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WEIGHTED INEQUALITIES FOR A CLASS OF MATRIX OPERATORS: THE CASE $p \leqslant q$

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. We prove a new discrete Hardy-type inequality $\|Af\|_{q,u} \leqslant C\|f\|_{p,v}$, where the matrix operator A is defined by $(Af)_i := \sum_{j=1}^{i} a_{i,j} f_j, \quad a_{i,j} \geqslant 0$. Moreover, we study the problem of compactness of the operator A, and the dual result is stated.

1. Introduction

Hardy's original motivation in the period 1915–1925 until he finally stated and proved this famous inequality in [6] was to find an elementary proof of (discrete) Hilbert's inequality (see [8]). After that almost all development has been performed for the continuous case (see e.g. the books [7] and [9] and the references there) and surprisingly little has been done for the discrete case (however, see Chapter 6 of [7] and our description below). In this paper we will prove a new discrete Hardy type inequality involving a kernel which is of much more general form than studied before.

Let $1 < p, q < \infty$ and $u = \{u_i\}_{i=1}^{\infty}$, $v = \{v_i\}_{i=1}^{\infty}$ be positive sequences of real numbers, which we in the sequel call weight sequences. Let $l_{p,v}$ denote the space of sequences of real numbers $f = \{f_i\}_{i=1}^{\infty}$ such that

$$\|f\|_{p,v} := \left(\sum_{i=1}^{\infty} |v_i f_i|^p \right)^{\frac{1}{p}} < \infty, \qquad 1 < p < \infty.$$

Moreover, $(a_{i,j})$ is a non-negative triangular matrix with entries $a_{i,j} \geqslant 0$ when $i \geqslant j \geqslant 1$ and $a_{i,j} = 0$ when $i < j$.

We will study inequalities of the following form

$$\|Af\|_{q,u} \leqslant C\|f\|_{p,v}, \qquad \forall f \in l_{p,v}, \tag{1.1}$$

where the matrix operator A is defined by

$$(Af)_i := \sum_{j=1}^{i} a_{i,j} f_j, \qquad i \geqslant 1. \tag{1.2}$$

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and C is positive finite constant not depending on f.

When $a_{i,j}=1$, $i \geqslant j \geqslant 1$, the operator (1.2) coincides with the discrete Hardy operator and the inequality (1.1) with the operator (1.2) was studied in [1]-[5] and [13] for the cases $0<q<\infty$, $1 \leqslant p<\infty$. The case $a_{i,j}=\alpha_i\beta_j$, $i \geqslant j \geqslant 1$, where $\alpha=\{\alpha_i\}_{i=1}^{\infty}$ and $\beta=\{\beta_i\}_{i=1}^{\infty}$ are positive sequences, was considered a.o. in [14].

In [11], [12] necessary and sufficient conditions for the validity (1.1) was obtained for the case $1<p,q<\infty$ under the assumption that there exists $d \geqslant 1$ such that the inequality

$$\frac{1}{d}(a_{i,k}+a_{k,j}) \leqslant a_{i,j} \leqslant d(a_{i,k}+a_{k,j}), \qquad i \geqslant k \geqslant j \geqslant 1. \tag{1.3}$$

holds.

The sequence $\{a_i\}_{i=1}^{\infty}$ is called the almost non-decreasing (non-increasing) sequence, if there exists $c>0$ such that $ca_i \geqslant a_k$ $(a_k \leqslant ca_j)$ for all $i \geqslant k \geqslant j \geqslant 1$.

Let $(a_{i,j})$ is a non-negative triangular matrix, i.e. the entries of matrix $a_{i,j} \geqslant 0$ when $i \geqslant j \geqslant 1$ and $a_{i,j}=0$ when $i<j$. If $a=\{a_i\}_{i=1}^{\infty}$ is non-decreasing sequence and $\alpha \geqslant 0$, then the inequality (1.3) holds e.g. for $a_{i,j}=(a_i-a_j)^{\alpha}$, $i \geqslant j \geqslant 1$, but in the case $a_{i,j}=(a_i-b_j)^{\alpha}$ when $i \geqslant j \geqslant 1$, where $b=\{b_j\}_{j=1}^{\infty}$ is a arbitrary sequence, such that $a_i \geqslant \max\limits_{1 \leqslant j \leqslant i} b_j$ the inequality (1.3) does not hold, in particular, when

$$a_{i,j}=\left(ln\frac{a_i}{b_j}\right)^{\alpha}, \text{ where } \frac{a_i}{b_j} \geqslant 1, \ i \geqslant j \geqslant 1$$

and

$$a_{i,j}=\left(\sum_{\tau=m_j}^{n_i} c_\tau\right)^{\alpha}, \text{ when } i \geqslant j \geqslant 1,$$

where $\{m_j\}_{j=1}^{\infty}$ is arbitrary sequence of integer numbers, $\{n_i\}_{i=1}^{\infty}$ is non-decreasing sequence of integer numbers, such that $n_i \geqslant \max\limits_{1 \leqslant j \leqslant i} m_j$ and $\{c_i\}_{i=-\infty}^{+\infty}$ is sequence of non-negative numbers. To investigate such and more general cases we will consider the inequality (1.1) under following assumption, which is strictly weaker than (1.3):

