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SOME INEQUALITIES INVOLVING A FRACTAL OPERATOR OF FUNCTIONS ON THE SPHERE

M. A. NAVASCUES

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Abstract. This paper undertakes the construction of fractal versions of the classical spherical harmonics. Some inequalities satisfied by the coefficients of the iterated function systems defining the fractal functions provide sufficient conditions for the existence of new Hilbert bases of functions on the sphere. This fact confirms their properties of good approximation. The methodology used implies the definition of an operator mapping standard functions into their fractal analogues. The transformation is linear and bounded and some upper bounds of its norm are also established.

1. Introduction

The study of sherical signals has important applications nowadays in climatology, celestial mechanics, satellite technology, etc. Freeden et al., in the reference [6], wonder about the suitability of using smooth functions to model less smooth variables as, for instance, geostrophic flows and they solve the question appealing to the theorem of extension of Helly. We approach the same problem from a different perspective and we define fractal versions of the spherical harmonics by means of fractal interpolation. This type of approximation does not imply the use of smooth functions. If an order of regularity is required to solve a given problem, one must impose specific additional conditions to the system (see for instance [3], [11]).

In this paper, we assign a fractal analogue for every function with integrable square on the sphere. This association is performed via a linear and bounded operator. The text studies the properties of the transformation and, in particular, some upper bounds of its norm are established.

Other inequalities satisfied by the coefficients of the iterated function system defining the fractal functions become a key in order to provide Hilbert bases of functions on the sphere.

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2. α-Fractal functions

Let $t_0 < t_1 < ... < t_N$ be real numbers, and $I = [t_0, t_N] = [a, b]$ the closed interval that contains them. Let a set of data points $\{(t_n, x_n) \in I \times R: \ n = 0, 1, 2, ..., N\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n, \ \ n \in \{1, 2, ..., N\}$ be contractive homeomorphisms such that:

$$L_n(t_0) = t_{n-1}, \ L_n(t_N) = t_n \tag{1}$$

$$|L_n(c_1) - L_n(c_2)| \leqslant l \ |c_1 - c_2| \quad \forall \ c_1, c_2 \ \in I \tag{2}$$

for some $0 \leqslant l < 1$.

Let $F = I \times R$ and N continuous mappings, $F_n : F \to R$, be given satisfying:

$$F_n(t_0, x_0) = x_{n-1}, \ \ F_n(t_N, x_N) = x_n, \ \ n = 1, 2, ..., N \tag{3}$$

$$|F_n(t, x) - F_n(t, y)| \leqslant r|x - y|, \quad t \in I, \quad x, y \in R, \quad 0 \leqslant r < 1. \tag{4}$$

Now define functions $w_n(t, x) = (L_n(t), F_n(t, x)), \ \forall \ n = 1, 2, ..., N$.

THEOREM 2.1. *(Barnsley* [1]*) The Iterated Function System (IFS)* $\{F, \ w_n : n = 1, 2, ..., N\}$ *defined above admits a unique attractor* G. G *is the graph of a continuous function* $g : I \to R$ *which obeys* $g(t_n) = x_n$ *for* $n = 0, 1, 2, ..., N$.

The previous function is called a Fractal Interpolation Function (FIF) corresponding to $\{(L_n(t), F_n(t, x))\}_{n=1}^{N}$. The map g is unique satisfying the functional equation ([1]):

$$g(t) = F_n(L_n^{-1}(t), g \circ L_n^{-1}(t)), \quad n = 1, 2, ..., N, \ \ t \in I_n = [t_{n-1}, t_n]. \tag{5}$$

The most widely studied fractal interpolation functions so far are defined by the IFS

$$\begin{cases} L_n(t) = a_n t + b_n \\ F_n(t, x) = \alpha_n x + q_n(t) \end{cases} \tag{6}$$

where $-1 < \alpha_n < 1$, $\forall n = 1, 2, \ldots, N$. α_n is called a vertical scaling factor of the transformation w_n and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is the scale vector of the IFS. Following the equalities (1),

$$a_n = \frac{t_n - t_{n-1}}{t_N - t_0}, \qquad b_n = \frac{t_N t_{n-1} - t_0 t_n}{t_N - t_0}. \tag{7}$$

Let $f \in \mathscr{C}(I)$ be a continuous function. We consider the case

$$q_n(t) = f \circ L_n(t) - \alpha_n b(t) \tag{8}$$

where b is continuous and such that $b(t_0) = x_0$; $b(t_N) = x_N$.
It is easy to check that the condition (3) is fulfilled. The set of data points is here $\{(t_n, x_n = f(t_n)) \in I \times R: \ n = 0, 1, 2, ..., N\}$. Using this IFS one can define fractal analogues of any continuous function (see Fig. 1 and 2, and references [9], [10]).

