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MONOTONICITY AND CONVEXITY OF S–MEANS

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Abstract. For real α, r, s and positive x, y we define S-means by

$$S(\alpha;r,s;x,y) = \frac{E(r,s;x^{\alpha+1},y^{\alpha+1})}{E(r,s;x^{\alpha},y^{\alpha})},$$

where E is the Stolarsky mean. S contains Gini, Heronian and many other known means.
In this paper we investigate convexity properties of $S(\alpha)$ and obtain new inequalities between Gini, Heronian and Stolarsky means.
The results lead to new inequalities for generalized Heronian means and reveal new properties of Stolarsky means.

1. Introduction

The S-means, defined for real α, r, s and positive $x, y,$ are given by the formula

$$S(\alpha;r,s;x,y) = \frac{E(r,s;x^{\alpha+1},y^{\alpha+1})}{E(r,s;x^{\alpha},y^{\alpha})}, \tag{1.1}$$

where

$$E(r,s) = E(r,s;x,y) = \begin{cases} \left(\frac{r}{s}\frac{y^s-x^s}{y^r-x^r}\right)^{1/(s-r)} & sr(s-r)(x-y)\neq 0, \\ \left(\frac{1}{r}\frac{y^r-x^r}{\log y-\log x}\right)^{1/r} & r(x-y)\neq 0,\ s=0, \\ e^{-1/r}\left(y^{y^r}/x^{x^r}\right)^{1/(y^r-x^r)} & r=s,\ r(x-y)\neq 0, \\ \sqrt{xy} & r=s=0,\ x-y\neq 0, \\ x & x=y \end{cases} \tag{1.2}$$

is the Stolarsky mean. They have been introduced in [10], where the problem of comparison between $S(\alpha;a,b;x,y)$ and $S(\alpha;c,d;x,y)$ was solved. In this paper we investigate monotonicity and convexity of $S(\alpha)$.
The S-means are interesting as this family contains some well-known families of two-parameter means. Clearly $S(0)$ is the Stolarsky mean. Further

$$S(1;r,s;x,y) = G(r,s) = G(r,s;x,y) = \left(\frac{x^s+y^s}{x^r+y^r}\right)^{1/(s-r)}$$

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is the Gini mean. Similarly, we have the Heronian

$$S(1/2;r,s;x,y) = H(r,s) = H(r,s;x,y) = \left(\frac{x^s + (\sqrt{xy})^s + y^s}{x^r + (\sqrt{xy})^r + y^r} \right)^{1/(s-r)}$$

and the centroidal

$$S(2;r,s;x,y) = T(r,s) = T(r,s;x,y) = \left(\frac{x^{2s} + (xy)^s + y^{2s}}{x^s + y^s} \Big/ \frac{x^{2r} + (xy)^r + y^{2r}}{x^r + y^r} \right)^{1/(s-r)}$$

means.

Observe that S-means are homogeneous of order 1 with respect to x and y and symmetric both in r,s and x,y.

We shall investigate monotonicity and logarithmic convexity of S-means with respect to all the variables in section 2. Section 3 deals with the properties of

$$V(\alpha;r,s;x,y) = S\left(\alpha; \frac{r}{2\alpha+1}, \frac{s}{2\alpha+1};x,y\right). \tag{1.3}$$

In the final section we use just obtained results to investigate some new properties of Stolarsky means and establish some inequalities for generalized Heronian means.

In [8] we introduced the weighted extended mean values (see also [9]) defined as

$$F(r,s;a,b;x,y) = \frac{E(r,s;ax,by)}{E(r,s;a,b)}. \tag{1.4}$$

We see that

$$S(\alpha;r,s;x,y) = F(r,s;x^\alpha,y^\alpha;x,y),$$

therefore, obviously, some properties of the weighted extended mean values will be used here.

In this paper we use standard notation A,G,L for the arithmetic, geometric and logarithmic means. We also omit some or all arguments if there is no risk of confusion.

Let us recall some known properties of convex functions and some facts from [9].

PROPERTY 1.1. If f is continuous, then it is convex (resp. concave) if and only if for every $h>0$ $f(x+h)-f(x)$ increases (resp. decreases) in x. For negative h the monotonicity of $f(x+h)-f(x)$ reverses.

PROPERTY 1.2. If f is continuous, then it is convex (resp. concave) if and only if for every x_0 the function $h(t) = f(x_0-t)+f(x_0+t)$ increases (resp. decreases) for $t>0$.