ASSUMPTION 1.1. There exists $d \geqslant 1$, a sequence of positive numbers $\{\omega_k\}_{k=1}^{\infty}$ and a non-negative matrix $(b_{i,j})$, where $b_{i,j}$ is almost non-decreasing in i and almost non-increasing in j such that

$$\frac{1}{d}(b_{i,k}\omega_j+a_{k,j}) \leqslant a_{i,j} \leqslant d(b_{i,k}\omega_j+a_{k,j}), \tag{1.4}$$

for all $i \geqslant k \geqslant j \geqslant 1$.

We remark that in particular the above stated examples satisfy Assumption 1.1. We also note that from (1.4) it follows that

$$da_{i,j} \geqslant b_{i,k}\omega_j, \tag{1.5}$$

$$da_{i,j} \geqslant a_{k,j}, \tag{1.6}$$

when $i \geqslant k \geqslant j \geqslant 1$.

A continuous analogue of (1.3)–(1.4) even in a slightly more general form was considered R.Oinarov in [10].

Moreover, we study the problem of compactness of the operator (1.2) from $l_{p,v}$ into $l_{q,u}$.

Convention: The symbol $M \ll K$ means that $M \leqslant cK$, where $c > 0$ is a constant depending only on unessential parameters. If $M \ll K \ll M$, then we write $M \approx K$.

For the proof of our main results we need the following well-known result for the discrete weighted Hardy inequality (see [13], Theorem 7) and the criteria on precompactness of sets in l_p (see [15], p. 32). For better presentation let us state these results here:

THEOREM 1.1. *Let $1 < p \leqslant q < \infty$. Then the inequality*

$$\left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i}\omega_j f_j\right)^q u_i^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{i=1}^{\infty}|v_i f_i|^p\right)^{\frac{1}{p}}, \qquad 0 \leqslant f \in l_{p,v} \tag{1.7}$$

holds if and only if

$$H_1 := \sup_{n \geqslant 1}\left(\sum_{i=n}^{\infty} u_i^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{n}\omega_j^{p'} v_j^{-p'}\right)^{\frac{1}{p'}} < \infty.$$

Moreover, $H_1 \approx C$, where C is the best constant in (1.7).

THEOREM 1.2. *Let T be a set from l_p, $1 \leqslant p < \infty$. The set T is compact if and only if T is bounded and for all $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for all $x = \{x_i\}_{i=1}^{\infty} \in T$ the following inequality*

$$\sum_{i=N}^{\infty}|x_i|^p < \varepsilon$$

holds.

2. Main results

Our first result reads:

THEOREM 2.1. *Let $1 < p \leqslant q < \infty$ and the entries of the matrix $(a_{i,j})$ satisfy Assumption 1.1. Then the inequality (1.1) holds if and only if $F = \max\{F_0, F_1\} < \infty$, where*

$$F_0 = \sup_{n \geqslant 1}\left(\sum_{i=n}^{\infty} b_{i,n}^q u_i^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{n}\omega_j^{p'} v_j^{-p'}\right)^{\frac{1}{p'}},$$

and

$$F_1 = \sup_{n\geqslant 1}\left(\sum_{i=n}^{\infty} u_i^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{n} a_{n,j}^{p'} v_j^{-p'}\right)^{\frac{1}{p'}}.$$

Moreover $F \approx C$, where C is the best constant in (1.1)

Proof. Necessity. Let us assume that (1.1) holds for a finite constant C. Let $r \geqslant 1$ and take a test sequence $\widetilde{f}_r = \{\widetilde{f}_{r,s}\}_{s=1}^{\infty}$ such that $\widetilde{f}_{r,s} = \omega_s^{p'-1} v_s^{-p'}$, $1 \leqslant s \leqslant r$, and $\widetilde{f}_{r,s} = 0$, $s > r$.