In particular, we consider in this paper the case

$$b = vf \tag{9}$$

where $v : I \to R$ is fixed, continuous, $v(t_0) = v(t_N) = 1$ and $v(t)$ non-constant. If we consider the norm (in the space of continuous functions $\mathscr{C}(I)$):

$$\|f\|_{\mathscr{L}^2} = (\int_a^b |f|^2 dt)^{1/2} \tag{10}$$

then

$$\|vf\|_{\mathscr{L}^2} = (\int_a^b |vf|^2 dt)^{1/2} \leqslant \|v\|_\infty \|f\|_{\mathscr{L}^2} \tag{11}$$

where

$$\|v\|_\infty = \max\{|v(t)| : t \in I\}. \tag{12}$$

DEFINITION 2.2. Let $\Delta : a = t_0 < t_1 < \ldots < t_N = b$, where $N > 1$, be a partition of the interval $I = [a,b]$. A scale vector associated to Δ is an $\alpha \in (-1,1)^N$.

DEFINITION 2.3. Let f^α be the continuous function defined by the IFS (6), (7), (8) and (9). f^α is called α-fractal function associated to f with respect to $b = vf$ and the partition Δ.

According to (5), f^α satisfies the fixed point equation:

$$f^\alpha(t) = f(t) + \alpha_n (f^\alpha - vf) \circ L_n^{-1}(t), \qquad \forall t \in I_n. \tag{13}$$

f^α interpolates to f at t_n as, using (1), (13) and Barnsley's theorem:

$$f^\alpha(t_n) = f(t_n) + \alpha_n (f^\alpha - vf)(t_N) = f(t_n), \qquad \forall n = 0,1,\ldots,N. \tag{14}$$

Let us call α-fractal operator $\mathscr{V}^\alpha = \mathscr{V}^\alpha_\Delta$ with respect to Δ, to the transformation which assigns f^α to the function f ($\mathscr{V}^\alpha(f) = f^\alpha$).

THEOREM 2.4. *(a) $\mathscr{V}^\alpha$: $\mathscr{C}[a,b] \to \mathscr{C}[a,b]$ is linear and bounded with respect to the $\mathscr{L}^2$-norm.*
(b) If $\alpha = 0$, $\mathscr{V}^\alpha = Identity$.
(c) The following inequalities hold

$$\|\mathscr{V}^\alpha\|_2 \leqslant \frac{1 + |\alpha|_\infty \|v\|_\infty}{1 - |\alpha|_\infty}, \tag{15}$$

$$\|I - \mathscr{V}^\alpha\|_2 \leqslant \frac{(1 + \|v\|_\infty)|\alpha|_\infty}{1 - |\alpha|_\infty} \tag{16}$$

where $|\alpha|_\infty = \max\{|\alpha_n| : n = 1,2,\ldots,N\}$ and $\|\mathscr{V}^\alpha\|_2$ is the norm of the operator $\mathscr{V}^\alpha$ defined as

$$\|\mathscr{V}^\alpha\|_2 = \max\{\|\mathscr{V}^\alpha(f)\|_{\mathscr{L}^2} : \|f\|_{\mathscr{L}^2} = 1,\ f \in \mathscr{C}[a,b]\}.$$

Proof. (a) The linearity is proved as in [9] and [10]. The item (b) is deduced from (13). For the boundness, let us consider, according to the equation (13),

$$\|\mathscr{V}^{\alpha}(f)-f\|_{\mathscr{L}^2}^2=\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}|\alpha_n|^2|(\mathscr{V}^{\alpha}(f)-vf)\circ L_n^{-1}(t)|^2dt.$$

