Mathematical
Inequalities
& Applications
Volume 12, Number 4 (2009), 905–915

OVERVIEW OF MIXED MEANS, OPERATOR NORMS OF AVERAGING OPERATORS AND MAXIMAL FUNCTIONS, AND SOME NEW RESULTS

IVAN PERIĆ

Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. We overview the so-called mixed-means inequalities, that is, inequalities for mixed power means for averaging operators which average functions over several scaled families of subsets of $\mathbb{R}^n$, such as rectangles, balls, spheres and similar. A general case of such inequalities related to rectangles with sides parallel to coordinate hyperplanes and ellipsoids centered at the origin is proved. Motivation for considering these families can be found in considering collection of subsets of $\mathbb{R}^n$ which differentiate suitable functions on $\mathbb{R}^n$. Guided by this motivation we distinguish centered and uncentered cases. As a direct consequence of the obtained mixed-means inequalities, the Hardy type inequalities, that is, the operator norms of the averaging operators on L^p spaces are deduced. An interesting and important feature of these norms is that they are lower bounds for operator norms of appropriate maximal functions. Further, they can give asymptotic behavior of the operator norms of maximal functions for large n and fixed $p > 1$.

1. Introduction

The basic idea underlying investigations of integral mixed-means inequalities for power means in the series of papers [2, 4, 5, 6, 12] can be found in a remark of A. E. Ingham (see [9]), that the one-dimensional Hardy's inequality is a simple consequence of the inequality

$$\left[\frac{1}{b}\int_0^b\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx\right]^{\frac{1}{p}} \leqslant \frac{1}{b}\int_0^b\left(\frac{1}{t}\int_0^t f^p(x)dx\right)^{\frac{1}{p}} dt,\ p\geqslant 1,\ f\geqslant 0, \tag{1}$$

which can be easily proved by Minkowski's inequality (see [9]). Inequality (1) can be written as the mixed-means inequality

$$M_p\left[M_1\left[f;(0,x)\right],(0,b)\right] \leqslant M_1\left[M_p\left[f;(0,x)\right],(0,b)\right], \tag{2}$$

where $M_p[f;\Omega] = \left(\frac{1}{\mu(\Omega)}\int_\Omega f^p(x)d\mu(x)\right)^{\frac{1}{p}}$, $p\in\mathbb{R}$, $p\neq 0$, and $M_0[f;\Omega] = \exp\left(\frac{1}{\mu(\Omega)}\int_\Omega \log f(x)d\mu(x)\right)$, $f\geqslant 0$.

Mathematics subject classification (2000): 26D10, 26D15.
Keywords and phrases: Integral power means, mixed-means inequalities, averaging operators, scaled family of rectangles, balls, spheres, Hardy's inequality, maximal functions, lower bounds for operator norms.

Paper MIA-12-73

In deducing mixed-means inequalities of type (2) in more general settings, it is of crucial importance to find a suitable averaging operator. M. Christ and L. Grafakos introduced in [1] the following averaging operator particularly suitable for deriving mixed-means inequalities:

$$\left(T_\delta^c f\right)(\mathbf{x}) = \frac{1}{|B(\mathbf{x},\delta|\mathbf{x}|)|}\int_{B(\mathbf{x},\delta|\mathbf{x}|)} f(\mathbf{y})d\mathbf{y},\ f\in L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

where $\delta > 0$, $B(\mathbf{x},r)$ is the ball in $\mathbb{R}^n$ centered at $\mathbf{x}\in\mathbb{R}^n$ and of radius $r>0$, $|\mathbf{x}|$ is the Euclidean norm of $\mathbf{x}\in\mathbb{R}^n$ and $|A|$ is the Lebesgue measure of a measurable set $A\subset\mathbb{R}^n$. In the same paper they proved Hardy's inequality for the operator T_δ^c, using Young's inequality for the convolution on the group $(\mathbb{R}^+, dt/t)$.

An important property of the operator norm (on L^p spaces) of T_δ^c, deduced from Hardy's inequality for this operator, is that it is a lower bound for the operator norm of the Hardy-Littlewood (centered) maximal function

$$(M_c f)(\mathbf{x}) = \sup_{r>0}\frac{1}{|B(\mathbf{x},r)|}\int_{B(\mathbf{x},r)} f(\mathbf{y})d\mathbf{y}.$$

Our motivation for the investigations is that a maximal function can be defined for various collections $\mathscr{C}$ of sets, $\mathscr{C}=\{C : C\subseteq\mathbb{R}^n\}$, by

$$(M_{\mathscr{C}} f)(\mathbf{x}) = \sup_{C\in\mathscr{C},x\in C}\frac{1}{|C|}\int_C f(\mathbf{x}-\mathbf{y})d\mathbf{y}.$$

We deal with scaled families of sets defined in a similar way as the family of balls in the definition of the operator T_δ^c. Having in mind different behaviors of the operator norm of maximal functions in centered and in uncentered case (see for example [13]), it is natural to introduce the operator

$$\left(T_\delta^{unc} f\right)(\mathbf{x}) = \frac{1}{|B(\delta\mathbf{x},|1-\delta|\ |\mathbf{x}|)|}\int_{B(\delta\mathbf{x},|1-\delta|\ |\mathbf{x}|)} f(\mathbf{y})d\mathbf{y},\ f\in L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

where $\delta\in\mathbb{R}$, $\delta\neq 1$, and $S(\mathbf{a},r)$ denotes the $(n-1)-$dimensional sphere in $\mathbb{R}^n$ centered at $\mathbf{a}\in\mathbb{R}^n$ and of radius $r>0$. Using the same method as in [6] it can be proved that