Applying the test sequence to the right hand side of (1.1), we have that

$$\|\widetilde{f}_r\|_{p,v} = \left(\sum_{s=1}^{\infty} |v_s \widetilde{f}_{r,s}|^p\right)^{\frac{1}{p}} = \left(\sum_{s=1}^{r} \omega_s^{p'} v_s^{-p'}\right)^{\frac{1}{p}}. \tag{2.1}$$

Substituting $\widetilde{f}_r$ in the left hand side of the inequality (1.1) and using (1.5) we have that

$$\begin{aligned}
\|A\widetilde{f}_r\|_{q,u} &= \left(\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i} a_{i,s}\widetilde{f}_{r,s}\right)^q u_i^q\right)^{\frac{1}{q}} \\
&\geqslant \left(\sum_{i=r}^{\infty}\left(\sum_{s=1}^{r} a_{i,s}\omega_s^{p'-1} v_s^{-p'}\right)^q u_i^q\right)^{\frac{1}{q}} \\
&\geqslant \frac{1}{d}\left(\sum_{i=r}^{\infty} b_{i,r}^q u_i^q\right)^{\frac{1}{q}}\left(\sum_{s=1}^{r} \omega_s^{p'} v_s^{-p'}\right).
\end{aligned} \tag{2.2}$$

From (1.1), (2.1) and (2.2) it follows that

$$\left(\sum_{i=r}^{\infty} b_{i,r}^q u_i^q\right)^{\frac{1}{q}}\left(\sum_{s=1}^{r} \omega_s^{p'} v_s^{-p'}\right)^{\frac{1}{p'}} \ll C$$

for all $r \geqslant 1$. Therefore

$$F_0 \ll C. \tag{2.3}$$

Now we assume that $\widehat{f}_r = \{\widehat{f}_{r,s}\}_{s=1}^{\infty}$, where $\widehat{f}_{r,s} = a_{r,s}^{p'-1} v_s^{-p'}$, $1 \leqslant s \leqslant r$, and $\widehat{f}_{r,s} = 0$, $s > r$, and apply this sequence to (1.1). Substituting $\widehat{f}_r$ in the left hand side of the inequality (1.1) and using (1.6) we find that

$$\|A\widehat{f}_r\|_{q,u} = \left(\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i} a_{i,s}\widehat{f}_{r,s}\right)^q u_i^q\right)^{\frac{1}{q}} \gg \left(\sum_{i=r}^{\infty} u_i^q\right)^{\frac{1}{q}}\left(\sum_{s=1}^{r} a_{r,s}^{p'} v_s^{-p'}\right). \tag{2.4}$$

For the right hand side of (1.1) it yields that

$$\|\widehat{f}_r\|_{p,v} = \left(\sum_{s=1}^{\infty} |v_s \widehat{f}_{r,s}|^p\right)^{\frac{1}{p}} = \left(\sum_{s=1}^{r} a_{r,s}^{p'} v_s^{-p'}\right)^{\frac{1}{p}}. \tag{2.5}$$

According to (1.1), (2.4), (2.5) and since $r \geqslant 1$ is arbitrary we have that $F_1 \ll C$, which together with (2.3) gives that

$$F = \max\{F_0,\ F_1\} \ll C. \tag{2.6}$$

The proof of the necessity is complete.

Sufficiency. Let $F < \infty$ and $0 \leqslant f \in l_{p,v}$.

For all $i \geqslant 1$ we define the following set of positive numbers:

$$T_i = \{k \in Z : (d+1)^k \leqslant (Af)_i\},$$

where d is the constant from (1.4), Z is the set of integers and we assume that $k_i = \max T_i$. Then

$$(d+1)^{k_i} \leqslant (Af)_i < (d+1)^{k_i+1}, \qquad i \geqslant 1. \tag{2.7}$$

Let $m_1 = 1$ and $M_1 = \{i \in N : k_i = k_1 = k_{m_1}\}$. Suppose that m_2 is such that $\sup M_1 + 1 = m_2$. Obviously $m_2 > m_1$ and if the set M_1 is upper bounded, then $m_2 < \infty$ and $m_2 - 1 = \max M_1 = \sup M_1$. Let us inductively define numbers $1 = m_1 < m_2 < \ldots < m_s < \infty$, $s \geqslant 1$. To define m_{s+1} we assume that $m_{s+1} = \sup M_s + 1$, where $M_s = \{i \in N : k_i = k_{m_s}\}$.