The change of variable $\tilde{t}=L_n^{-1}(t)$ provides

$$\|\mathscr{V}^{\alpha}(f)-f\|_{\mathscr{L}^2}^2=\sum_{n=1}^{N}a_n|\alpha_n|^2\int_a^b|(\mathscr{V}^{\alpha}(f)-vf)(\tilde{t})|^2d\tilde{t},$$

$$\|\mathscr{V}^{\alpha}(f)-f\|_{\mathscr{L}^2}^2=\sum_{n=1}^{N}a_n|\alpha_n|^2\|\mathscr{V}^{\alpha}(f)-vf\|_{\mathscr{L}^2}^2, \tag{17}$$

$$\|\mathscr{V}^{\alpha}(f)-f\|_{\mathscr{L}^2}^2\leqslant|\alpha|_{\infty}^2\,\|\mathscr{V}^{\alpha}(f)-vf\|_{\mathscr{L}^2}^2\sum_{n=1}^{N}a_n, \tag{18}$$

but if $T=b-a$, using (7),

$$\sum_{n=1}^{N}a_n=\frac{1}{T}\sum_{n=1}^{N}(t_n-t_{n-1})=1 \tag{19}$$

then

$$\|\mathscr{V}^{\alpha}(f)-f\|_{\mathscr{L}^2}\leqslant|\alpha|_{\infty}\,\|\mathscr{V}^{\alpha}(f)-vf\|_{\mathscr{L}^2}, \tag{20}$$

$$\|\mathscr{V}^{\alpha}(f)-f\|_{\mathscr{L}^2}\leqslant|\alpha|_{\infty}\,(\|\mathscr{V}^{\alpha}(f)-f\|_{\mathscr{L}^2}+\|f-vf\|_{\mathscr{L}^2}) \tag{21}$$

and

$$\|\mathscr{V}^{\alpha}(f)-f\|_{\mathscr{L}^2}\leqslant\frac{|\alpha|_{\infty}}{1-|\alpha|_{\infty}}\,\|f-vf\|_{\mathscr{L}^2}. \tag{22}$$

Moreover,

$$\|\mathscr{V}^{\alpha}(f)\|_{\mathscr{L}^2}-\|f\|_{\mathscr{L}^2}\leqslant\|\mathscr{V}^{\alpha}(f)-f\|_{\mathscr{L}^2}\leqslant\frac{|\alpha|_{\infty}}{1-|\alpha|_{\infty}}\,\|f-vf\|_{\mathscr{L}^2} \tag{23}$$

according to (10),

$$\|\mathscr{V}^{\alpha}(f)\|_{\mathscr{L}^2}-\|f\|_{\mathscr{L}^2}\leqslant\|\mathscr{V}^{\alpha}(f)-f\|_{\mathscr{L}^2}\leqslant\frac{|\alpha|_{\infty}}{1-|\alpha|_{\infty}}(1+\|v\|_{\infty})\|f\|_{\mathscr{L}^2}. \tag{24}$$

As a consequence,

$$\|\mathscr{V}^{\alpha}\|_2\leqslant 1+\frac{|\alpha|_{\infty}}{1-|\alpha|_{\infty}}(1+\|v\|_{\infty})=\frac{1+|\alpha|_{\infty}\|v\|_{\infty}}{1-|\alpha|_{\infty}} \tag{25}$$

where $\|\mathscr{V}^{\alpha}\|_2$ is the norm of the operator with respect to the $\mathscr{L}^2$-norm in $\mathscr{C}[a,b]$. Using (24),

$$\|I-\mathscr{V}^{\alpha}\|_2\leqslant\frac{(1+\|v\|_{\infty})|\alpha|_{\infty}}{1-|\alpha|_{\infty}}. \tag{26}$$

Consequence: According to Theorem 2.4 (b), the collection of maps f^{α} constitutes a family of continuous functions containing f as a particular case (for $\alpha=0$).

