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INEQUALITIES ON REAL ROOTS OF POLYNOMIALS

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Dedicated to Professor Josip Pecarić's
60th Birthday

Abstract. We survey the most used bounds for positive roots of polynomials and discuss their efficiency. We obtain new inequalities on roots of polynomials. Then we give new inequalities on roots of orthogonal polynomials, obtained from the differential equations satisfied by these polynomials.

1. Introduction

Polynomials are among the most frequently used mathematical objects. They appear in all branches of Mathematics and have strong computational features. In this paper we discuss inequalities concerning the real roots of univariate polynomials with real coefficients.

We present new methods for estimating bounds for positive roots. In particular the estimation of lower bounds is a key step in real root isolation. This is realized by obtaining new inequalities satisfied by the largest positive roots.

Several bounds exist for the absolute values of the roots of a univariate polynomial with complex coefficients. These bounds are expressed as functions of the degree and of the coefficients, and naturally they can be used also for the roots (real or complex) of polynomials with real coefficients.

However, for the real roots of polynomials with real coefficients there also exist some specific bounds. We briefly survey here the most often used bounds for positive roots and discuss their efficiency in particular cases, emphasizing the classes of orthogonal polynomials. We then obtain new inequalities on the positive roots of polynomials. We also give new inequalities on roots of orthogonal polynomials derived from the differential equations satisfied by these polynomials.

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2. Bounds for polynomial roots

The computation of bounds for roots of polynomials is important for computation of roots and for estimation of polynomial sizes.

Several bounds exist for the absolute values of the roots of a univariate polynomial with complex coefficients. They are expressed as functions of the degree and of the coefficients (see [10], ch. 2).

Bounds for real roots of polynomials

The computation of the real roots of univariate polynomials with real coefficients is based on their isolation. To isolate the real positive roots, it is sufficient to estimate the smallest positive root. This can be frequently achieved through the consideration of the reciprocal polynomial and the computation of accurate estimates for positive dominant roots.

For obtaining bounds of the largest positive roots we can use the known bounds for univariate polynomials with complex coefficients. But there also exist some specific bounds. The classical bound for the largest positive root was given by the following

THEOREM 1. (Lagrange, [7]) *Let* $P(X) = a_0X^d + \cdots + a_mX^{d-m} - a_{m+1}X^{d-m-1} \pm \cdots \pm a_d \in \mathbb{R}[X]$, *with all* $a_i \geqslant 0$, $a_0, a_{m+1} > 0$. *Let*

$$A = \max\left\{a_i ; \text{coeff}\,(X^{d-i}) < 0\right\}.$$

The number

$$1 + \left(\frac{A}{a_0}\right)^{1/(m+1)}$$

is an upper bound for the positive roots of P.

Theorem 1 returns only numbers larger than one. For polynomials with subunitary real roots (like Legendre orthogonal polynomials) it is recommended to use the more recent bounds of Kioustelidis [4] and Ştefănescu [12].

The bound of Kioustelidis (1986)

J. B. Kioustelidis [4] gives the following upper bound for the positive real roots:

THEOREM 2. (Kioustelidis [4]) *Let* $P(X) = X^d - b_1X^{d-m_1} - \cdots - b_kX^{d-m_k} + g(X)$, *with* $g(X)$ *having positive coefficients and* $b_1 > 0, \ldots, b_k > 0$. *The number*

$$K(P) = 2 \cdot \max\{b_1^{1/m_1}, \ldots, b_k^{1/m_k}\}$$

is an upper bound for the positive roots of P.

A bound of Ştefănescu (2005, 2008)

For polynomials with an even number of variations of sign, we proposed in [12] another bound. Our method can be applied also to polynomials having at least one sign variation and gives the following

THEOREM 3. *Let* $P(X) \in \mathbb{R}[X]$ *and suppose that* P *has at least one sign variation. If*

$$P(X) = c_1 X^{d_1} - b_1 X^{m_1} + c_2 X^{d_2} - b_2 X^{m_2} + \cdots + c_k X^{d_k} - b_k X^{m_k} + g(X),$$

with $g(X) \in \mathbb{R}_+[X]$, $c_i > 0$, $b_i > 0$, $d_i > m_i$ *for all* i, *the number*

$$S(P) = \max\left\{ \left(\frac{b_1}{c_1}\right)^{1/(d_1-m_1)}, \ldots, \left(\frac{b_k}{c_k}\right)^{1/(d_k-m_k)} \right\}$$

is an upper bound for the positive roots of P.

