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TWO–SIDED ESTIMATES OF PROJECTION OPERATORS NORM, WITH APPLICATIONS TO DEFORMABLE MODELS

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. This paper is devoted to give estimates involving the norm for classes of projection operators (used, for example, in the theory of deformable models) and to obtain theorems concerning the convergence or the superdense unbounded divergence corresponding to these operators.

1. Introduction

Let C be the Banach space of all continuous functions $f:[-1,1]\to\mathbb{R}$, endowed with the uniform norm and let $\mathscr{P}_n$ be the space of all polynomials of degree at most n. If $r\geqslant 1$ is an integer, denote by C^r the space of all functions $f:[-1,1]\to\mathbb{R}$ which are continuous together with their derivatives up to the order r; we admit $C^0=C$.

The linear and continuous operators $U_n:C\to\mathscr{P}_n$, $n\geqslant 0$, are said to be **projection operators** (shortly: *projections*) of C into $\mathscr{P}_n$ if the equality $U_nf=f$ holds for each $n\geqslant 0$ and $f\in\mathscr{P}_n$.

The fundamental purpose of this paper is to obtain inequalities involving the norm for classes of projection operators, such as least squares projections of integral type and discrete best approximation operators, which are used in the theory of deformable models. Based on these inequalities, we shall prove theorems concerning the convergence or the superdense unbounded divergence of the corresponding family of projections. Consequently, in the second section of the paper, we shall present, shortly, general estimates involving the norm of projection operators.

The following *principle of the condensation of singularities* will be used in the third section.

THEOREM 1.1. [3] *If X is a Banach space, Y a normed space and $(A_n)_{n\geqslant 0}$ is a sequence of continuous linear operators from X into Y so that the set of norms $\{\|A_n\|:n\geqslant 0\}$ is unbounded, then the set of unbounded divergence associated to the family $\{A_n:n\geqslant 0\}$, i.e.*

$$UD(X)=\{x\in X:\ \limsup_{n\to\infty}\|A_nx\|=\infty\}\tag{1.1}$$

is superdense in X.

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The set $UD(X)$ is named, also, the set of *singularities* of the family $\{A_n : n \geqslant 0\}$. We recall that a subset of a topological space T is said to be *superdense* in T if it is residual (i.e. its complement is of first Baire category), uncountable and dense in T.

In what follows, we shall denote by M_k, $k \geqslant 1$, some positive constants which do not depend on n. We shall write, usually, $a_n \sim b_n$ if the sequences of real numbers (a_n) and (b_n), with $b_n \neq 0$, $\forall\, n \geqslant 0$, satisfy the inequalities $0 < M_1 \leqslant |a_n/b_n| \leqslant M_2$, $\forall\, n \geqslant 0$ [10], [11].

2. Estimates involving the norms of projection operators

Suppose that $U_n : C \to \mathscr{P}_n$, $n \geqslant 0$, are projection operators. Writing $U_n f - f = U_n(f-P) + P - f$, we deduce:

$$\|U_n f - f\| \leqslant (\|U_n\| + 1)\|f - P\|; \quad f \in C;\ P \in \mathscr{P}_n. \tag{2.1}$$

It is known that for $f \in C^r[a,b]$, the distance from n-th degree polynomials is bounded by

$$const(r) \cdot \left(\frac{b-a}{n-1}\right)^r \omega\left(f^{(r)}; \frac{b-a}{2(n-r-1)}\right) \tag{2.2}$$

where $const(r) = 6(3e^r)/(1+r)$, [7].

Now, it follows from (2.1) and (2.2):

$$\|U_n f - f\| \leqslant M_3(\|U_n\| + 1)n^{-r}\omega\left(f^{(r)}; \frac{1}{n}\right); \quad f \in C^r,\ r \geqslant 0 \tag{2.3}$$

where M_3 depends on r.

Further, let us point out, a lower bound for the norm of the projection operators U_n, $n \geqslant 0$. If $T_n : C \to \mathscr{P}_n$, $n \geqslant 0$, are linear and continuous operators and

$$\Delta_n = \sup\{\|T_n f - f\| : f \in \mathscr{P}_n,\ \|f\| \leqslant 1\}, \quad n \geqslant 0,$$

then the following inequalities

$$\|T_n\| \geqslant \frac{2}{\pi^2}(1 - \Delta_n)\ln n + O(1), \quad n \geqslant 1 \tag{2.4}$$

hold, [8]. The relations (2.4) generalize the famous Theorem of Lozinski and Harsiladze, which asserts that for any family of projection operators $(U_n)_{n \geqslant 0}$, i.e. $\Delta_n = 0$, $\forall\, n \geqslant 0$, the inequalities

$$\|U_n\| \geqslant \frac{2}{\pi^2}\ln n, \quad n \geqslant 1, \tag{2.5}$$

are valid, [9].