LEMMA 2.5. *([8]) If L is a linear operator from a Banach space into itself such that $\|L\| < 1$, then $(I-L)^{-1}$ exists and is bounded. Moreover,*

$$(I-L)^{-1} = I + L + L^2 + \dots \tag{27}$$

THEOREM 2.6. *If $|\alpha|_\infty < 1/(2+\|v\|_\infty)$, $\mathscr{V}^\alpha$ has a bounded inverse and*

$$\|(\mathscr{V}^\alpha)^{-1}\|_2 \leqslant \frac{1+|\alpha|_\infty}{1-|\alpha|_\infty\|v\|_\infty}. \tag{28}$$

Proof. According to the inequality (26) and the hypothesis given

$$\|I - \mathscr{V}^\alpha\|_2 < 1. \tag{29}$$

The previous lemma assures that $\mathscr{V}^\alpha = I - (I - \mathscr{V}^\alpha)$ has a bounded inverse. Let us denote $f^\alpha = \mathscr{V}^\alpha(f)$. In this case, the inequality (20) implies that $\forall f$,

$$\|f\|_{\mathscr{L}^2} \leqslant |\alpha|_\infty \, \|f^\alpha - vf\|_{\mathscr{L}^2} + \|f^\alpha\|_{\mathscr{L}^2}, \tag{30}$$

$$\|f\|_{\mathscr{L}^2} \leqslant |\alpha|_\infty \, (\|f^\alpha\|_{\mathscr{L}^2} + \|v\|_\infty\|f\|_{\mathscr{L}^2}) + \|f^\alpha\|_{\mathscr{L}^2}, \tag{31}$$

$$(1-|\alpha|_\infty\|v\|_\infty)\|f\|_{\mathscr{L}^2} \leqslant (1+|\alpha|_\infty)\|f^\alpha\|_{\mathscr{L}^2}. \tag{32}$$

By hypothesis

$$|\alpha|_\infty < \frac{1}{2+\|v\|_\infty} < \frac{1}{\|v\|_\infty}$$

and

$$1 - |\alpha|_\infty\|v\|_\infty > 0$$

then (32)

$$\|f\|_{\mathscr{L}^2} \leqslant \frac{1+|\alpha|_\infty}{1-|\alpha|_\infty\|v\|_\infty}\|f^\alpha\|_{\mathscr{L}^2} \tag{33}$$

and the inequality (28) is proved.

3. A fractal operator of $\mathscr{L}^2(S)$

3.1. Fractal spherical harmonics

A homogeneous polynomial V of degree n in the variables x, y, z satisfying the Laplace equation $\Delta V = 0$ is called a Laplace or harmonic polynomial of degree n. If we consider spherical coordinates (ρ, θ, φ) for $P \in R^3$ and

$$\xi = \sin(\varphi)\cos(\theta); \quad \eta = \sin(\varphi)\sin(\theta); \quad \zeta = \cos(\varphi)$$

(θ is the longitude and φ is the colatitude), then

$$V(x,y,z) = \rho^n V(\xi, \eta, \zeta)$$

The function

$$Y(\theta,\varphi) = V(\sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi))$$

is called the Laplace function or spherical harmonic of order n.
Two spherical harmonics of different degree (or order) are orthogonal over the sphere:

$$\int_S Y_n(P)Y_m(P)dS = 0; \quad n \neq m$$

where dS is the element of area of the sphere S. It is well known that the set of spherical harmonics of order n, $\mathscr{H}_n$, is a linear subspace of the functions on the sphere with dimension $2n+1$, and one of its orthogonal bases is:

$$\begin{cases} U_n^0(Q) = P_n(\cos\varphi) \\ U_n^m(Q) = P_n^m(\cos\varphi)\cos(m\theta) \\ V_n^m(Q) = P_n^m(\cos\varphi)\sin(m\theta) \end{cases} \tag{34}$$

if $Q = (\theta,\varphi)$, $m = 1,2,\ldots,n$. P_n is the n-th Legendre polynomial and P_n^m is the Ferrers or associated Legendre polynomial of degree n and order m defined as ([13])

$$P_n^m(t) = (1-t^2)^{\frac{m}{2}} P_n^{(m)}(t)$$

for $m = 1,2,\ldots,n$. These polynomials satisfy the equalities ([13]):

$$\int_{-1}^{1} P_n^m(t)P_r^m(t)dt = 0, \quad n \neq r,$$

$$\int_{-1}^{1} (P_n^m(t))^2 dt = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}.$$