Let $N_0 = \{s \in N : m_s < \infty\}$. Further, we assume that $k_{m_s} = n_s$, $s \in N_0$. From the definition of m_s and from (2.7) it follows that, for $s \in N_0$,

$$(d+1)^{n_s} \leqslant (Af)_i < (d+1)^{n_s+1}, \quad m_s \leqslant i \leqslant m_{s+1} - 1 \tag{2.8}$$

and

$$N = \bigcup_{s \in N_0} [m_s, m_{s+1}).$$

Therefore

$$\|Af\|_{q,u}^q = \sum_{s \in N_0} \sum_{j=m_s}^{m_{s+1}-1} (Af)_j^q u_j^q. \tag{2.9}$$

We assume that $\sum\limits_{j=m_s}^{m_{s+1}-1} = 0$, if $m_s = \infty$. Then we can rewrite the expression (2.9) in the following form:

$$\begin{aligned} \|Af\|_{q,u}^q = \sum_{s \in N_0} \sum_{j=m_s}^{m_{s+1}-1} (Af)_j^q u_j^q = \sum_{j=m_1}^{m_2-1} (Af)_j^q u_j^q \\ + \sum_{j=m_2}^{m_3-1} (Af)_j^q u_j^q + \sum_{s \geqslant 3} \sum_{j=m_s}^{m_{s+1}-1} (Af)_j^q u_j^q. \end{aligned} \tag{2.10}$$

Since $m_1 = 1 \in N_0$ and by using (2.8) we find that

$$\sum_{j=m_1}^{m_2-1} (Af)_j^q u_j^q \leqslant \sum_{j=1}^{m_2-1} (d+1)^{(n_1+1)q} u_j^q \leqslant (d+1)^q (d+1)^{n_1 q} \sum_{j=1}^{\infty} u_j^q$$
$$\leqslant (d+1)^q (Af)_1^q \sum_{j=1}^{\infty} u_j^q \ll \left(\sum_{s=1}^{1} a_{1,s}^{p'} v_s^{-p'}\right)^{\frac{q}{p'}} \sum_{j=1}^{\infty} u_j^q \|f\|_{p,v}^q \tag{2.11}$$
$$\leqslant F_1^q \|f\|_{p,v}^q.$$

If $m_2 = \infty$, then $m_i = \infty$ for all $i \geqslant 2$ and, according to (2.10) and (2.11), we find that

$$\|Af\|_{q,u} \ll F \|f\|_{p,v}. \tag{2.12}$$

If $m_2 < \infty$, i.e. $2 \in N_0$, then arguing as before in (2.11), we obtain that

$$\sum_{j=m_2}^{m_3-1} (Af)_j^q u_j^q \leqslant (d+1)^q (d+1)^{n_2 q} \sum_{j=m_2}^{\infty} u_j^q$$
$$\ll (Af)_{m_2}^q \sum_{j=m_2}^{\infty} u_j^q = \left(\sum_{i=1}^{m_2} a_{m_2,i} f_i\right)^q \sum_{i=m_2}^{\infty} u_j^q \tag{2.13}$$
$$\leqslant \left(\sum_{i=1}^{m_2} a_{m_2,i}^{p'} v_i^{p'}\right)^{\frac{q}{p'}} \sum_{j=m_2}^{\infty} u_j^q \left(\sum_{i=1}^{m_2} |v_i f_i|^p\right)^{\frac{q}{p}} \leqslant F_1^q \|f\|_{p,v}^q.$$

If $m_3 = \infty$, then by combing (2.10), (2.11) and (2.13) we get (2.12).

For $s \geqslant 3$ and $s \in N_0$, by using (2.8), (1.4) and $n_{s-2} + 1 \leqslant n_s - 1$, which follows from the inequality $n_{s-2} < n_{s-1} < n_s$, we can estimate the value $(d+1)^{n_s - 1}$ as follows:

$$(d+1)^{n_s-1} = (d+1)^{n_s} - d(d+1)^{n_s-1} \leqslant (d+1)^{n_s} - d(d+1)^{n_{s-2}+1}$$
$$< (Af)_{m_s} - d\,(Af)_{m_{s-1}-1} = \sum_{i=1}^{m_s} a_{m_s,i} f_i - d \sum_{i=1}^{m_{s-1}-1} a_{m_{s-1}-1,i} f_i$$
$$= \sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i + \sum_{i=1}^{m_{s-1}-1} [a_{m_s,i} - d a_{m_{s-1}-1,i}] f_i \tag{2.14}$$
$$\leqslant \sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i + \sum_{i=1}^{m_{s-1}-1} [d(b_{m_s,m_{s-1}-1}\omega_i + a_{m_{s-1}-1,i}) - d a_{m_{s-1}-1,i}] f_i$$
$$= \sum_{i=m_{s-1}}^{m_s} a_{m_s,i} f_i + d b_{m_s,m_{s-1}-1} \sum_{i=1}^{m_{s-1}-1} \omega_i f_i.$$