$$\begin{aligned} &M_p\left[M_1\left[f;B(\delta\mathbf{x},|1-\delta|\ |\mathbf{x}|)\right];B(R)\right] \\ &\leqslant M_1\left[M_p\left[f;B(|\mathbf{x}|)\right];B(\delta R\mathbf{e},|1-\delta|R)\right], \end{aligned} \tag{3}$$

where $p\geqslant 1$, $R>0$, $B(R)=B(\mathbf{0},R)$, $\delta\in\mathbb{R}$, $\delta\neq 1$, f is a non-negative function on $B((|\delta|+|1-\delta|)R)$, and $\mathbf{e}\in\mathbb{R}^n$, $|\mathbf{e}|=1$, is arbitrary. From this, in the standard manner, Hardy's inequality can be deduced and

$$\|T_\delta^{unc}\|_p = \frac{1}{|B(\delta\mathbf{e},|1-\delta|)|}\int_{B(\delta\mathbf{e},|1-\delta|)}|\mathbf{x}|^{-\frac{n}{p}}d\mathbf{x}.$$

It can be shown that $\|T_{1/2}^{unc}\|_p$ has exponential growth for fixed $p>1$ and large n.

Analogues of operators T_δ^c and T_δ^{unc} for the case of averaging functions over hyperspheres in $\mathbb{R}^n$ were investigated in [12], where appropriate mixed-means inequalities were proved and the operator norms were deduced. Asymptotic behavior of these norms was also investigated.

In [3], an interested reader can find additional information on mixed-means inequalities.

In this paper we investigate mixed-means inequalities for scaled families of rectangles in $\mathbb{R}^n$ with sides parallel to coordinate hyperplanes, ℓ_1- balls (non-scaled) in $\mathbb{R}^n$ centered at the origin and ellipsoids centered at the origin (non-scaled) with axes parallel to coordinate axes. In this sense, the case of rectangles is completely solved. The general problem of establishing mixed-means inequalities for operators defined analogously as T_δ^c and T_δ^{unc} for scaled $\ell_\alpha-$balls, $\alpha \neq 2$, and scaled ellipsoids, remains open. Notice that in the contrast to balls and spheres, these sets are not rotationally invariant.

2. Mixed means and Hardy-type inequalities for rectangles in $\mathbb{R}^n$

2.1. Mixed means for rectangles in $\mathbb{R}^n$

The mixed-means inequalities for non-scaled rectangles are given in [4].

For given $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^n$, we denote by $B_\infty(\mathbf{b}_1, \mathbf{b}_2)$ the rectangle in $\mathbb{R}^n$ with sides parallel to coordinate hyper-planes and with the main diagonal determined by $\mathbf{b}_1$ and $\mathbf{b}_2$. In the case $\mathbf{b}_1 = -\mathbf{b}_2 = \mathbf{b}$, we write $B_\infty(\pm\mathbf{b})$. Set $\mathbf{1} = (1, \ldots, 1)$.

For $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we set $\mathbf{x} \circ \mathbf{y} = (x_1 y_1, \ldots, x_n y_n)$, $\max\{\mathbf{x}, \mathbf{y}\} = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$ and $\frac{1}{\mathbf{x}} = \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right)$ if $x_i \neq 0$ for $i = 1, \ldots, n$.

THEOREM 1. (General case) *Let* $\mathbf{b} \in \mathbb{R}_+^n$, $p \geqslant 1$, $\boldsymbol{\delta}_1 = (\delta_{11}, \ldots, \delta_{1n}) \in \mathbb{R}^n$, $\boldsymbol{\delta}_2 = (\delta_{21}, \ldots, \delta_{2n}) \in \mathbb{R}^n$, $\delta_{1i} \neq \delta_{2i}$, $i = 1, \ldots n$. *If* $f : B_\infty(\pm(\max\{|\delta_{1i}|, |\delta_{2i}|\}) \circ \mathbf{b}) \to \mathbb{R}$ *is a non-negative function, then*

$$\left[\frac{1}{|B_\infty(\pm\mathbf{b})|}\int_{B_\infty(\pm\mathbf{b})}\left(\frac{1}{|B_\infty(\boldsymbol{\delta}_1\circ\mathbf{x},\boldsymbol{\delta}_2\circ\mathbf{x})|}\int_{B_\infty(\boldsymbol{\delta}_1\circ\mathbf{x},\boldsymbol{\delta}_2\circ\mathbf{x})} f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}}$$
$$\leqslant \frac{1}{|B_\infty(\boldsymbol{\delta}_1\circ\mathbf{b},\boldsymbol{\delta}_2\circ\mathbf{b})|}\int_{B_\infty(\boldsymbol{\delta}_1\circ\mathbf{b},\boldsymbol{\delta}_2\circ\mathbf{b})}\left(\frac{1}{|B_\infty(\pm\mathbf{y})|}\int_{B_\infty(\pm\mathbf{y})} f^p(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}} d\mathbf{y}. \tag{4}$$

Inequality (4) *is sharp and equality holds for functions of the form* $f(\mathbf{x}) = C\prod_{i=1}^n |x_i|^{\alpha_i}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $C \geqslant 0$.