PROPERTY 1.3. f is convex (resp. concave) if and only if the function $\frac{f(s)-f(t)}{s-t}$ increases (resp. decreases) in s and t.

For real t and positive $A, B \neq 1$ let

$$g(t,A,B) = \frac{A^t \log^2 A}{(A^t-1)^2} - \frac{B^t \log^2 B}{(B^t-1)^2}. \tag{1.5}$$

LEMMA 1.1. ([9], Lemma 2.2)

(a) $g(t,A,B) = g(\pm t, A^{\pm 1}, B^{\pm 1})$ *for arbitrary choice of signs,*

(b) g *is increasing in* t *on* $(0,\infty)$ *if* $\log^2 A - \log^2 B > 0$, *and decreasing otherwise.*

2. Monotonicity and logaritmic convexity of S-means

THEOREM 2.1. $S(\alpha,r,s;x,y)$ *is increasing in variables* r *and* s *if* $\alpha > -\frac{1}{2}$, *and decreasing if* $\alpha < -\frac{1}{2}$.

Proof. Theorem 2 in [8] states that function $F(r,s;a,b;x,y)$ increases in r and s if $(x-y)(a^2x-b^2y) > 0$, and decreases otherwise. For S-means this condition reads $(x-y)(x^{2\alpha+1}-y^{2\alpha+1}) > 0$, and this is equivalent to $2\alpha+1 > 0$. □

THEOREM 2.2. $S(\alpha,r,s;x,y)$ *is increasing in* α *if* $r+s > 0$, *and decreasing if* $r+s < 0$.

Proof. Since S-means are homogeneous we can assume that $y = 1$. By Theorem 3 in [8], $F(r,s;a,b;x,y)$ increases in a if $(x-y)(r+s) > 0$, and decreases otherwise, so

$$\begin{aligned}
&\operatorname{sgn}(S(\alpha,r,s;x,1) - S(\beta,r,s;x,1)) \\
&\qquad = \operatorname{sgn}(F(r,s;x^\alpha,1;x,1) - F(r,s;x^\beta,1;x,1)) \\
&\qquad = \operatorname{sgn}((x^\alpha - x^\beta)(x-1)(r+s)) \\
&\qquad = \operatorname{sgn}(x^\beta(x^{\alpha-\beta}-1)(x-1)(r+s)) \\
&\qquad = \operatorname{sgn}(\alpha-\beta)(r+s).
\end{aligned}$$

because $\operatorname{sgn}((x^{\alpha-\beta}-1)(x-1)) = \operatorname{sgn}(\alpha-\beta)$. □

In view of $S(0) \leqslant S(1/2) \leqslant S(1) \leqslant S(2)$ we immediately get

COROLLARY 2.3.

$$E(r,s) \leqslant H(r,s) \leqslant G(r,s) \leqslant T(r,s)$$

if $r+s > 0$. *For* $r+s < 0$ *the inequalities reverse.*

The above-mentioned inequality between Stolarsky and Gini means was obtained by Neuman and Páles [4].
It is worth noting that in general S-means are not monotone in x and y. Indeed, since for $r,s>0$ we have $\lim_{x\to 0}E(r,s,x,1)>0$, we see that for $\alpha>0 \quad \lim_{x\to 0}S(\alpha;r,s;x,1)=1=S(\alpha;r,s;1,1)$.

Stolarsky means $E(r,s;x,y)$ are positive and continuous in all variables, hence so are S-means. Therefore, we can consider only the general case ($r\neq s,\ rs\neq 0$) and have the other cases follow by continuity.
Moreover, due to homogeneity of S as a function of x and y, we can consider only the case of $y=1$.
Given that

$$\frac{\partial^2}{\partial\alpha^2}\log|b^\alpha-1|=-\frac{b^\alpha\log^2 b}{(b^\alpha-1)^2}$$

and

$$S(\alpha;r,s;x,1)=\left(\frac{x^{(\alpha+1)s}-1}{x^{\alpha s}-1}\Big/\frac{x^{(\alpha+1)r}-1}{x^{\alpha r}-1}\right)^{1/s-r}$$

we see that

$$\frac{\partial^2}{\partial\alpha^2}\log S=\frac{g(\alpha+1,x^r,x^s)-g(\alpha,x^r,x^s)}{s-r}, \tag{2.1}$$

where g is defined by (1.5). This leads to the following

THEOREM 2.4. *If $r+s>0$, then $S(\alpha)$ is log-concave for $\alpha>-\frac{1}{2}$ and log-convex for $\alpha<-\frac{1}{2}$.*
If $r+s<0$, then $S(\alpha)$ is log-convex for $\alpha>-\frac{1}{2}$ and log-concave for $\alpha<-\frac{1}{2}$.