By now using (2.8) and (2.14), we can estimate the summand on the left hand side in (1.1) for $s \geqslant 3$ in the following way:

$$\begin{aligned}\sum_{s\geqslant 3}\sum_{j=m_s}^{m_{s+1}-1}(Af)_j^q u_j^q &< \sum_{s\geqslant 3}\sum_{j=m_s}^{m_{s+1}-1}(d+1)^{(n_s+1)q}u_j^q\\ &= (d+1)^{2q}\sum_{s\geqslant 3}(d+1)^{(n_s-1)q}\sum_{j=m_s}^{m_{s+1}-1}u_j^q\\ &\ll \sum_{s\geqslant 3}\left(\sum_{i=m_{s-1}}^{m_s}a_{m_s,i}f_i + db_{m_s,m_{s-1}-1}\sum_{i=1}^{m_{s-1}-1}\omega_i f_i\right)^q\sum_{j=m_s}^{m_{s+1}-1}u_j^q \qquad (2.15)\\ &\ll \sum_{s\geqslant 3}\left(\sum_{i=m_{s-1}}^{m_s}a_{m_s,i}f_i\right)^q\sum_{j=m_s}^{m_{s+1}-1}u_j^q\\ &\quad+\sum_{s\geqslant 3}b_{m_s,m_{s-1}-1}^q\left(\sum_{i=1}^{m_{s-1}-1}\omega_i f_i\right)^q\sum_{j=m_s}^{m_{s+1}-1}u_j^q := S_1+S_2.\end{aligned}$$

Estimate of S_1 and S_2:
To estimate S_1 we apply Hölder's and Jensen's inequalities and find that

$$\begin{aligned}S_1 &= \sum_{s\geqslant 3}\left(\sum_{i=m_{s-1}}^{m_s}a_{m_s,i}f_i\right)^q\sum_{j=m_s}^{m_{s+1}-1}u_j^q\\ &\leqslant \sum_{s\geqslant 3}\left(\sum_{i=m_{s-1}}^{m_s}a_{m_s,i}^{p'}v_i^{-p'}\right)^{\frac{q}{p'}}\left(\sum_{i=m_{s-1}}^{m_s}|f_iv_i|^p\right)^{\frac{q}{p}}\sum_{j=m_s}^{m_{s+1}-1}u_j^q\\ &\leqslant \left[\sup_{k\geqslant 1}\left(\sum_{i=1}^{k}a_{k,i}^{p'}v_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{j=k}^{\infty}u_j^q\right)^{\frac{1}{q}}\right]^q\sum_{s\geqslant 3}\left(\sum_{i=m_{s-1}}^{m_s}|f_iv_i|^p\right)^{\frac{q}{p}} \qquad (2.16)\\ &\leqslant F_1^q\left(\sum_{s\geqslant 3}\sum_{i=m_{s-1}}^{m_s}|f_iv_i|^p\right)^{\frac{q}{p}} \ll F_1^q\|f\|_{p,v}^q.\end{aligned}$$

We introduce the sequence $\{\Delta_i\}_{i=1}^{\infty}$ such that $\Delta_i = b_{m_s,m_{s-1}-1}^q\sum_{j=m_s}^{m_{s+1}-1}u_j^q$, $i=m_{s-1}-1$, $s\geqslant 3$ and $\Delta_i = 0$, $i\neq m_{s-1}-1$, $s\geqslant 3$. Hence, we can rewrite S_2 in the following form:

$$S_2 = \sum_{s\geqslant 3}\left(\sum_{j=1}^{m_{s-1}-1}\omega_j f_j\right)^q b_{m_{s-1},m_{s-1}-1}^q\sum_{j=m_s}^{m_{s+1}-1}u_j^q = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{i}\omega_j f_j\right)^q\Delta_i. \qquad (2.17)$$

Thus, in view of Theorem 1.1, we have that

$$S_2 \ll \widetilde{H_1^q}\|f\|_{p,v}^q, \qquad (2.18)$$

where

$$\widetilde{H_1}^q = \sup_{k\geqslant 1}\left(\sum_{i=k}^{\infty}\Delta_i\right)^{\frac{1}{q}}\left(\sum_{j=1}^{k}\omega_j^{p'}v_j^{-p'}\right)^{\frac{1}{p'}}. \tag{2.19}$$

Since, by Assumption 1.1, $b_{i,j}$ is almost non-decreasing in i and almost non-increasing in j, we find that

$$\begin{aligned}\sum_{i=k}^{\infty}\Delta_i &= \sum_{m_{s-1}-1\geqslant k} b^q_{m_s,m_{s-1}-1}\sum_{j=m_s}^{m_{s+1}-1}u_j^q \\ &\ll \sum_{m_{s-1}-1\geqslant k}\sum_{j=m_s}^{m_{s+1}-1} b^q_{j,k}u_j^q \leqslant \sum_{j=k}^{\infty} b^q_{j,k}u_j^q.\end{aligned} \tag{2.20}$$