Proof. By using obvious transformations and Minkowski's inequality, we have

$$\left[\frac{1}{2^n\prod_{i=1}^n b_i}\int_{B_\infty(\pm\mathbf{b})}\left(\frac{1}{\prod_i|\delta_{2i}-\delta_{1i}|\,|x_i|}\int_{B_\infty(\boldsymbol{\delta}_1\circ\mathbf{x},\boldsymbol{\delta}_2\circ\mathbf{x})}f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}}$$
$$=\left(\frac{1}{2^n\prod_{i=1}^n b_i}\right)^{\frac{1}{p}}\frac{1}{\prod_i|\delta_{2i}-\delta_{1i}|}\left[\int_{B_\infty(\pm\mathbf{b})}\left(\int_{B_\infty(\boldsymbol{\delta}_1\circ\mathbf{1},\boldsymbol{\delta}_2\circ\mathbf{1})}f(\mathbf{x}\circ\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}}$$
$$\leqslant\left(\frac{1}{2^n\prod_{i=1}^n b_i}\right)^{\frac{1}{p}}\frac{1}{\prod_i|\delta_{2i}-\delta_{1i}|}\int_{B_\infty(\boldsymbol{\delta}_1\circ\mathbf{1},\boldsymbol{\delta}_2\circ\mathbf{1})}\left(\int_{B_\infty(\pm\mathbf{b})}f^p(\mathbf{x}\circ\mathbf{y})d\mathbf{x}\right)^{\frac{1}{p}}d\mathbf{y}$$
$$=\left(\frac{1}{2^n\prod_{i=1}^n b_i}\right)^{\frac{1}{p}}\frac{1}{\prod_i|\delta_{2i}-\delta_{1i}|\,b_i}\int_{B_\infty(\boldsymbol{\delta}_1\circ\mathbf{b},\boldsymbol{\delta}_2\circ\mathbf{b})}\left(\int_{B_\infty(\pm\mathbf{b})}f^p\left(\frac{1}{\mathbf{b}}\circ\mathbf{x}\circ\mathbf{y}\right)d\mathbf{x}\right)^{\frac{1}{p}}d\mathbf{y}$$
$$=\frac{1}{\prod_i|\delta_{2i}-\delta_{1i}|\,b_i}\int_{B_\infty(\boldsymbol{\delta}_1\circ\mathbf{b},\boldsymbol{\delta}_2\circ\mathbf{b})}\left(\frac{1}{2^n\prod_i|y_i|}\int_{B_\infty(\pm\mathbf{y})}f^p(\mathbf{x})\,d\mathbf{x}\right)^{\frac{1}{p}}d\mathbf{y},$$

by which inequality (4) is proved.

In the following corollaries we give the precise meaning of the main cases of the scaled families which we are interested in: origin-centered case, centered case and uncentered case.

COROLLARY 1. (Origin-centered case) *Let* $\mathbf{b}\in\mathbb{R}_+^n$, $p\geqslant 1$. *If* $f:B_\infty(\pm\mathbf{b})\to\mathbb{R}$ *is a non-negative function, then*

$$\left[\frac{1}{|B_\infty(\pm\mathbf{b})|}\int_{B_\infty(\pm\mathbf{b})}\left(\frac{1}{|B_\infty(\pm\mathbf{x})|}\int_{B_\infty(\pm\mathbf{x})}f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}}$$
$$\leqslant\frac{1}{|B_\infty(\pm\mathbf{b})|}\int_{B_\infty(\pm\mathbf{b})}\left(\frac{1}{|B_\infty(\pm\mathbf{y})|}\int_{B_\infty(\pm\mathbf{y})}f^p(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}}d\mathbf{y}. \tag{5}$$

Inequality (5) *is sharp and equality holds for functions of the form* $f(\mathbf{x})=C\prod_{i=1}^n|x_i|^{\alpha_i}$, $\alpha_1,\ldots,\alpha_n\in\mathbb{R}$, $C\geqslant 0$.

COROLLARY 2. (Uncentered case) *Let* $\mathbf{b}\in\mathbb{R}_+^n$, $p\geqslant 1$, $\delta\in\mathbb{R}$, $\delta\neq 1$. *If* $f:B_\infty(\pm\delta\mathbf{b})\to\mathbb{R}$ *is a non-negative function, then*

$$\left[\frac{1}{|B_\infty(\pm\mathbf{b})|}\int_{B_\infty(\pm\mathbf{b})}\left(\frac{1}{|B_\infty(\delta\mathbf{x},\mathbf{x})|}\int_{B_\infty(\delta\mathbf{x},\mathbf{x})}f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}}$$
$$\leqslant\frac{1}{|B_\infty(\delta\mathbf{b},\mathbf{b})|}\int_{B_\infty(\delta\mathbf{b},\mathbf{b})}\left(\frac{1}{|B_\infty(\pm\mathbf{y})|}\int_{B_\infty(\pm\mathbf{y})}f^p(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}}d\mathbf{y}. \tag{6}$$

Inequality (6) *is sharp and equality holds for functions of the form* $f(\mathbf{x})=C\prod_{i=1}^n|x_i|^{\alpha_i}$, $\alpha_1,\ldots,\alpha_n\in\mathbb{R}$, $C\geqslant 0$.

COROLLARY 3. (Centered case) *Let* $\mathbf{b} \in \mathbb{R}^n_+,\ p \geqslant 1,\ \delta > 0$. *If* $f : B_\infty(\pm(1+\delta)\mathbf{b}) \to \mathbb{R}$ *is a non-negative function, then*

$$\left[\frac{1}{|B_\infty(\pm\mathbf{b})|}\int_{B_\infty(\pm\mathbf{b})}\left(\frac{1}{|B_\infty((1-\delta)\mathbf{x},(1+\delta)\mathbf{x})|}\int_{B_\infty((1-\delta)\mathbf{x},(1+\delta)\mathbf{x})} f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}}$$
$$\leqslant \frac{1}{|B_\infty((1-\delta)\mathbf{b},(1+\delta)\mathbf{b})|}\int_{B_\infty((1-\delta)\mathbf{b},(1+\delta)\mathbf{b})}\left(\frac{1}{|B_\infty(\pm\mathbf{y})|}\int_{B_\infty(\pm\mathbf{y})} f^p(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}} d\mathbf{y}. \tag{7}$$

Inequality (7) is sharp and equality holds for functions of the form $f(\mathbf{x}) = C\prod_{i=1}^n |x_i|^{\alpha_i}$, $\alpha_1,\ldots,\alpha_n \in \mathbb{R},\ C \geqslant 0$.