NOTATION.

- The bound of Lagrange will be denoted by $L_1(P)$.
- The bound of Kioustelidis from Theorem 2 is denoted by $K(P)$.

EXAMPLE. Let $P(X) = 2X^7 - 3X^4 - X^3 - 2X + 1 \in \mathbb{R}[X]$. We use two representations of P:

$$P(X) = P_1(X) = (X^7 - 3X^4) + (0.5X^7 - X^3) + (0.5X^7 - 2X) + 1,$$
$$P(X) = P_2(X) = (1.1X^7 - 3X^4) + (0.4X^7 - X^3) + (0.5X^7 - 2X) + 1.$$

and obtain the bounds

$$S_1(P) = 1.442, \quad S_2(P) = 1.397.$$

Note that the largest positive root of P is 1.295.

Other bounds give

$$K(P) = 2.289, \quad L_1(P) = 2.404.$$

Note that both $S_1(P)$ and $S_2(P)$ are smaller than $L_1(P)$ and $K(P)$.

We obtain a more general bound in the next

THEOREM 4. *Let*

$$P(X) = a_1 X^{d_1} + a_2 X^{d_2} + \cdots + a_s X^{d_s} - b_1 X^{e_1} - b_2 X^{e_2} - \cdots - b_s X^{e_t} \in \mathbb{R}[X],$$

where $a_i >$, $b_j > 0$, $d_1 = \deg(P)$ *and* $d_1 > d_2 \geqslant \cdots \geqslant d_s$. *An upper bound for the positive roots of* P *is given by*

$$\max_{\substack{1 \leqslant i \leqslant s \\ 1 \leqslant j \leqslant t \\ \beta_j \neq 0}} \left(\frac{\gamma_{ji} b_j}{\beta_j a_i}\right)^{\dfrac{1}{d_i - e_j}}$$

for any $\beta_j \geqslant 0$, $\gamma_{jk} \geqslant 0$ *such that*

$$\begin{aligned} &\textstyle\sum_{j=1}^{t} \beta_j \;= 1\,, \\ &\textstyle\sum_{i=1}^{s} \gamma_{ji}\,, = 1 \quad \textit{with} \quad \gamma_{ji} = 0 \quad \textit{if} \quad d_i < e_j\,. \end{aligned}$$

Proof. For $x > 0$ we have

$$\begin{aligned} P(x) &= \sum_{i=1}^{s} a_i x^{d_i} - \sum_{j=1}^{t} b_j x^{e_j} \\ &= \left(\sum_{j=1}^{t} \beta_j\right) \cdot \left(\sum_{i=1}^{s} a_i x^{d_i}\right) - \left(\sum_{j=1}^{t} b_j x^{e_j}\right) \\ &= \sum_{j=1}^{t} \left(\sum_{i=1}^{s} \beta_j\, a_i x^{d_i} - b_j x^{e_j}\right) \qquad (1) \\ &= \sum_{j=1}^{t} \left(\sum_{i=1}^{s} \beta_j\, a_i x^{d_i} - \left(\sum_{i=1}^{s} \gamma_{ji}\right) b_j x^{e_j}\right) \\ &= \sum_{j=1}^{t} x^{e_j} \sum_{i=1}^{s} \left(\beta_j\, a_i x^{d_i - e_j} - \gamma_{ji}\, b_j\right). \end{aligned}$$

It follows that $P(x) > 0$ as soon as

$$x > \left(\frac{\gamma_{ji} b_j}{\beta_j a_i}\right)^{\dfrac{1}{d_i - e_j}}$$

for all $j = 1, 2, \ldots, t$, $i = 1, 2, \ldots, s$, $\beta_j \neq 0$ ..