3. Convergence and superdense unbounded divergence for projection operators

The relations (2.3) lead to the following statement.

THEOREM 3.1. *Suppose that* $(U_n)_{n\geqslant 0}$ *are projection operators and* $r \geqslant 0$ *is a given integer.*

1°. *If* $\|U_n\| = O(n^r)$, *then the sequence* $(U_n)_{n\geqslant 0}$ *is convergent on the space* C^r, *i.e.* $\lim_{n\to\infty} U_n f = f$, $\forall\, f \in C^r$.

2°. *If* $\|U_n\| = O(\ln n)$, *then the sequence* $(U_n)_{n\geqslant 0}$ *is convergent on the subset* $DL(C)$ *of all functions* $f \in C$ *which satisfy a Dini-Lipschitz condition* $\lim_{\delta\searrow 0} \omega(f;\delta)\ln\delta = 0$.

Regarding the topological characterization of the set of unbounded divergence associated to a family of projection operators, we get:

THEOREM 3.2. *If* $\{U_n : n \geqslant 0\}$ *is a family of projection operators, then the corresponding set of unbounded divergence* $UD(C)$, *described by* (1.1), *is superdense in the Banach space* $(C, \|\cdot\|)$.

Proof. Let us take $X = C$, $Y = \mathscr{P}_n$ and $A_n = U_n$ in Theorem 1.1. The inequalities (2.5) provide the unboundedness of the set $\{\|U_n\| : n \geqslant 0\}$.

4. Projection operators in the theory of deformable models

This section is devoted to emphasize important classes of projection operators, which are used in the theory of deformable models, in order to approximate some curves or surfaces. A 2D deformable model is based on a parametric curve $(\gamma) : v = v(s) = (x(s), y(s))$, $-1 \leqslant s \leqslant 1$, but it is often represented by a discrete set of points $v_i = (x_i, y_i)$, named *snaxels*, [4], [5]. We associate to this curve the so-called energy-functional $E(v)$ defined by an integral involving a function $\phi(v,g)$ which measures the distance between the "reconstruction" $g(s)$ and the data $v(s)$. For the sake of simplicity, we consider an explicit cartesian curve, i.e. $v = (x,y)$, with $x(s) = s$ and $y(s) = f(s)$, $s \in [-1,1]$. Our goal is to find that "reconstruction" g, belonging to a given space of functions (usually a class of polynomials), which minimizes $\phi(f,g)$.

A. Least-Squares Projections of integral type

Let $w : [-1,1] \to \mathbb{R}$ be a weight-function and denote by $(e_n)_{n\geqslant 0}$ the sequence of orthonormal polynomials with respect to w. It is known that for each $f \in C$ and $n \in \mathbb{N}$, the polynomial $g = U_n f$ for which the infimum of the set

$$\left\{ \int_{-1}^{1} w(x)[P(x) - f(x)]^2 dx : \; P \in \mathscr{P}_n \right\}$$

is attained, is given by

$$U_n f = \sum_{k=0}^{n} \langle f, e_k \rangle \cdot e_k, \tag{4.1}$$

where $\langle f, e_k \rangle = \int_{-1}^{1} w(x)f(x)e_k(x)dx$, [2], [6].

It is a simple exercise to obtain the inequalities

$$\|U_n f\| \leqslant \left(\sum_{k=0}^{n} \tau_k \|e_k\| \right) \|f\|, \quad f \in C,\ n \geqslant 0 \tag{4.2}$$

with $\tau_k = \int_{-1}^{1} w(x)|e_k(x)|dx$.