2.2. Hardy-type inequalities for rectangles in $\mathbb{R}^n$

The general Hardy-type inequality for rectangles is given in the following theorem. We omit the proof, since it is analogous to the proofs given in [4, 5, 6, 12].

THEOREM 2. (General case) Let $\mathbf{b} \in \mathbb{R}^n_+$, $p > 1$, $\boldsymbol{\delta}_1 = (\delta_{11},\ldots,\delta_{1n}) \in \mathbb{R}^n$, $\boldsymbol{\delta}_2 = (\delta_{21},\ldots,\delta_{2n}) \in \mathbb{R}^n$, $\delta_{1i} \neq \delta_{2i}$, $i = 1,\ldots n$. If $f : B_\infty(\pm(\max\{|\delta_{1i}|,|\delta_{2i}|\}) \circ \mathbf{b}) \to \mathbb{R}$ is a non-negative function, then

$$\left[\int_{B_\infty(\pm\mathbf{b})}\left(\frac{1}{|B_\infty(\boldsymbol{\delta}_1\circ\mathbf{x},\boldsymbol{\delta}_2\circ\mathbf{x})|}\int_{B_\infty(\boldsymbol{\delta}_1\circ\mathbf{x},\boldsymbol{\delta}_2\circ\mathbf{x})} f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}}$$
$$\leqslant C_\infty(n,p;\boldsymbol{\delta}_1,\boldsymbol{\delta}_2)\left(\int_{B_\infty(\pm\max\{\boldsymbol{\delta}_1,\boldsymbol{\delta}_2\}\circ\mathbf{b})} f^p(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}}, \tag{8}$$

where

$$C_\infty(n,p;\boldsymbol{\delta}_1,\boldsymbol{\delta}_2) = \frac{1}{|B_\infty(\boldsymbol{\delta}_1\circ\mathbf{1},\boldsymbol{\delta}_2\circ\mathbf{1})|}\int_{B_\infty(\boldsymbol{\delta}_1\circ\mathbf{1},\boldsymbol{\delta}_2\circ\mathbf{1})}\left(\prod_{i=1}^n |y_i|\right)^{-\frac{1}{p}} d\mathbf{y} \tag{9}$$

is the best possible constant.

Notice that

$$C_\infty(n,p;\boldsymbol{\delta}_1,\boldsymbol{\delta}_2) = \prod_{i=1}^n\left(\frac{1}{|\delta_{2i}-\delta_{1i}|}\int_{(\delta_{1i},\delta_{2i})} |y_i|^{-\frac{1}{p}} dy_i\right). \tag{10}$$

We emphasize some cases where the best possible constants can be easily computed:

1. Rectangles centered at the origin: $\delta_{1i} = -1, \delta_{2i} = 1$, $i = 1,\ldots,n$, and

$$C_\infty(n,p;-1,1) = \left(\int_0^1 |y|^{-\frac{1}{p}}\right)^n = \left(\frac{p}{p-1}\right)^n.$$

2. Centered case: $\delta_{1i} = 1-\delta, \delta_{2i} = 1+\delta,\ i = 1,\dots,n,\ \delta > 0$, and
$$C_\infty(n,p;1-\delta,1+\delta) = \left(\frac{1}{2\delta}\int_{1-\delta}^{1+\delta} |y|^{-\frac{1}{p}}\right)^n.$$
It is easy to see that $\max_{\delta>0} C_\infty(1,p;1-\delta,1+\delta)$ is achieved for $1<\delta<2$. For $p=2,\ \delta_{\max} = \frac{2}{\sqrt{3}}$ and $\max_{\delta>0} C_\infty(1,p;1-\delta,1+\delta) = \frac{\sqrt[4]{27}}{\sqrt{2}} \approx 1.611185 < 2 = \frac{2}{2-1}$. This is conjectured to be the operator norm of the centered Hardy-Littlewood maximal function in [7]. The problem remains unsolved.

3. Uncentered case: $\delta_{1i} = \delta, \delta_{2i} = 1,\ i=1,\dots,n,\ \delta\in\mathbb{R},\ \delta\neq 1$, and
$$C_\infty(n,p;\delta,1) = \left(\frac{1}{|1-\delta|}\int_{(\delta,1)} |y|^{-\frac{1}{p}}\right)^n.$$
Again, it is easy to see that $\max_{\delta\in\mathbb{R}} C_\infty(1,p;\delta,1)$ is achieved for $-1<\delta<0$. For $p=2,\ \delta_{\max} = 2\sqrt{2}-3$ and $\max_{\delta>0} C_\infty(1,p;\delta,1) = 1+\sqrt{2} \approx 2.414214 > 2 = \frac{2}{2-1}$. This is proved, in [8], to be the operator norm of the uncentered maximal function.

3. Mixed means for ℓ_1-balls in $\mathbb{R}^n$

In order to obtain mixed-means inequalities for ℓ_1-balls, the first idea is to use results for ℓ_2-balls, but in the process of reducing the mixed-means inequality for ℓ_1-balls centered at the origin to the mixed-means inequality for ℓ_2-balls investigated in [5], we obtain weights that were not considered in [5].

Set $B_1(R) = \{\mathbf{x}\in\mathbb{R}^n : |\mathbf{x}|_1 = \sum_{i=1}^n |x_i| \leqslant R\}$.