The family

$$\{U_n^0, U_n^m, V_n^m; n = 0,1,2,\ldots, m = 1,2,\ldots,n\}$$

is an orthogonal and complete system of $\mathscr{L}^2(S)$. As a consequence, there is an orthonomal basis $\{Y_{nk}\}$ of spherical harmonics and every $f \in \mathscr{L}^2(S)$ can be expressed as

$$f = \sum_{n=0}^{\infty}\sum_{k=0}^{2n} c_{nk}Y_{nk}.$$

The expansion of a function $f \in \mathscr{L}^2(S)$ in terms of the elements of this system is called sometimes Laplace series of f.

In the following we extend the operator $\mathscr{V}^\alpha$ to the functions on the sphere $S \subset R^3$.

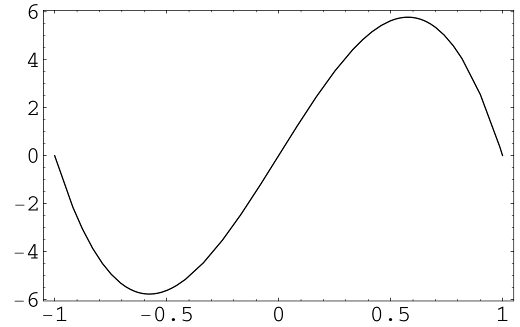

Figure 1. Graph of the associated Legendre polynomial P_3^2 of degree 3 and order 2.

LEMMA 3.1. *$\forall n = 0, 1, \ldots;$ $\forall m = 1, 2, \ldots, n,$*

$$\|U_n^0\|_{\mathscr{L}^2(S)} = \sqrt{2\pi}\|P_n\|_{\mathscr{L}^2},$$

$$\|U_n^m\|_{\mathscr{L}^2(S)} = \|V_n^m\|_{\mathscr{L}^2(S)} = \sqrt{\pi}\|P_n^m\|_{\mathscr{L}^2}.$$

Proof. For instance ([13]),

$$\|U_n^m\|^2_{\mathscr{L}^2(S)} = \int_0^{2\pi}\int_0^{\pi} |P_n^m(\cos\varphi)|^2 \cos^2(m\theta)\sin(\varphi)d\varphi d\theta,$$
$$\|U_n^m\|^2_{\mathscr{L}^2(S)} = \pi\int_{-1}^{1} |P_n^m(t)|^2 dt = \pi\|P_n^m\|^2_{\mathscr{L}^2}.$$

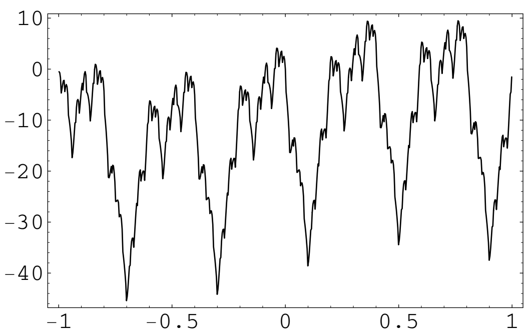

Figure 2. Graph of the α-fractal function of the associated Legendre polynomial P_3^2 of degree 3 and order 2, $\Delta : -1 < -3/5 < -1/5 < 1/5 < 3/5 < 1$, $v(t) = (12 + 24t - 11t^2 - 24t^3)$ and $\alpha_n = 0.3 \quad \forall n = 1, \ldots, 5.$

PROPOSITION 3.2. *There exists an operator $\mathscr{S}_n^\alpha : \mathscr{H}_n \to \mathscr{L}^2(S)$, where $\mathscr{H}_n$ is the space of spherical harmonics of order n, linear, injective and such that*

$$\|\mathscr{S}_n^\alpha\|_2 \leqslant \|\mathscr{V}^\alpha\|_2 \tag{35}$$

where $\|\cdot\|_2$ *is the operator norm with respect to the norm* $\|\cdot\|_{\mathscr{L}^2(S)}$ *defined as*

$$\|f\|_{\mathscr{L}^2(S)} = (\int_S f^2(P)dS)^{1/2}$$

and $\mathscr{V}^\alpha$ *is the operator of* $\mathscr{C}[a,b]$ *studied in the previous section.*