Further, let us take as $w(x)$ the *Jacobi-weight*

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha > -1,\ \beta > -1$$

and denote by $P_n^{(\alpha,\beta)}$, $e_n^{(\alpha,\beta)}$ the corresponding Jacobi polynomials and orthonormal Jacobi polynomials, respectively. In order to estimate the norm of U_n, we get for τ_k of (4.2), by means of Cauchy-Schwarz-Buniakowski inequality:

$$\begin{aligned} \tau_k^2 &\leqslant \int_{-1}^{1} w(x)dx \int_{-1}^{1} w(x)e_k^2(x)dx = \int_{-1}^{1} w(x)dx \\ &= \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}dx. \end{aligned}$$

With the substitution $x = 2t-1$, we obtain [9], [11]:

$$\tau_k^2 \leqslant 2^{\alpha+\beta+1} \cdot \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} = M_4. \tag{4.3}$$

Regarding $e_k^{(\alpha,\beta)}$, $k \geqslant 0$, we have [9]:

$$e_k^{(\alpha,\beta)} = \delta_k P_k^{(\alpha,\beta)};$$

$$\delta_k^2 = \frac{\Gamma(\alpha+\beta+2k+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \binom{\alpha+\beta+2k}{k}^{-1} \tag{4.4}$$

where

$$\binom{m}{n} = \frac{m(m-1)\dots(m-n+1)}{n!}, \quad m \in \mathbb{R},\ n \in \mathbb{N}.$$

In accordance with the estimates $\binom{n+\alpha}{n} \sim n^{\alpha}$ and $\Gamma(n+\alpha+1) \sim n!n^{\alpha}$, $\alpha > -1$, we obtain by standard computations:

$$\delta_k \sim \sqrt{k}. \tag{4.5}$$

Now, the relations (4.4) and (4.5), together with the estimate

$$\|P_k^{(\alpha,\beta)}\| \sim k^{\mu},$$

with $\mu = \max\left\{\max\{\alpha,\beta\}, -\frac{1}{2}\right\}$, [11], give

$$\|e_k^{(\alpha,\beta)}\| \sim k^{\mu+1/2}$$

which, combined with (4.2) and (4.3), leads to the inequalities:

$$\|U_n f\| \leqslant M_5 n^{\mu+3/2}\|f\|; \quad n \geqslant 0,\ f \in C.$$

So, we conclude:

$$\|U_n\| \leqslant M_5 n^{\mu+3/2}. \tag{4.6}$$

We are in a position to prove the following statement.

THEOREM 4.1. *Suppose that* $\{U_n : n \geqslant 0\}$ *is the family of the least-squares projections of integral-type, defined by (*4.1*).*

1° *The set of unbounded divergence* $UD(C)$ *of the family* $\{U_n : n \geqslant 0\}$ *is superdense in the Banach space* $(C, \|\cdot\|)$.

2° *If* w *is the Jacobi-weight and* $r \geqslant \mu + 3/2$*, then the sequence* $(U_n)_{n\geqslant 0}$ *is convergent on* C^r.

Proof. The assertion 1° is a direct consequence of Theorem 3.2, since $(U_n)_{n\geqslant 0}$ are projection operators. The assertion 2° follows from (4.6) and Theorem 3.1 (1°), because the hypothesis $r \geqslant \mu + \frac{3}{2}$ implies $\|U_n\| = O(n^r)$.

B. Projection operators of discrete type

The most classic problem in the theory of deformable models is the so-called "best fitting", given the position of a curve at a collection of points $(s_i, f(s_i))$, $1 \leqslant i \leqslant m$, at known values of the parameter s_i, [4]. We can interpret physically this problem as a spring connecting a point $(U_n f)(s_i)$ of the curve (γ) and a given point $A_i(s_i, f(s_i))$. From mathematical point of view, writing x_i instead of s_i, this problem can be described as follows.

B1. Least squares projections of discrete type

Given $f \in C$, the points $\{x_0, x_1, x_2, x_3, \ldots, x_m\} \subseteq [-1,1]$ and the weights w_j, $0 \leqslant j \leqslant m$, let us find a polynomial $g = U_n f \in \mathscr{P}_n$, $n < m$, so that

$$\sum_{j=0}^{m} w_j[(U_n f)(x_j) - f(x_j)]^2 = \inf\left\{\sum_{j=0}^{m} w_j[P(x_j) - f(x_j)]^2 : P \in \mathscr{P}_n\right\}$$

Similarly to the integral case, we consider a set of orthonormal polynomials p_n, $n \geqslant 0$, with respect to the weights w_j, $0 \leqslant j \leqslant m$ (such as Chebyshev or Krawtchouk polynomials, [6]), and deduce

$$U_n f = \sum_{k=0}^{n} \langle f, p_k\rangle p_k, \quad n \geqslant 0 \tag{4.7}$$

with $\langle f, p_k \rangle = \sum_{j=0}^{m} w_j f(x_j) p_k(x_j)$.