By combining (2.18), (2.19) and (2.20), we obtain that

$$S_2 \ll F_0^q\|f\|^q_{p,v}. \tag{2.21}$$

Thus, from (2.10), (2.11), (2.13), (2.15), (2.16) and (2.21) it follows that

$$\|Af\|_{q,u} \ll F\|f\|_{p,v}, \qquad f\geqslant 0,$$

i.e the inequality (1.1) is valid and we see that the best constant in (1.1) $C \ll F$, which together with (2.6) gives that $C \approx F$.
The proof is complete.

The inequality (1.1) holds if and only if the following dual inequality

$$\|A^*g\|_{p',v^{-1}} \leqslant C\|g\|_{q',u^{-1}}, \qquad g\in l_{q',u^{-1}} \tag{2.22}$$

holds for the conjugate operator

$$(A^*g)_j = \sum_{i=j}^{\infty} a_{i,j}g_i, \qquad j\geqslant 1. \tag{2.23}$$

Moreover, the best constants in (1.1) and (2.22) coincides.

Indeed,

$$\begin{aligned}C &= \sup_{0\neq f\in l_{p,v}}\frac{\|Af\|_{q,u}}{\|f\|_{p,v}} = \sup_{0\neq f\in l_{p,v}}\sup_{0\neq g\in l_{q',u^{-1}}}\frac{\sum_{i=1}^{\infty} g_i(Af)_i}{\|f\|_{p,v}\|g\|_{q',u^{-1}}} \\ &= \sup_{0\neq g\in l_{q',u^{-1}}}\sup_{0\neq f\in l_{p,v}}\frac{\sum_{j=1}^{\infty}(A^*g)_jf_j}{\|g\|_{q',u^{-1}}\|f\|_{p,v}} = \sup_{0\neq g\in l_{q',u^{-1}}}\frac{\|A^*g\|_{p',v^{-1}}}{\|g\|_{q',u^{-1}}}.\end{aligned}$$

Therefore using Theorem 2.1 with p', q', v^{-1} and u^{-1} replaced by q, p, u and v, respectively, we obtain the following dual version of Theorem 2.1:

THEOREM 2.2. *Let* $1 < p \leqslant q < \infty$ *and the entries of the matrix* $(a_{i,j})$ *satisfy Assumption 1.1. Then the inequality*

$$\|A^* f\|_{q,u} \leqslant C\|f\|_{p,v}, \qquad \forall f \in l_{p,v} \tag{2.24}$$

holds if and only if $F^* = \max\{F_0^*, F_1^*\} < \infty$, *where*

$$F_0^* = \sup_{k\geqslant 1}\left(\sum_{i=k}^{\infty} b_{i,k}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{j=1}^{k}\omega_j^q u_j^q\right)^{\frac{1}{q}}$$

and

$$F_1^* = \sup_{k\geqslant 1}\left(\sum_{i=k}^{\infty} v_i^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{j=1}^{k} a_{k,j}^q u_j^q\right)^{\frac{1}{q}}.$$

Moreover $F^* \approx C$, *where* C *is the best constant in (*2.24*).*

Now we state our compactness result for the operator (1.2) from $l_{p,v}$ into $l_{q,u}$.

THEOREM 2.3. *Let* $1 < p \leqslant q < \infty$ *and the entries of the matrix* $(a_{i,j})$ *satisfy Assumption 1.1. Then the operator (*1.2*) is compact from* $l_{p,v}$ *into* $l_{q,u}$ *if and only if*

$$\lim_{r\to\infty}(F_0)_r = 0, \tag{2.25}$$

$$\lim_{r\to\infty}(F_1)_r = 0. \tag{2.26}$$

where

$$(F_0)_r = \left(\sum_{i=r}^{\infty} b_{i,r}^q u_i^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{r}\omega_j^{p'} v_j^{-p'}\right)^{\frac{1}{p'}},$$

$$(F_1)_r = \left(\sum_{i=r}^{\infty} u_i^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{r} a_{r,j}^{p'} v_j^{-p'}\right)^{\frac{1}{p'}}.$$

Proof. Necessity. Let the operator (1.2) be compact from $l_{p,v}$ into $l_{q,u}$. For all $r \geqslant 1$ we introduce the following sequence:

$$g_r = \{g_{r,j}\}_{j=1}^{\infty}: \quad g_{r,j} = \frac{f_{r,j}}{\|v f_r\|_{l_p}},$$

where $f_r = \{f_{r,j}\}_{j=1}^{\infty}$: $\quad f_{r,j} = \begin{cases} a_{r,j}^{p'-1} v_j^{-p'}, & 1 \leqslant j \leqslant r, \\ 0, & j > r. \end{cases}$

