY 9. *There is no nontrivial case such that*

$$\mathscr{P}^{\mathscr{L}}_{m;\nu}=\mathscr{B}_{p,p-q;\mu},\ \textit{for}\ q\neq 0.$$

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