THEOREM 3. *Let $R>0,\ p\geqslant 1$. If $f : B_1(R)\to\mathbb{R}$ is a non-negative function, then*
$$\left[\frac{1}{|B_1(R)|}\int_{B_1(R)}\left(\frac{1}{|B_1(|\mathbf{x}|_1)|}\int_{B_1(|\mathbf{x}|_1)} f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}} \leqslant \frac{1}{|B_1(R)|}\int_{B_1(R)}\left(\frac{1}{|B_1(|\mathbf{y}|_1)|}\int_{B_1(|\mathbf{y}|_1)} f^p(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}} d\mathbf{y}. \tag{11}$$
Inequality (11) is sharp and equality holds for functions of the form $f(\mathbf{x}) = C|\mathbf{x}|_1^\lambda$, $\lambda\in\mathbb{R}$, $C\geqslant 0$.

The following lemma is useful in proving Theorem 3. The proof of this lemma is an exercise in elementary inequalities.

LEMMA 1. *Inequality (11) is equivalent to the inequality*
$$\left[\frac{1}{|B_1^+(R)|}\int_{B_1^+(R)}\left(\frac{1}{|B_1^+(|\mathbf{x}|_1)|}\int_{B_1^+(|\mathbf{x}|_1)} f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}}$$

$$\leqslant \frac{1}{\left|B_1^+(R)\right|}\int_{B_1^+(R)}\left(\frac{1}{\left|B_1^+(|\mathbf{y}|_1)\right|}\int_{B_1^+(|\mathbf{y}|_1)} f^p(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}} d\mathbf{y}, \tag{12}$$

where $B_1^+(R) = \{\mathbf{x}\in\mathbb{R}^n:\ |\mathbf{x}|_1 = \sum_{i=1}^n x_i \leqslant R,\ x_1,\dots,x_n \geqslant 0\}$.

Proof of Theorem 3. We prove B_1^+ version of Theorem 3. Since $\left|B_1^+(R)\right| = R^n/n!$ and after obvious transformations we have

$$\begin{aligned}&\left[\frac{1}{\left|B_1^+(R)\right|}\int_{B_1^+(R)}\left(\frac{1}{\left|B_1^+(|\mathbf{x}|_1)\right|}\int_{B_1^+(|\mathbf{x}|_1)} f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}}\\ &= \frac{(n!)^{\frac{1}{p}+1}}{R^{\frac{n}{p}}}\left[\int_{|\mathbf{x}|_1\leqslant R}\left(\int_{|\mathbf{y}|_1\leqslant 1} f(|\mathbf{x}|_1\mathbf{y})\,d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}}.\end{aligned} \tag{13}$$

Set $y_i = u\phi_i^2$, $i = 1,\dots,n$, where $\phi = (\phi_1,\dots,\phi_n) = \left(\sin\varphi_{n-1}\overline{\phi}, \cos\varphi_{n-1}\right) \in S^{n-1}$, $\overline{\phi} \in S^{n-2}$. The Jacobian of this transformation is $J\frac{(y_1,\dots,y_n)}{(u,\varphi_1,\dots,\varphi_{n-1})} = 2^{n-1}u^{n-1}\prod_{i=1}^n \phi_i J\phi$. Now, the right-hand side of (13) is equal to

$$\frac{(n!)^{\frac{1}{p}+1}}{R^{\frac{n}{p}}}\left[\int_{B_1^+(R)}\left(\int_{S_+^{n-1}}\int_0^1 f\left(u|\mathbf{x}|_1\left(\phi_1^2,\dots,\phi_n^2\right)\right)2^{n-1}u^{n-1}\prod_{i=1}^n \phi_i\, du d\phi\right)^p d\mathbf{x}\right]^{\frac{1}{p}}. \tag{14}$$

By applying Jensen's inequality

$$\left[(n-1)!\int_{S_+^{n-1}} F(\phi)2^{n-1}\prod_{i=1}^n \phi_i d\phi\right]^p \leqslant (n-1)!\int_{S_+^{n-1}} F^p(\phi)2^{n-1}\prod_{i=1}^n \phi_i d\phi,$$

we get that (14) is not greater than

$$\frac{(n!)^{\frac{1}{p}+1}(n-1)!^{\frac{1}{p}-1}}{R^{\frac{n}{p}}}\left[\int_{B_1^+(R)}\int_{S_+^{n-1}}\left(\int_0^1 f\left(u|\mathbf{x}|_1\left(\phi_1^2,\dots,\phi_n^2\right)\right)u^{n-1}\,du\right)^p 2^{n-1}\prod_{i=1}^n \phi_i d\phi d\mathbf{x}\right]^{\frac{1}{p}}. \tag{15}$$

By using Minkowski's inequality, transformation $x_i' = \sum_{j=1}^i x_j \in [x_{i-1}', R]$, $i = 1,\dots,n$, $x_0' = 0$, where D_n denotes the transformed domain, (15) is not greater than

$$\begin{aligned}&\frac{(n!)^{\frac{1}{p}+1}(n-1)!^{\frac{1}{p}-1}}{R^{\frac{n}{p}}}\int_0^1\left(\int_{D_n}\int_{S_+^{n-1}} f^p\left(ux_n\left(\phi_1^2,\dots,\phi_n^2\right)\right)2^{n-1}\prod_{i=1}^n \phi_i d\phi\ d\mathbf{x}\right)^{\frac{1}{p}} u^{n-1}du\\ &= \frac{(n!)^{\frac{1}{p}+1}}{R^{\frac{n}{p}}(n-1)!}\int_0^1\left(\int_{x_n=0}^R\int_{S_+^{n-1}} x_n^{n-1} f^p\left(ux_n\left(\phi_1^2,\dots,\phi_n^2\right)\right)2^{n-1}\prod_{i=1}^n \phi_i\, d\phi dx_n\right)^{\frac{1}{p}} u^{n-1}du\end{aligned}$$