Proof. Let us start defining the image of the elements of the basis:

$$(U_n^0)^\alpha(\theta,\varphi) = \mathscr{S}_n^\alpha(U_n^0)(\theta,\varphi) = P_n^\alpha(\cos\varphi),$$
$$(U_n^m)^\alpha(\theta,\varphi) = \mathscr{S}_n^\alpha(U_n^m)(\theta,\varphi) = (P_n^m)^\alpha(\cos\varphi)\cos(m\theta),$$
$$(V_n^m)^\alpha(\theta,\varphi) = \mathscr{S}_n^\alpha(V_n^m)(\theta,\varphi) = (P_n^m)^\alpha(\cos\varphi)\sin(m\theta),$$

where $(P_n^m)^\alpha(\cos\varphi) = \mathscr{V}^\alpha(P_n^m)(\cos\varphi)$ and $\mathscr{V}^\alpha$ is the operator defined in Section 2. We consider here the interval $I=[-1,1]$ and any partition Δ of I in order to define the fractal analogues. By linearity we can extend $\mathscr{S}_n^\alpha$ to the rest of $\mathscr{H}_n$ in obvious way. Let us denote

$$\mathscr{H}_n^\alpha = \mathscr{S}_n^\alpha(\mathscr{H}_n).$$

$\mathscr{H}_n^\alpha$ is spanned by $\{(U_n^0)^\alpha,(U_n^m)^\alpha,(V_n^m)^\alpha; m=1,2,\dots,n\}$. These fractal elements are mutually orthogonal as well. For instance,

$$((U_n^m)^\alpha,(U_n^j)^\alpha)_{\mathscr{L}^2(S)}=\int_0^{2\pi}\int_0^{\pi}(P_n^m)^\alpha(\cos\varphi)(P_n^j)^\alpha(\cos\varphi)\cos(m\theta)\cos(j\theta)\sin\varphi d\varphi d\theta=0$$

if $m\neq j$, due to the orthogonality of $\cos(m\theta),\cos(j\theta)$. Using arguments similar to those of Lemma 3.1,

$$\|(U_n^m)^\alpha\|_{\mathscr{L}^2(S)} = \sqrt{\pi}\|(P_n^m)^\alpha\|_{\mathscr{L}^2} \leqslant \sqrt{\pi}\|\mathscr{V}^\alpha\|_2\|P_n^m\|_{\mathscr{L}^2}.$$

Using the same Lemma

$$\|(U_n^m)^\alpha\|_{\mathscr{L}^2(S)} \leqslant \|\mathscr{V}^\alpha\|_2\|U_n^m\|_{\mathscr{L}^2(S)}. \tag{36}$$

In the same way,

$$\|(V_n^m)^\alpha\|_{\mathscr{L}^2(S)} \leqslant \|\mathscr{V}^\alpha\|_2\|V_n^m\|_{\mathscr{L}^2(S)}. \tag{37}$$

For an arbitrary element Y_n of $\mathscr{H}_n$

$$Y_n = a_{n0}U_n^0 + \sum_{m=1}^n (a_{nm}U_n^m + b_{nm}V_n^m),$$
$$\mathscr{S}_n^\alpha(Y_n) = a_{n0}(U_n^0)^\alpha + \sum_{m=1}^n (a_{nm}(U_n^m)^\alpha + b_{nm}(V_n^m)^\alpha).$$

The orthogonality of $(U_n^m)^\alpha,(V_n^m)^\alpha$ implies that

$$\|\mathscr{S}_n^\alpha(Y_n)\|^2_{\mathscr{L}^2(S)} = \|a_{n0}(U_n^0)^\alpha\|^2_{\mathscr{L}^2(S)} + \sum_{m=1}^n (\|a_{nm}(U_n^m)^\alpha\|^2_{\mathscr{L}^2(S)} + \|b_{nm}(V_n^m)^\alpha\|^2_{\mathscr{L}^2(S)})$$

applying (36) and (37)