It is obvious that $\|g_r\|_{p,v} = 1$. Since the operator (1.2) is compact from $l_{p,v}$ into $l_{q,u}$, it yields that the set $\{uA\varphi, \|\varphi\|_{p,v} = 1\}$ is precompact in l_q. Hence from criteria on precompactness of the sets in l_p (see Theorem 1.2) we conclude that

$$\lim_{r\to\infty}\sup_{\|\varphi\|_{p,v}=1}\left(\sum_{i=r}^{\infty} u_i^q (A\varphi)_i^q\right)^{\frac{1}{q}} = 0. \tag{2.27}$$

Moreover, by using (1.6) we have that

$$\begin{aligned}
\sup_{\|\varphi\|_{p,v}=1}\left(\sum_{i=r}^{\infty}u_i^q\,(A\varphi)_i^q\right)^{\frac{1}{q}} &\geqslant \left(\sum_{i=r}^{\infty}u_i^q\,(Ag_r)_i^q\right)^{\frac{1}{q}} = \left(\sum_{i=r}^{\infty}u_i^q\left(\sum_{j=1}^{i}a_{i,j}\frac{f_{r,j}}{\|vf_r\|_p}\right)^q\right)^{\frac{1}{q}}\\
&\geqslant \left(\sum_{i=r}^{\infty}u_i^q\left(\sum_{j=1}^{r}a_{i,j}\frac{f_{r,j}}{\|vf_r\|_p}\right)^q\right)^{\frac{1}{q}}\\
&\geqslant \frac{1}{d}\left(\sum_{i=r}^{\infty}u_i^q\left(\sum_{j=1}^{r}a_{r,j}^{p'}v_j^{-p'}\right)^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{r}a_{r,j}^{p'}v_j^{-p'}\right)^{-\frac{1}{p}}\\
&= \frac{1}{d}\left(\sum_{i=r}^{\infty}u_i^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{r}a_{r,j}^{p'}v_j^{-p'}\right)^{\frac{1}{p'}} = \frac{1}{d}(F_1)_r.
\end{aligned} \tag{2.28}$$

Obviously, (2.26) follows from (2.27) and (2.28).

To prove (2.25) for all $r \geqslant 1$ we introduce the following sequence

$$\widetilde{g}_r = \{\widetilde{g}_{r,j}\}_{j=1}^{\infty}: \ \ \widetilde{g}_{r,j} = \frac{\widetilde{f}_{r,j}}{\|v\widetilde{f}_r\|_{l_p}},$$

where $\widetilde{f}_r = \{\widetilde{f}_{r,j}\}_{j=1}^{\infty}$: $\quad \widetilde{f}_{r,j} = \begin{cases} \omega_j^{p'-1}v_j^{-p'}, & 1 \leqslant j \leqslant r,\\ 0, & j > r.\end{cases}$

Using (1.5) in (2.27) we find that

$$\begin{aligned}
\sup_{\|\varphi\|_{p,v}=1}\left(\sum_{i=r}^{\infty}u_i^q\,(A\varphi)_i^q\right)^{\frac{1}{q}} &\geqslant \left(\sum_{i=r}^{\infty}u_i^q\left(\sum_{j=1}^{i}a_{i,j}\frac{\widetilde{f}_{r,j}}{\|v\widetilde{f}_r\|_p}\right)^q\right)^{\frac{1}{q}}\\
&\geqslant \frac{1}{d}\left(\sum_{i=r}^{\infty}u_i^q\left(\sum_{j=1}^{i}b_{i,r}\omega_j\widetilde{f}_{r,j}\right)^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{r}\omega_j^{p'}v_j^{-p'}\right)^{-\frac{1}{p}}\\
&= \frac{1}{d}\left(\sum_{i=r}^{\infty}b_{i,r}^q u_i^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{r}\omega_j^{p'}v_j^{-p'}\right)^{\frac{1}{p'}} = \frac{1}{d}(F_0)_r.
\end{aligned} \tag{2.29}$$

According to (2.27) and (2.29) we find that (2.25) holds and the proof of necessity is complete.