$$= \frac{n}{R^n}\int_0^R \left(\frac{n!}{u^n}\int_{t=0}^{u}\int_{S_+^{n-1}} f^p\left(t\left(\phi_1^2,\dots,\phi_n^2\right)\right)2^{n-1}t^{n-1}\prod_{i=1}^{n}\phi_i \, dt d\phi\right)^{\frac{1}{p}} u^{n-1}du$$

$$= \frac{n}{R^n}\int_{u=0}^R \left(\frac{1}{\left|B_1^+(u)\right|}\int_{B_1(u)} f^p\left(y_1,\dots,y_n\right)\right)^{\frac{1}{p}} u^{n-1}du$$

$$= \frac{1}{\left|B_1^+(R)\right|}\int_{B_1^+(R)} \left(\frac{1}{\left|B_1^+\left(|\mathbf{x}|_1\right)\right|}\int_{B_1^+(|\mathbf{x}|_1)} f^p(\mathbf{y})\,d\mathbf{y}\right)^{\frac{1}{p}} d\mathbf{x},$$

where the last equality holds since the volume of the $(n-1)-$dimensional l_1-ball of the radius u is $u^{n-1}/(n-1)!$.

By using the same arguments as in [4, 5, 6, 12], it follows that the operator norm of the operator $(SQf)(\mathbf{x}) = \frac{1}{|B_1(|\mathbf{x}|_1)|}\int_{B_1(|\mathbf{x}|_1)} f(\mathbf{y})d\mathbf{y}$ is

$$\|SQ\|_p = \frac{1}{|B_1(1)|}\int_{B_1(1)} |\mathbf{x}|_1^{-\frac{n}{p}}\, d\mathbf{x} = \frac{p}{p-1}.$$

4. Mixed means for ellipsoids in $\mathbb{R}^n$

For an $\mathbf{x}\in\mathbb{R}^n$, set $\mathrm{Ell}(\mathbf{x}) = \left\{\mathbf{y}\in\mathbb{R}^n\,;\sum_{i=1}^n \frac{y_i^2}{x_i^2}\leqslant n\right\}$. Then

$$|\mathrm{Ell}(\mathbf{x})| = n^{\frac{n}{2}}\,|B(1)|\prod_{i=1}^{n}|x_i|.$$

Notice that $\mathbf{x}$ is an element of the boundary of $\mathrm{Ell}(\mathbf{x})$.

The proof of a mixed-means inequality for ellipsoids defined in this way, is much easier than the proof for balls, even in the case of balls centered at the origin, and it is more related to the case of rectangles. This is especially seen in the asymptotic behavior of the operator norm of the operator which averages functions over the ellipsoids.

THEOREM 4. *Let $R>0$, $p\geqslant 1$. If $f: B(\sqrt{n}R)\to\mathbb{R}$ is a non-negative function, then*

$$\left[\frac{1}{|B(R)|}\int_{B(R)}\left(\frac{1}{|\mathrm{Ell}(\mathbf{x})|}\int_{\mathrm{Ell}(\mathbf{x})} f(\mathbf{y})dy\right)^p d\mathbf{x}\right]^{\frac{1}{p}}$$
$$\leqslant \frac{1}{|B(R)|}\int_{B(R)}\left(\frac{1}{|\mathrm{Ell}(\mathbf{y})|}\int_{\mathrm{Ell}(\mathbf{y})} f^p(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}} d\mathbf{y}. \tag{16}$$

Inequality (16) is sharp and equality holds for functions of the form $f(\mathbf{x}) = C\prod_{i=1}^n |x_i|^{\alpha_i}$, where $C\geqslant 0$, $\alpha_1,\dots,\alpha_n\in\mathbb{R}$.

Proof. By using obvious transformations and Minkowski's inequality we have:

$$\begin{aligned}
&\left[\frac{1}{|B(R)|}\int_{B(R)}\left(\frac{1}{|\mathrm{Ell}\,(\mathbf{x})|}\int_{\mathrm{Ell}(\mathbf{x})} f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}} \\
&= \left[\frac{1}{R^n\,|B(1)|}\int_{B(R)}\left(\frac{1}{n^{\frac{n}{2}}\prod_{i=1}^n |x_i|\ |B(1)|}\int_{\mathrm{Ell}(\mathbf{x})} f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}} \\
&= \left(\frac{1}{R^n\,|B(1)|}\right)^{\frac{1}{p}} \frac{1}{|B(1)|}\left[\int_{B(R)}\left(\int_{B(1)} f\left(\sqrt{n}\,\mathbf{x}\circ\mathbf{y}\right)d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}} \\
&= \left(\frac{1}{|B(1)|}\right)^{\frac{1}{p}+1} \frac{1}{R^n}\left[\int_{B(1)}\left(\int_{B(R)} f\left(\sqrt{n}\,\mathbf{x}\circ\mathbf{y}\right)d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}} \\
&\leqslant \left(\frac{1}{|B(1)|}\right)^{\frac{1}{p}+1} \frac{1}{R^n}\int_{B(R)}\left(\int_{B(1)} f^p\left(\sqrt{n}\,\mathbf{x}\circ\mathbf{y}\right)d\mathbf{x}\right)^{\frac{1}{p}} d\mathbf{y} \\
&= \left(\frac{1}{|B(1)|}\right)^{\frac{1}{p}+1} \frac{1}{R^n}\int_{B(R)}\left(\frac{1}{n^{\frac{n}{2}}\prod_{i=1}^n |y_i|}\int_{\mathrm{Ell}(\mathbf{y})} f^p(\mathbf{x})\,d\mathbf{x}\right)^{\frac{1}{p}} d\mathbf{y} \\
&= \frac{1}{|B(R)|}\int_{B(R)}\left(\frac{1}{|\mathrm{Ell}\,(\mathbf{y})|}\int_{\mathrm{Ell}(\mathbf{y})} f^p(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}} d\mathbf{y}. \qquad (17)
\end{aligned}$$

In the following theorem we show how symmetry in mixed-means inequalities can be lost, if we mix ”shapes” of sets in these means.