Proof. By Lemma 1.1(a) we can replace α and $\alpha+1$ in the right hand side of the formula (2.1) with their absolute values. Since $\alpha>-\frac{1}{2}$ is equivalent to $|\alpha+1|>|\alpha|$, after applying Lemma 1.1(b) we get

$$\begin{aligned}\operatorname{sgn}\frac{\partial^2}{\partial\alpha^2}\log S&=\operatorname{sgn}\frac{g(|\alpha+1|,x^r,x^s)-g(|\alpha|,x^r,x^s)}{s-r}\\&=\operatorname{sgn}\left(\left(\alpha+\tfrac{1}{2}\right)\frac{\log^2(x^r)-\log^2(x^s)}{s-r}\right)\\&=-\operatorname{sgn}\left(\left(\alpha+\tfrac{1}{2}\right)(r+s)\right),\end{aligned}$$

which completes the proof. □

An easy calculation reveals the following properties of $S(\alpha)$:

$$S(-\tfrac{1}{2};r,s;x,y)=\sqrt{xy} \tag{2.2}$$

$$S(-\tfrac{1}{2}-\alpha)S(-\tfrac{1}{2}+\alpha)=S^2(-\tfrac{1}{2})=xy, \tag{2.3}$$

which enables us to prove the

THEOREM 2.5. *If* $r+s>0$ *then*

- *if* $\alpha_0 > -\frac{1}{2}$, *then* $S(\alpha_0-t)S(\alpha_0+t)$ *is decreasing for positive* t,
- *if* $\alpha_0 < -\frac{1}{2}$, *then* $S(\alpha_0-t)S(\alpha_0+t)$ *is increasing for positive* t.

For $r+s<0$ *the monotonicities reverse.*

Proof. From (2.2) and (2.3) we see that the function $\log S(\alpha-\frac{1}{2})-\log S(-\frac{1}{2})$ is odd, so the theorem follows from Lemma 1.6 in [9] and Theorem 2.4. □

3. Properties of $V(\alpha;r,s;x,y)$

What we are aiming to prove in this section is that

THEOREM 3.1. *If* $r+s>0(<0)$, *then* $V(\alpha)$ *increases (decreases) and is concave (convex) for* $\alpha>-1/2$.

NOTE: a simple calculation shows that the function V is symmetric with respect to the line $\alpha=-1/2$.

For $\alpha>-1/2$ let $\overline{\alpha}=\frac{-\alpha}{2\alpha+1}$. The function $\alpha\to\overline{\alpha}$ maps the half-line $(-1/2,\infty)$ onto itself, is decreasing, $\overline{\overline{\alpha}}=\alpha$ and $(2\alpha+1)(2\overline{\alpha}+1)=1$.

The function S satisfies the identity

$$S(\alpha;r,s;x,y)=(xy)^{-\alpha}S^{2\alpha+1}\left(\tfrac{-\alpha}{2\alpha+1};(2\alpha+1)r,(2\alpha+1)s\right). \tag{3.1}$$

To show it, let $\mu=2\alpha+1$ and $\nu=-\alpha/(2\alpha+1)$. Then $-\alpha=\mu\nu$, and $\alpha+1=\mu(\nu+1)$. Using the identities $E(r,s;x^t,y^t)=E^t(tr,ts;x,y)$ and $E(r,s;1/x,1/y)=E(r,s;x,y)/xy$, we obtain

$$\begin{aligned}
S(\alpha;r,s;x,y)&=\frac{E(r,s;x^{\alpha+1},y^{\alpha+1})}{E(r,s;x^{\alpha},y^{\alpha})}=\frac{E(r,s;x^{\alpha+1},y^{\alpha+1})}{(xy)^{\alpha}E(r,s;x^{-\alpha},y^{-\alpha})}\\
&=(xy)^{-\alpha}\frac{E(r,s;x^{\mu(\nu+1)},y^{\mu(\nu+1)})}{E(r,s;x^{\mu\nu},y^{\mu\nu})}\\
&=(xy)^{-\alpha}S(\nu;r,s;x^{\mu},y^{\mu})\\
&=(xy)^{-\alpha}S^{2\alpha+1}\left(\frac{-\alpha}{(2\alpha+1)};(2\alpha+1)r,(2\alpha+1)s;x,y\right).
\end{aligned}$$