$$\|\mathscr{S}_n^\alpha(Y_n)\|^2_{\mathscr{L}^2(S)} \leqslant \|\mathscr{V}^\alpha\|_2^2(\|a_{n0}U_n^0\|^2_{\mathscr{L}^2(S)} + \sum_{m=1}^{n}(\|a_{nm}U_n^m\|^2_{\mathscr{L}^2(S)} + \|b_{nm}V_n^m\|^2_{\mathscr{L}^2(S)})),$$
$$\|\mathscr{S}_n^\alpha(Y_n)\|^2_{\mathscr{L}^2(S)} \leqslant \|\mathscr{V}^\alpha\|_2^2\|Y_n\|^2_{\mathscr{L}^2(S)},$$

(due to the orthogonality of the classical basis). As a consequence $\mathscr{S}_n^\alpha$ is bounded and

$$\|\mathscr{S}_n^\alpha\|_2 \leqslant \|\mathscr{V}^\alpha\|_2.$$

Additionally, since the system $\{(U_n^0)^\alpha, (U_n^m)^\alpha, (V_n^m)^\alpha; m = 1,2,\dots,n\}$ is orthogonal and thus linearly independent, the operator $\mathscr{S}_n^\alpha$ is injective.

PROPOSITION 3.3. *$\mathscr{H}_n^\alpha = \mathscr{S}_n^\alpha(\mathscr{H}_n)$ is a reproducing kernel space.*

Proof. Let us consider an orthonormal basis of $\mathscr{H}_n^\alpha$ composed by α-fractal functions $\{X_{nk}^\alpha; j = 0,1,\dots,2n\}$. Let us define $H^\alpha : S\times S \to R$ such that $\forall (P,Q) \in S\times S$

$$H^\alpha(P,Q) = \sum_{k=0}^{2n} X_{nk}^\alpha(P)X_{nk}^\alpha(Q)$$

(here H^α is only a notation and not a linearly transformed of H). It is evident that

$$X_{nk}^\alpha(P) = \int_S H^\alpha(P,Q)X_{nk}^\alpha(Q)dQ$$

consequently H^α reproduces every element of $\mathscr{H}_n^\alpha$, that is to say, $\forall f \in \mathscr{H}_n^\alpha$,

$$f(P) = \sum_{k=0}^{2n} c_{nk}X_{nk}^\alpha(P) = \sum_{k=0}^{2n} c_{nk}\int_S H^\alpha(P,Q)X_{nk}^\alpha(Q)dQ = \int_S H^\alpha(P,Q)f(Q)dQ.$$

COROLLARY 3.4. *The point evaluation functional $F_P(f) = f(P)$, for $P\in S$, is linear and bounded in $\mathscr{H}_n^\alpha$. Its Riesz representer is $H^\alpha(P,\cdot)$ and*

$$\|F_P\|_2 = \|H^\alpha(P,\cdot)\|_{\mathscr{L}^2(S)}. \tag{38}$$

Proof. According to the previous proposition, $\forall f \in \mathscr{H}_n^\alpha$,

$$F_P(f) = f(P) = (f, H^\alpha(P,\cdot))_{\mathscr{L}^2(S)}. \tag{39}$$

The boundedness of F_P follows from the application of the Cauchy-Schwarz inequality to the inner product in (39). The Riesz representer of F_P is $H^\alpha(P,\cdot)$ and the equality (38) follows from the theory of Hilbert spaces.

LEMMA 3.5. *([8]) Let M be a dense set in a Banach space E. Then, each element $y \in E$, $y \neq 0$, can be expressed as*

$$y = \sum_{k=1}^{\infty} y_k \tag{40}$$

so that $y_k \in M$ and

$$\|y_k\| \leqslant \frac{3\|y\|}{2^k}.$$

THEOREM 3.6. *There exists an operator $\mathscr{S}^\alpha : \mathscr{L}^2(S) \to \mathscr{L}^2(S)$ linear and bounded such that $\mathscr{S}^\alpha$ restricted to $\mathscr{H}_n$ is $\mathscr{S}_n^\alpha$. Moreover,*