THEOREM 5. *Let $R>0$, $p\geqslant 1$ and $\delta_1,\ \delta_2\in\mathbb{R}$, $\delta_1\neq\delta_2$. If $f: B_\infty\left(\pm\max\{\delta_1,\delta_2\}\,\mathbf{R}\right)\to\mathbb{R}$ is a nonnegative function, then*

$$\begin{aligned}
&\left[\frac{1}{|B(R)|}\int_{B(R)}\left(\frac{1}{|B_\infty(\delta_1\mathbf{x},\delta_2\mathbf{x})|}\int_{B_\infty(\delta_1\mathbf{x},\delta_2\mathbf{x})} f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}} \\
&\leqslant \frac{1}{|B_\infty(\delta_1\mathbf{R},\delta_2\mathbf{R})|}\int_{B_\infty(\delta_1\mathbf{R},\delta_2\mathbf{R})}\left(\frac{1}{|\overline{\mathrm{Ell}}(\mathbf{y})|}\int_{\overline{\mathrm{Ell}}(\mathbf{y})} f^p(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}} d\mathbf{y}, \qquad (18)
\end{aligned}$$

where $\mathbf{R}=R(1,\ldots,1)$, $\overline{\mathrm{Ell}}\,(\mathbf{x})=\left\{\mathbf{y}\in\mathbb{R}^n\,;\sum_{i=1}^n \frac{y_i^2}{x_i^2}\leqslant 1\right\},\ \ \mathbf{x}\in\mathbb{R}^n.$

Proof. By using obvious changes of variables, we have:

$$\begin{aligned}
&\left[\frac{1}{|B(R)|}\int_{B(R)}\left(\frac{1}{|B_\infty(\delta_1\mathbf{x},\delta_2\mathbf{x})|}\int_{B_\infty(\delta_1\mathbf{x},\delta_2\mathbf{x})} f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}} \\
&= \left[\frac{1}{R^n\,|B(1)|}\int_{B(R)}\left(\frac{1}{|\delta_2-\delta_1|^n\prod_{i=1}^n|x_i|}\int_{B_\infty(\delta_1\mathbf{x},\delta_2\mathbf{x})} f(\mathbf{y})d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}} \\
&= \left(\frac{1}{R^n\,|B(1)|}\right)^{\frac{1}{p}} \frac{1}{|\delta_2-\delta_1|^n}\left[\int_{B(R)}\left(\int_{B_\infty(\delta_1\mathbf{1},\delta_2\mathbf{1})} f(\mathbf{x}\circ\mathbf{y})\,d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}}
\end{aligned}$$

$$= \left(\frac{1}{|B(1)|}\right)^{\frac{1}{p}} \frac{1}{R^n |\delta_2-\delta_1|^n} \left[\int_{B(1)} \left(\int_{B_\infty(\delta_1\mathbf{R},\delta_2\mathbf{R})} f(\mathbf{x}\circ\mathbf{y})\, d\mathbf{y}\right)^p d\mathbf{x}\right]^{\frac{1}{p}}$$

$$\leqslant \left(\frac{1}{|B(1)|}\right)^{\frac{1}{p}} \frac{1}{R^n |\delta_2-\delta_1|^n} \int_{B_\infty(\delta_1\mathbf{R},\delta_2\mathbf{R})} \left(\int_{B(1)} f^p(\mathbf{x}\circ\mathbf{y})\, d\mathbf{x}\right)^{\frac{1}{p}} d\mathbf{y}$$

$$= \left(\frac{1}{|B(1)|}\right)^{\frac{1}{p}} \frac{1}{R^n |\delta_2-\delta_1|^n} \int_{B_\infty(\delta_1\mathbf{R},\delta_2\mathbf{R})} \left(\frac{1}{\prod_{i=1}^n |y_i|} \int_{\overline{\mathrm{Ell}}(\mathbf{y})} f^p(\mathbf{x})\, d\mathbf{x}\right)^{\frac{1}{p}} d\mathbf{y}$$

$$= \frac{1}{|B_\infty(\delta_1\mathbf{R},\delta_2\mathbf{R})|} \int_{B_\infty(\delta_1\mathbf{R},\delta_2\mathbf{R})} \left(\frac{1}{|\overline{\mathrm{Ell}}(\mathbf{y})|} \int_{\overline{\mathrm{Ell}}(\mathbf{y})} f^p(\mathbf{x}) d\mathbf{x}\right)^{\frac{1}{p}} d\mathbf{y}.$$

4.1. Hardy-type inequality for ellipsoids

By using analogous arguments as in [4, 5, 6, 12], we obtain from (16) that for nonnegative function $f \in L^p(\mathbb{R}^n)$

$$\left[\int_{\mathbb{R}^n} \left(\frac{1}{|\mathrm{Ell}(x)|} \int_{\mathrm{Ell}(x)} f(y) dy\right)^p dx\right]^{\frac{1}{p}} \leqslant C_{\mathrm{Ell}}(n,p)\, \|f\|_p, \;\; p > 1, \tag{19}$$

where

$$C_{\mathrm{Ell}}(n,p) = |B(R)|^{\frac{1}{p}-1} \int_{B(R)} |\mathrm{Ell}(y)|^{-\frac{1}{p}}\, dy = \frac{n^{-\frac{n}{2p}}}{|B(1)|} \int_{B(1)} \left(\prod_{i=1}^n |y_i|\right)^{-\frac{1}{p}} dy$$