Sufficiency. Assume that (2.25) and (2.26) hold. Then, by Theorem 2.1, the operator (1.2) is bounded from $l_{p,v}$ into $l_{q,u}$. Consequently, the set $\{uAf,\ \|f\|_{p,v} \leqslant 1\}$ is bounded in l_q. Let us show that this set is precompact in l_q. By the criteria on precompactness of the sets in l_q (see Theorem 1.2), the bounded set $\{uAf,\ \|f\|_{p,v} \leqslant 1\}$ is compact in l_q, if

$$\lim_{r\to\infty}\sup_{\|f\|_{p,v}\leqslant 1}\left(\sum_{i=r}^{\infty}u_i^q|\,(Af)_i\,|^q\right)^{\frac{1}{q}} = 0. \tag{2.30}$$

For all $r > 1$ we assume that $\widetilde{u} = \{\widetilde{u}_i\}_{i=1}^{\infty}$: $\widetilde{u}_i = \begin{cases} 0, & 1 \leqslant i \leqslant r-1 \\ u_i, & r \leqslant i \end{cases}$.
Then, by Theorem 2.1, we have that

$$\sup_{\|f\|_{p,v} \leqslant 1} \left(\sum_{i=r}^{\infty} u_i^q |(Af)_i^q \right)^{\frac{1}{q}} = \sup_{\|f\|_{p,v} \leqslant 1} \left(\sum_{i=1}^{\infty} \widetilde{u}_i^q |(Af)_i|^q \right)^{\frac{1}{q}} \ll \widetilde{F}(r), \tag{2.31}$$

where

$$\widetilde{F}(r) = \max\{\widetilde{F}_0(r), \widetilde{F}_1(r)\},$$

$$\widetilde{F}_0(r) = \sup_{n \geqslant 1} \left(\sum_{i=n}^{\infty} b_{i,n}^q \widetilde{u}_j^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^{n} \omega_j^{p'} v_j^{-p'} \right)^{\frac{1}{p'}},$$

$$\widetilde{F}_1(r) = \sup_{n \geqslant 1} \left(\sum_{i=n}^{\infty} \widetilde{u}_j^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^{n} a_{n,j}^{p'} v_j^{-p'} \right)^{\frac{1}{p'}}.$$

According to that $\widetilde{u}_i = 0$ if $1 \leqslant i \leqslant r-1$ we have that

$$\widetilde{F}_0(r) = \sup_{n \geqslant r} \left(\sum_{i=n}^{\infty} b_{i,n}^q u_i^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^{n} \omega_j^{p'} v_j^{-p'} \right)^{\frac{1}{p'}} = \sup_{n \geqslant r} (F_0)_n, \tag{2.32}$$

$$\widetilde{F}_1(r) = \sup_{n \geqslant r} \left(\sum_{i=n}^{\infty} u_i^q \right)^{\frac{1}{q}} \left(\sum_{j=1}^{n} a_{n,j}^{p'} v_j^{-p'} \right)^{\frac{1}{p'}} = \sup_{n \geqslant r} (F_1)_n. \tag{2.33}$$

From (2.25), (2.26), (2.32) and (2.33) we find that

$$\lim_{r\to\infty} \widetilde{F}_0(r) = \lim_{r\to\infty} \sup_{n \geqslant r} (F_0)_n = \overline{\lim_{r\to\infty}} (F_0)_r = \lim_{r\to\infty} (F_0)_r = 0,$$

$$\lim_{r\to\infty} \widetilde{F}_1(r) = \lim_{r\to\infty} \sup_{n \geqslant r} (F_1)_n = \overline{\lim_{r\to\infty}} (F_1)_r = \lim_{r\to\infty} (F_1)_r = 0.$$

Hence, by using (2.31) we obtain (2.30) and the proof is complete.

Since the compactness of the operator A from $l_{p,v}$ into $l_{q,u}$ is equivalent of the compactness of the operator A^* from $l_{q',u^{-1}}$ into $l_{p',v^{-1}}$, then if we change q' by p, p' by q, u^{-1} by v, and v^{-1} by u from Theorem 2.3 we have the following dual version of this theorem:

THEOREM 2.4. *Let $1 < p \leqslant q < \infty$ and the entries of the matrix $(a_{i,j})$ satisfy Assumption 1.1. Then the operator (2.23) is compact from $l_{p,v}$ into $l_{q,u}$ if and only if*

$$\lim_{r\to\infty} (F_0^*)_r = 0, \tag{2.34}$$

$$\lim_{r\to\infty} (F_1^*)_r = 0, \tag{2.35}$$

where

$$(F_0^*)_r = \left(\sum_{i=r}^{\infty} b_{i,r}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{j=1}^{r} \omega_j^q u_j^q\right)^{\frac{1}{q}},$$

$$(F_1^*)_r = \left(\sum_{i=r}^{\infty} v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{j=1}^{r} a_{r,j}^q u_j^q\right)^{\frac{1}{q}}.$$

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