The identity (3.1) can be written in the form

$$(xy)^{\frac{\alpha}{2\alpha+1}}S^{\frac{1}{2\alpha+1}}(\alpha;r,s)=S\left(\overline{\alpha};\frac{r}{2\overline{\alpha}+1},\frac{s}{2\overline{\alpha}+1}\right)=V(\overline{\alpha};r,s). \tag{3.2}$$

Now we can prove Theorem 3.1:

Proof. Let $-1/2<\alpha<\beta$. Then $-1/2<\overline{\beta}<\overline{\alpha}$, and we can write $\overline{\beta}$ as a convex combination of $-1/2$ and $\overline{\alpha}$

$$\overline{\beta}=-\tfrac{1}{2}\tfrac{2\overline{\alpha}-2\overline{\beta}}{2\overline{\alpha}+1}+\tfrac{2\overline{\beta}+1}{2\overline{\alpha}+1}\overline{\alpha}.$$

The log-concavity of S implies the inequality

$$S^{\frac{2\overline{\alpha}-2\overline{\beta}}{2\overline{\alpha}+1}}(-1/2;r,s)S^{\frac{2\overline{\beta}+1}{2\overline{\alpha}+1}}(\overline{\alpha};r,s)\leqslant S(\overline{\beta};r,s),$$

and since $S(-1/2;r,s,x,y)=\sqrt{xy}$, we have

$$(xy)^{\frac{\overline{\alpha}}{2\overline{\alpha}+1}}S^{\frac{1}{2\overline{\alpha}+1}}(\overline{\alpha};r,s)\leqslant(xy)^{\frac{\overline{\beta}}{2\overline{\beta}+1}}S^{\frac{1}{2\overline{\beta}+1}}(\overline{\beta};r,s)$$

and applying (3.2), we obtain

$$V(\alpha;r,s)\leqslant V(\beta;r,s), \tag{3.3}$$

so the monotonicity is proved (obviously if $r+s<0$ then S is log-convex, and the inequalities are reversed).
To show that V is concave, it is enough to prove that for fixed $\beta>-1/2$ the function

$$m(\alpha)=\frac{V(\alpha)-V(\beta)}{\alpha-\beta}$$

is decreasing for $\alpha>-1/2$. Let

$$n(\alpha)=\frac{\log S(\alpha;r,s;x,y)-\log S(\beta;r,s;x,y)}{\alpha-\beta}.$$

As $\log S$ is concave, n is decreasing, and applying once more (3.2) we have

$$\begin{aligned}n(\overline{\alpha})&=-\log(xy)+\frac{1}{\beta-\alpha}\left[\frac{V(\alpha)}{(2\overline{\beta}+1)}-\frac{V(\beta)}{(2\overline{\alpha}+1)}\right]\\&=-\log(xy)+\frac{(2\beta+1)V(\alpha)-(2\alpha+1)V(\beta)}{\beta-\alpha}\\&=-\log(xy)+2V(\beta)-(2\beta+1)\frac{V(\beta)-V(\alpha)}{\beta-\alpha}\\&=-\log(xy)+2V(\beta)-(2\beta+1)m(\alpha)\end{aligned}$$

This means that n and m are of the same monotonicity and the proof is complete. □

Applying Theorem 3.1, the inequalities $V(0)\leqslant V(1/2)\leqslant V(1)\leqslant V(2)$ yield the following

COROLLARY 3.2. *For $r+s>0$*

$$E(r,s)\leqslant H(r/2,s/2)\leqslant G(r/3,s/3)\leqslant T(r/5,s/5)$$

The above inequality between E and G was found by Czinder and Páles ([2]). It is optimal in the following sense

THEOREM 3.3. *If* $-1/2 \leqslant \alpha \leqslant \beta$, *then for small positive* ε *the means*

$$S\left(\alpha; \frac{r}{2\alpha+1}, \frac{s}{2\alpha+1}; x, y\right) \quad \textit{and} \quad S\left(\beta; \frac{r}{2\beta+1+\varepsilon}, \frac{s}{2\beta+1+\varepsilon}; x, y\right)$$

are not comparable.