$$= \frac{n^{-\frac{n}{2p}}}{|B(1)|} 2^n \int_{u=0}^1 \int_{\mathbb{S}_+^{n-1}} \left(u^n \prod_{i=1}^n \phi_i\right)^{-\frac{1}{p}} u^{n-1} du d\phi = \frac{n^{-\frac{n}{2p}} 2^n}{|B(1)|} \frac{p}{n(p-1)} \int_{\mathbb{S}_+^{n-1}} \left(\prod_{i=1}^n \phi_i\right)^{-\frac{1}{p}} d\phi$$

$$= \frac{n^{-\frac{n}{2p}} 2^n}{|B(1)|} \frac{p}{n(p-1)} \int_{(0,\pi/2)^n} \left(\prod_{i=1}^{n-1} \sin^i \varphi_i \cos\varphi_i\right)^{-\frac{1}{p}} \prod_{i=2}^{n-1} \sin^{i-1} \varphi_i d\varphi_1 \dots d\varphi_{n-1}$$

$$= \frac{p}{p-1} \frac{n^{-\frac{n}{2p}} \Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \frac{\Gamma^n\left(\frac{1}{2p'}\right)}{\Gamma\left(\frac{n}{2p'}\right)}.$$

It holds:

$$1 = \frac{1}{|S(\mathbf{0},1)|} \int_{S(\mathbf{0},1)} |\mathbf{x}|^{-\frac{n}{p}} ds(\mathbf{x}) < \frac{p}{p-1} = \frac{1}{|B(\mathbf{0},1)|} \int_{B(\mathbf{0},1)} |\mathbf{x}|^{-\frac{n}{p}} d\mathbf{x}$$

$$< C_{\mathrm{Ell}}(n,p) = \frac{1}{|\mathrm{Ell}(\mathbf{1})|} \int_{\mathrm{Ell}(\mathbf{1})} \left(\prod_{i=1}^n |x_i|\right)^{-\frac{1}{p}} d\mathbf{x}$$

$$< \left(\frac{p}{p-1}\right)^n = \frac{1}{|B_\infty(\pm\mathbf{1})|} \int_{B_\infty(\pm\mathbf{1})} \left(\prod_{i=1}^n |x_i|\right)^{-\frac{1}{p}} d\mathbf{x}. \tag{20}$$

The second inequality in (20) follows from the inequality $\left(\prod_{i=1}^n |x_i|\right)^{\frac{1}{n}} \leqslant \left(\frac{1}{n}\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$, where equality holds if and only if $|x_i| = |x_j|$ for every $i,j = 1,\ldots,n$. The third inequality holds since the function $\mathbf{x} \mapsto \left(\prod_{i=1}^n |x_i|\right)^{-\frac{1}{p}}$ is coordinatewise decreasing and $B_\infty(\pm\mathbf{1}) \subset \mathrm{Ell}(\mathbf{1})$.

REFERENCES

[1] M. CHRIST AND L. GRAFAKOS, *Best constants for two nonconvolution inequalities*, Proc. Amer. Math. Soc., **123**, 6 (1995), 1687–1693.

[2] A. ČIŽMEŠIJA AND J. PEČARIĆ, *Mixed means and Hardy's inequality*, Math. Inequal. Appl., **1**, 4 (1998), 491–506.

[3] A. ČIŽMEŠIJA AND J. PEČARIĆ, *Classical Hardy's and Carleman's inequalities and mixed means*, Survey on Classical Inequalities (Ed. T. M. Rassias), Kluwer Academic Publishers, Netherlands, 2000, 27–65.

[4] A. ČIŽMEŠIJA AND J. PEČARIĆ, *Multivariable mixed means and inequalities of Hardy and Levin-Cochran-Lee type*, Math. Inequal. Appl., **5**, 3 (2002), 397–415.

[5] A. ČIŽMEŠIJA, J. PEČARIĆ, AND I. PERIĆ, *Mixed means and inequalities of Hardy and Levin–Cochran–Lee type for multidimensional balls*, Proc. Amer. Math. Soc., **128**, 9 (2000), 2543–2552.

[6] A. ČIŽMEŠIJA, I. PERIĆ, *Mixed means over balls and annuli and lower bounds for operator norms of maximal functions*, J. Math. Anal. Appl., **291** (2004), 625–637.

[7] R. DROR, S. GANGULI, R. S. STRICHARTZ, *A search for best constants in the. Hardy-Littlewood maximal theorem*, J. Fourier Anal. Appl., **2** (1996), 473–486.

[8] L. GRAFAKOS, S. MONTGOMERY-SMITH, *Best constants for uncentered maximal functions*, Bull. London Math. Soc., **29** (1997), 60–64.

[9] G. HARDY, J. E. LITTLEWOOD, AND G. POLYA, *Inequalities*, 2nd edition, Cambridge University Press, Cambridge, 1967.

[10] E. H. LIEB AND M. LOSS, *Analysis*, Graduate Studies in Mathematics, Vol. 14, Amer. Math. Soc., Providence, RI, 1997.

[11] D. S. MITRINOVIĆ, J. E. PEČARIĆ, AND A. M. FINK, *Inequalities Involving Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1991.

[12] I. PERIĆ, *Mixed means for centered and uncentered averaging operators over spheres and related results*, BJMA, **2**, 2 (2008), 150–162.

[13] E. M. STEIN, *Harmonic analysis: Real–Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.

[14] E. M. STEIN, G. WEISS, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, NJ, 1971.

(Received November 6, 2008)

Ivan Perić
Faculty of Food Technology and Biotechnology
University of Zagreb, Pierottijeva 6
10000 Zagreb, Croatia
e-mail: iperic@pbf.hr