Analogous to the integral case, but performing larger computations, we can obtain estimates concerning the norm of $U_n f - f$, $n \geqslant 0$. In this paper, we restrict to emphasize the following divergence result, which follows from Theorem 3.2.

THEOREM 4.2. *The set of unbounded divergence* $UD(C)$*, associated to the family of projection operators* $\{U_n : n \geqslant 0\}$ *given by (4.7), is superdense in the Banach space* $(C, \|\cdot\|)$.

B2. Discrete best approximation projections

Given an integer $m = m(n) \geqslant n+1$, $n \in \mathbb{N}^*$, a node matrix $\mathscr{M} = \{x_m^k :\ 1 \leqslant k \leqslant m,\ m \geqslant 1\} \subseteq [-1,1]$ and a function f in C, we search a polynomial $U_n f \in \mathscr{P}_n$ satisfying the condition

$$\max\{|(U_n f)(x_m^k) - f(x_m^k)| :\ 1 \leqslant k \leqslant m\}$$
$$= \min\{\max\{|P(x_m^k) - f(x_m^k)| :\ 1 \leqslant k \leqslant m\} :\ P \in \mathscr{P}_n\}.$$

In this paper, we shall consider the cases $m = n+1$ and $m = n+2$.

If $m = n+1$, it is clear that the solution is given by the *Lagrange interpolation projections*, i.e.:

$$U_n f = L_n f = \sum_{k=1}^{n+1} f(x_{n+1}^k) l_{n+1}^k, \quad n \geqslant 0 \tag{4.8}$$

where l_{n+1}^k, $n \geqslant 1$, $1 \leqslant k \leqslant n+1$, are the fundamental polynomials of Lagrange interpolation with respect to the node matrix $\mathscr{M}$.

If $m = n+2$, $n \geqslant 0$, let us denote by $a_{n+1}(f) = a_{n+1}(\mathscr{M}; f)$ the leading-coefficient of Lagrange polynomial $L_{n+1} f$, which interpolates f at the points x_{n+2}^k, $1 \leqslant k \leqslant n+2$ and consider a function $\sigma_{n+2} \in C$ satisfying the relations $\sigma_{n+2}(x_{n+2}^k) = (-1)^k$, $1 \leqslant k \leqslant n+2$; it is easily seen that $a_{n+1}(\sigma_{n+2}) \neq 0$.

Firstly, let us prove the equality

$$U_n f = L_{n+1} f - \frac{a_{n+1}(f)}{a_{n+1}(\sigma_{n+2})} L_{n+1} \sigma_{n+2}; \quad n \geqslant 0,\ f \in C. \tag{4.9}$$

Denote by $T_n f$ the polynomial of the right-hand of (4.9). Clearly, $T_n f \in \mathscr{P}_n$, $\forall f \in C$ and $T_n P = P$, $\forall P \in \mathscr{P}_n$ (because $a_{n+1}(P) = 0$ and $L_{n+1} P = P$); moreover,

$$(T_n f)(x_{n+2}^k) - f(x_{n+2}^k) = (-1)^{k+1} \frac{a_{n+1}(f)}{a_{n+1}(\sigma_{n+2})}, \quad 1 \leqslant k \leqslant n+2.$$

Taking into account these relations we deduce, by using Theorem of Charles de la Vallée-Poussin, [6], [9], the equality $T_n f = U_n f$, so that (4.9) is true.