Which gives our bound. □

3. Applications

We compare the bounds given by Theorem 4 with the following bounds:

$$L_1(P) = 1 + \left(\frac{A}{a_0}\right)^{1/(m+1)} \qquad \text{Lagrange [7]}$$

$$L_2(P) = 1 + \frac{A}{a_0 + a_1 + \cdots + a_m} \qquad \text{Longchamp [9]}$$

$$K(P) = 2\max\{b_j^{1/(d-e_j)}\} \qquad \text{Kioustelidis [4]}$$

$$S_1(P) = 1 + \max\left\{\left(\frac{A}{2a_0}\right)^{1/(m+1)}, \left(\frac{A}{2(a_0+a_1)}\right)^{1/m}\right\} \qquad \text{Ştefănescu [13]}$$

The computations were done using the `gp-pari` package.

We consider the polynomial

$$P(X) = X^7 + X^6 - 3X^2 - 2X^2 - 3\,.$$

We note that our Theorem 2 from [12] cannot be applied. We use instead Theorem 4. We consider

1) $\beta_1 = \beta_2 = \beta_3 = \frac{1}{3}$, $\gamma_{ij} = \frac{1}{2}$.

Thus gives the bound 1.651. We denote it by $BP_1(P)$

2) Representing P as

$$P(X) = 0.5X^7 + 0.5X^7 + X^6 - 3X^3 - 2X^2 - 3$$

we take

i. $\beta_1 = \beta_2 = 0.5$, $\gamma_{1j} = 1$, $\gamma_{2j} = \gamma_{3j} = 0$ and denote the bound by $BP_2(P)$.

ii. $\beta_1 = 0.7$, $\beta_2 = 0.3$, $\gamma_{1j} = 1$, $\gamma_{2j} = \gamma_{3j} = 0$ and denote the bound by $BP_3(P)$.

So we obtain the bounds

L_1	L_2	K	BP_1	BP_2	BP_3	LPR
2.246	2.5	2.632	1.651	1.565	1.461	1.346

4. Inequalities on the largest zeros of orthogonal polynomials

Classical orthogonal polynomials have real coefficients and all their zeros are real, distinct, simple and located in the interval of orthogonality. For some classes of orthogonal polynomials (e. g. for Legendre polynomials) the largest zeros are very ”dense” as the degree increases.

The polynomials P_n, H_n and U_n

We reming the algebraic expression of the orthogonal polynomials of Legendre, Hermite and Gegenbauer:

$$P_n(X) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} X^{n-2k},$$

$$H_n(X) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{k!(n-2k)!} (2X)^{n-2k},$$

$$C_n^{\alpha}(X) = \frac{(-1)^n}{n!2^n} (1-X^2)^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}X^n} (1-X^2)^{\alpha+n}$$

Bounds for P_n and H_n can be deduced form our Theorem 3. Other bounds can be derived from Newton's inequality:

THEOREM 5. (Newton [11]) *If the polynomial*

$$P(X) = X^n + a_1 X^{n-1} + a_2 X^{n-2} + \cdots + a_n \in \mathbb{R}[X]$$

is hyperbolic (has only real roots), the number

$$Nw(P) = \sqrt{a_1^2 - 2a_2}.$$

is an upper bound for the positive roots.

The Hessian of Laguerre

Another approach for estimating the largest positive root of an orthogonal polynomial is the study of inequalities derived from the positivity of the Hessian associated to an orthogonal polynomial. They will allow us to obtain better bounds than estimations derived form those on genereal univariate polynomials with real coefficients. If we consider

$$f(X) = \sum_{j=1}^{n} a_j X^j,$$

a univariate polynomial with real coefficients, its *Hessian* (of Laguerre [8]) is

$$\mathrm{H}(f) = (n-1)^2 f'^2 - n(n-1) f f' \geqslant 0.$$

5. Applications of the inequality of Laguerre

The inequality of Lagurerre $\mathrm{H}(f) \geqslant 0$ is a strong tool for obtaining refined bounds for the dominant roots of orthogonal polynomials. We discuss applications to Legendre, Hermite and Gegenbauer (ultraspherical) polynomials.