Proof. If for two homogeneous, symmetric means $M(x,y) \leqslant N(x,y)$ holds, then the function $\log N(x,1) - \log M(x,1)$ has the local minimum at $x=1$, so its second derivative is nonnegative there. Given that

$$\frac{d^2}{dx^2} \log \left| \frac{x^p - 1}{x^q - 1} \right| \Bigg|_{x=1} = (p-q)\frac{p+q-6}{12}$$

and

$$\begin{aligned} &\log S(\beta; r/(2\beta+1+\varepsilon), s/(2\beta+1+\varepsilon); x, 1) \\ &\qquad = \frac{2\beta+1+\varepsilon}{s-r}\left(\log\left|\frac{x^{s(\beta+1)/(2\beta+1+\varepsilon)}-1}{x^{r(\beta+1)/(2\beta+1+\varepsilon)}-1}\right| - \log\left|\frac{x^{s\beta/(2\beta+1+\varepsilon)}-1}{x^{r\beta/(2\beta+1+\varepsilon)}-1}\right|\right), \end{aligned}$$

then for $x=1$ and $r+s$ we have

$$\begin{aligned} 12\frac{d^2}{dx^2}&\left[\log S\left(\beta; \tfrac{r}{(2\beta+1+\varepsilon)}, \tfrac{s}{(2\beta+1+\varepsilon)}; x, 1\right) - \log S\left(\alpha; \tfrac{r}{(2\alpha+1)}, \tfrac{s}{(2\alpha+1)}; x, 1\right)\right] \\ &= \frac{2\beta+1+\varepsilon}{s-r}\frac{(s-r)(\beta+1)}{2\beta+1+\varepsilon}\left(\frac{(s+r)(\beta+1)}{2\beta+1+\varepsilon} - 6\right) \\ &\quad - \frac{2\beta+1+\varepsilon}{s-r}\frac{(s-r)\beta}{2\beta+1+\varepsilon}\left(\frac{(s+r)\beta}{2\beta+1+\varepsilon} - 6\right) \\ &\quad - \frac{2\alpha+1}{s-r}\frac{(s-r)(\alpha+1)}{2\alpha+1}\left(\frac{(s+r)(\alpha+1)}{2\alpha+1} - 6\right) \\ &\quad + \frac{2\alpha+1}{s-r}\frac{(s-r)\alpha}{2\alpha+1}\left(\frac{(s+r)\alpha}{2\alpha+1} - 6\right) \\ &= (s+r)\left(\frac{2\beta+1}{2\beta+1+\varepsilon} - 1\right) < 0. \quad \square \end{aligned}$$

4. Applications

In this section we shall show some new properties of the Stolarsky means. Let us begin with a kind of Chebyshev's inequality

THEOREM 4.1. *If $r+s>0$ and the pairs $(x_1,y_1),(x_2,y_2)$ are similarly ordered (i.e. $(x_1-y_1)(x_2-y_2)\geqslant 0$), then*

$$E(r,s;x_1,y_1)E(r,s;x_2,y_2)\leqslant E(r,s;x_1x_2,y_1y_2).$$

Any change of sign in the assumptions toggles this inequality.

Proof. Let $a=y_1/x_1$ and $b=y_2/x_2$. Similar ordering means that a and b are both either greater or lesser than 1, hence there is an $\alpha>0$ such that $b=a^{\alpha}$. By Theorem 2.2, $S(0;r,s;1,a)\leqslant S(\alpha;r,s;1,a)$ or equivalently

$$E(r,s;1,a)\leqslant \frac{E(r,s;1,a^{\alpha+1})}{E(r,s;1,a^{\alpha})}=\frac{E(r,s;1,ab)}{E(r,s;1,b)}. \quad \square$$

Another consequence of monotonicity of S-means is the following

THEOREM 4.2. *If $r+s>0$, then the function $h(t)=\log E(r,s;x^t,y^t)$ is convex and $g(t)=E(tr,ts;x,y)$ is increasing. In case $r+s<0$ h is concave and g decreases.*

Proof. Assume $r+s>0$ and fix $z>0$. Then

$$h(t+z)-h(t)=\log\frac{E(r,s;x^{t+z},y^{t+z})}{E(r,s;x^t,y^t)}=\log\frac{E(r,s;x^{z(t/z+1)},y^{z(t/z+1)})}{E(r,s;x^t,y^t)}=\log S(t/z;r,s;x^z,y^z)$$

increases, so h is convex by Property 1.1 and Theorem 2.2. Now, by Property 1.3

$$\frac{h(t)-h(0)}{t}=\frac{\log E(r,s;x^t,y^t)}{t}=\log g(t)$$

increases. $\square$

Feng Qi and Chao-Ping Chen (see [7] and references therein) proved that if $0<a<b<c$ then $E(r,1;a,b)/E(r,1;a,c)$ decreases in r. We prove here a bit stronger result:

THEOREM 4.3. *The function $E(r,s;a,b)/E(r,s;a,c)$ increases in r (and consequently in s) if $(b-c)(bc-a^2)>0$, and decreases otherwise.*

Proof. Set $B=b/a, C=c/a$. Note that

$$E(r,s;x,y)=\left(\frac{L(x^r,y^r)}{L(x^s,y^s)}\right)^{1/(r-s)}.$$

We have

$$\log\frac{E(r,s;a,b)}{E(r,s;a,c)}=\log\frac{E(r,s;1,B)}{E(r,s;1,C)}=\frac{\log\frac{L(1,\exp(r\log B))}{L(1,\exp(r\log C))}-\log\frac{L(1,\exp(s\log B))}{L(1,\exp(s\log C))}}{r-s}. \tag{4.1}$$

On the other hand by Theorem 4.2 the function

$$\log\frac{L(1,\exp(t\log B))}{L(1,\exp(t\log C))} = (\log B-\log C)\log E(\log B,\log C;1,e^t)$$

is convex if $(\log B-\log C)(\log B+\log C)>0$, and concave otherwise, so the divided difference (4.1) is monotone by Property 1.3. To complete the proof, note that $\log B-\log C>0$ is equivalent to $b-c>0$, and $\log B+\log C>0$ is equivalent to $bc-a^2>0$. □

COROLLARY 4.4. *The quotient* $E(r,s;a,b)/E(r,s;c,d)$ *increases (decreases) in* $r,\ s$ *if and only if*

$$\frac{a}{b}+\frac{b}{a}>(<)\frac{c}{d}+\frac{d}{c}. \tag{4.2}$$

Proof. Since

$$\frac{E(r,s;a,b)}{E(r,s;c,d)}=\frac{aE(r,s;a,b)}{cE(r,s;a,ad/c)}$$

the function incerases iff $(b-ad/c)(abd/c-a^2)>$ and this inequality is equivalent to (4.2). □

Let us concentrate now on generalized Heronian means defined by

$$H_n(x,y)=\frac{x+x^{\frac{n-1}{n}}y^{\frac{1}{n}}+\cdots+x^{\frac{1}{n}}y^{\frac{n-1}{n}}+y}{n+1} \tag{4.3}$$

If x and y are the n-dimensional volumes of the bases of an $(n+1)$-dimensional frustum of height h, then its $(n+1)$-dimensional volume equals $hH_n(x,y)$. The means are named after Heron of Alexandria who, almost 2000 years ago, discovered the formula for the volume of frustum of a pyramide.
It is easy to see that $H_n(x,y)=S(1/n;1,0;x,y)$, so by Theorem 2.2

$$L=H_\infty\leqslant\ldots\leqslant H_{n+1}\leqslant H_n\leqslant\ldots\leqslant H_1=A.$$

Since $\frac{1}{n}=\frac{n-1}{2n}\frac{1}{n-1}+\frac{n+1}{2n}\frac{1}{n+1}$, Theorem 2.4 implies

$$H_n^{2n}\geqslant H_{n-1}^{n-1}H_{n+1}^{n+1}\quad\text{or}\quad\frac{H_{n+1}^{n+1}}{H_n^n}\leqslant\frac{H_n^n}{H_{n-1}^{n-1}}. \tag{4.4}$$

Similarly we obtain inequalities

$$H_n^n\geqslant LA^{n-1}\quad\text{and}\quad H_{n+1}^{n+1}\geqslant H_n^nL.$$

This last inequality gives the lower bound for inequality (4.4).

Consider now for $n\geqslant 2$

$$H_{-n}(x,y)=S(-1/n;0,1;x,y)=\frac{x^{\frac{n-1}{n}}y^{\frac{1}{n}}+\cdots+x^{\frac{1}{n}}y^{\frac{n-1}{n}}}{n-1}=\frac{(n+1)H_n-2A}{n-1}. \tag{4.5}$$

The inequalities $G \leqslant H_{-n} \leqslant L$ combined with (4.5) result in

$$\frac{(n-1)G+2A}{n+1} \leqslant H_n \leqslant \frac{(n-1)L+2A}{n+1} \tag{4.6}$$

To obtain better estimate for H_n, observe that log-concavity of S-means implies $L^2 \geqslant H_n H_{-n}$. Applying (4.5) and solving the resulting quadratic inequality, we get

$$H_n \leqslant \frac{A+\sqrt{A^2+(n^2-1)L^2}}{n+1}.$$

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