Now, let us estimate the norm of the linear operators $U_n f$ of (4.9). We deduce from (4.9):

$$(U_n f)(x) = \sum_{k=1}^{n+2} \left[f(x_{n+2}^k) + \frac{a_{n+1}(f)}{a_{n+1}(\sigma_{n+2})} (-1)^{k+1} \right] l_{n+2}^k(x), \quad |x| \leqslant 1,$$

which leads to the inequality

$$|(U_nf)(x)| \leqslant \sum_{k=1}^{n+2}\left(\|f\| + \frac{|a_{n+1}(f)|}{|a_{n+1}(\sigma_{n+2})|}\right)|l_{n+2}^k(x)|, \quad |x| \leqslant 1. \tag{4.10}$$

Denoting $u_{n+2}(x) = (x - x_{n+2}^1)(x - x_{n+2}^2)\dots(x - x_{n+2}^{n+2})$, we obtain:

$$a_{n+1}(f) = \sum_{k=1}^{n+2}\frac{f(x_{n+2}^k)}{u_{n+2}'(x_{n+1}^k)}; \quad a_{n+1}(\sigma_{n+2}) = \sum_{k=1}^{n+2}\frac{(-1)^k}{u_{n+2}'(x_{n+2}^k)}$$

and $\operatorname{sign} u_{n+2}'(x_{n+2}^k) = (-1)^{n-k}$, $1 \leqslant k \leqslant n+2$, so:

$$|a_{n+1}(f)| \leqslant |a_{n+1}(\sigma_{n+2})| \cdot \|f\|; \quad f \in C,\ n \geqslant 1. \tag{4.11}$$

The relations (4.10) and (4.11) give:

$$\|U_nf\| \leqslant 2\lambda_{n+1}\|f\|; \quad n \geqslant 0,\ f \in C \tag{4.12}$$

where

$$\lambda_{n+1} = \max\left\{\sum_{k=1}^{n+2}|l_{n+2}^k(x)| : \ |x| \leqslant 1\right\}, \quad n \geqslant 0,$$

are the *Lebesgue constants*.

The inequalities (4.12) prove the continuity of U_n, $n \geqslant 0$, and together with the standard relations [2], [9]:

$$\lambda_n = \|L_n\|; \quad n \geqslant 0 \tag{4.13}$$

give:

$$\|U_n\| \leqslant 2\|L_{n+1}\|, \quad n \geqslant 0. \tag{4.14}$$

Now, by using Theorems 3.1 and 3.2, we can state

THEOREM 4.3. *If $\mathscr{M}$ is an arbitrary node matrix of $[-1,1]$ and $\{U_n : n \geqslant 0\}$ is the family of projections defined by (4.8) or (4.9), then the corresponding set of unbounded divergence $UD(C)$ is superdense in the Banach space $(C, \|\cdot\|)$.*

THEOREM 4.4. *Let $\mathscr{M}^\alpha$, $\alpha > -1$, be the Jacobi ultraspherical node matrix (namely, its rows are the roots of the ultraspherical Jacobi polynomials $P_n^{(\alpha)}$) and let $\{U_n : n \geqslant 0\}$ be the family of projections given by (4.8) or (4.9).*

1° *The sequence $(U_n)_{n \geqslant 0}$ is convergent on the space C^r in the following situations:*

(i) $\alpha \leqslant -\dfrac{1}{2}$, $r \geqslant 1$;

(ii) $\alpha > -\dfrac{1}{2}$, $r \geqslant \alpha + \dfrac{1}{2}$.

2° *If $\alpha \leqslant -\dfrac{1}{2}$, then the sequence $(U_n)_{n \geqslant 0}$ is convergent on the subset $DL(C) \subseteq C$ of all functions $f \in C$ which satisfy a Dini-Lipschitz condition $\lim\limits_{\delta \searrow 0}\omega(f;\delta)\ln\delta = 0$.*

Proof. Indeed, we deduce from (4.13) and (4.14):

$$\|U_n\| = \lambda_n \text{ for } (4.8) \text{ and } \|U_n\| \leqslant 2\lambda_{n+1} \text{ for } (4.9). \tag{4.15}$$

On the other hand, we obtain from [1]:

$$\lambda_n = \begin{cases} O(\ln n + \|P_n^{(\alpha)}\|\sqrt{n}), & \text{if } \alpha > -\frac{1}{2} \\ O(\ln n), & \text{if } \alpha \leqslant -\frac{1}{2} \end{cases} \tag{4.16}$$

The relations (4.15) and (4.16), combined with the estimate $\|P_n^{(\alpha)}\| \sim n^{\alpha}$, $\alpha \geqslant -\dfrac{1}{2}$, [11], prove the assertion 1°, because $\|U_n\| = O(n^r)$. Moreover, if $r = 0$ and $\alpha \leqslant -\dfrac{1}{2}$, the second relation of (4.16) is used. Now, apply Theorem 3.1, which completes the proof.

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