An inequality of Laguerre

Let $f \in \mathbb{R}[X]$ be a polynomial of degree $n \geqslant 2$ which satisfies the differential equation

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = 0, \tag{2}$$

with p, q and r univariate polynomials with real coefficients, $p(x) \neq 0$. We recall the following

THEOREM 6. (Laguerre [8]) *If all the roots of f are simple and real, we have*

$$4(n-1)\Big(p(\alpha)r(\alpha) + p(\alpha)q'(\alpha) - p'(\alpha)q(\alpha)\Big) - (n+2)q(\alpha)^2 \geqslant 0 \tag{3}$$

for any root α of f.

The inequality (3) can be applied successfully for finding upper bounds for the roots of orthogonal polynomials.

EXAMPLE. Consider the Legendre polynomial P_n, which satisfies the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

From (2) it follows that $La(n) = (n-1)\sqrt{\dfrac{n+2}{n(n^2+2)}}$ is a bound for the roots of P_n. We have thus the following bounds for the largest zeros of Legendre polynomials:

n	La(P)	LPR
8	0.96334	0.96028
15	0.98922	0.98799
55	0.99917	0.99906
100	0.99975	0.99971

EXAMPLE. Consider the Hermite polynomial H_n, which satisfies the differential equation

$$y'' - 2xy' + 2ny = 0.$$

From (2) it follows that $He(n) = (n-1)\sqrt{\dfrac{2}{n+2}}$ is a bound for the roots of H_n. We have the following bounds for the largest zeros of Hermite polynomials:

n	He(P)	LPR
3	1.264	1.224
8	3.130	2.930
12	4.156	3.889
50	9.609	9.182

A new inequality

THEOREM 7. *Let $f \in \mathbb{R}[X]$ be a polynomial of degree $n \geqslant 2$ that satisfies the second order differential equation*

$$p(x)y'' + q(x)y' + r(x)y = 0, \tag{4}$$

with p, q and r univariate polynomials with real coefficients, $p(x) \neq 0$.
If all the roots of f are simple and real we have

$$8(n-3)q_2(\alpha)^2 + 9(n-2)q(\alpha)q_3(\alpha) \geqslant 0,$$

where

$$q_2 = q^2 + p'q - pq' - pr,$$
$$q_3 = (2p'+q)\left(-q^2 - p'q + pq' - pr\right) - pq\,(p'' + 2q' + r) - p^2\,(q'' + 2r').$$

for any root α of f.

We derive a new bound for Hermite polynomials:

$$Se(H_n) = \sqrt{\frac{2n^2+n+6+\sqrt{(2n^2+n+6+32(n+6)(n^3-5n^2+7n-3)}}{4(n+6)}}.$$

Considering also the Bound of Laguerre

$$He(H_n) = (n-1)\sqrt{\frac{2}{n+2}}$$

we obtain

n	He(H_n)	Se(H_n)	LPR
3	1.264	1.224	1.224
25	6.531	6.382	6.164
60	10.596	10.478	10.159
120	15.236	15.146	14.776
200	19.801	19.729	19.339

Other bounds for the largest positive roots of Hermite polynomials are:

$Bott(H_n) = \sqrt{2n - 2\sqrt[3]{\frac{n}{3}}}$ O. Bottema [2]

$Venn(H_n) = \sqrt{2(n+1) - 2(5/4)^{2/3}(n+1)^{1/3}}$ S. C. Van Venn [15]

$Kras(H_n) = \sqrt{2n-2}$ I. Krasikov [5]

$FoKr(H_n) = \sqrt{\frac{4n - 3n^{1/3} - 1}{2}}$ W. H. Foster–I. Krasikov [3]

but our bound $Se(H,n)$ is sharper.

Bounds for zeros of Gegenbauer polynomials

Theorem 7 can be applied also to other orthogonal polynomials. For example, for Gegenbauer polynomials we obtain

n	α	Bu(P)	LPR
5	1	0.873	0.866
5	2	0.810	0.798
8	1.5	0.927	0.919
50	2.5	0.996	0.995
80	2.5	0.998	0.998

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