$$\|\mathscr{S}^\alpha\|_2 \leqslant 3\|\mathscr{V}^\alpha\|_2. \tag{41}$$

Proof. The spherical harmonics are dense in $\mathscr{C}(S)$ with respect to the uniform norm ([7], [12]). Since S is compact, the set is dense with respect to the $\mathscr{L}^2$-norm. At the same time, the continuous functions are dense in $\mathscr{L}^2(S)$ (consider, for instance, the n-th sums of the Laplace series to approximate an element of $\mathscr{L}^2(S)$). Let us apply the previous Lemma for $M = \mathscr{H}$, set of spherical harmonics of any order, and $E = \mathscr{L}^2(S)$. Any $f \in \mathscr{L}^2(S)$, $f \neq 0$, can be expressed as

$$f = \sum_{k=1}^{\infty} Y_k \tag{42}$$

so that $Y_k \in \mathscr{H}_{n_k}$ and

$$\|Y_k\|_{\mathscr{L}^2(S)} \leqslant \frac{3\|f\|_{\mathscr{L}^2(S)}}{2^k}.$$

Let us define

$$\mathscr{S}^\alpha(f) = \sum_{k=1}^{\infty} \mathscr{S}_{n_k}^\alpha(Y_k) \tag{43}$$

where $\mathscr{S}_{n_k}^\alpha$ is the operator defined in Proposition 3.2 such that

$$\|\mathscr{S}_{n_k}^\alpha\|_2 \leqslant \|\mathscr{V}^\alpha\|_2.$$

The expansion of $f \in \mathscr{L}^2(S)$ with respect to an orthonormal basis of sherical harmonics is unique and thus, the sum (42) is unique. Therefore $\mathscr{S}^\alpha$ does not depend on the choice of Y_k. Let us see that the sum is convergent,

$$\|\mathscr{S}_{n_k}^\alpha(Y_k)\|_{\mathscr{L}^2(S)} \leqslant \|\mathscr{V}^\alpha\|_2 \|Y_k\|_{\mathscr{L}^2(S)} \leqslant \|\mathscr{V}^\alpha\|_2 \frac{3\|f\|_{\mathscr{L}^2(S)}}{2^k}.$$

Since the series on the right hand side is convergent, the sum

$$\sum_{k=1}^{\infty} \mathscr{S}_{n_k}^\alpha(Y_k)$$

is absolutely convergent. In a Banach space the absolute convergence implies convergence ([5]). As a consequence $\mathscr{S}^\alpha$ is well defined.

Moreover,

$$\|\mathscr{S}^\alpha(f)\|_{\mathscr{L}^2(S)} \leqslant \sum_{k=1}^{\infty} \|\mathscr{S}^\alpha_{n_k}(Y_k)\|_{\mathscr{L}^2(S)} \leqslant \|\mathscr{V}^\alpha\|_2 \sum_{k=1}^{\infty} \frac{3\|f\|_{\mathscr{L}^2(S)}}{2^k}$$

and $\forall f \in \mathscr{L}^2(S)$,

$$\|\mathscr{S}^\alpha(f)\|_{\mathscr{L}^2(S)} \leqslant 3\|\mathscr{V}^\alpha\|_2 \|f\|_{\mathscr{L}^2(S)}.$$

As a consequence,

$$\|\mathscr{S}^\alpha\|_2 \leqslant 3\|\mathscr{V}^\alpha\|_2.$$

If $Y \in \mathscr{H}_n$, the sum (42) consists of a single term and using (43),

$$\mathscr{S}^\alpha(Y) = \mathscr{S}^\alpha_n(Y),$$

therefore the restriction of $\mathscr{S}^\alpha$ to $\mathscr{H}_n$ is $\mathscr{S}^\alpha_n$.

An immediate consequence of this theorem is

COROLLARY 3.7. *If*

$$f = \sum_{n=0}^{\infty} \sum_{k=0}^{2n} c_{nk} Y_{nk}$$

in $\mathscr{L}^2$-sense, then

$$\mathscr{S}^\alpha(f) = \sum_{n=0}^{\infty} \sum_{k=0}^{2n} c_{nk} \mathscr{S}^\alpha(Y_{nk})$$

(in the same sense).

Proof. The result follows from the linearity and continuity of $\mathscr{S}^\alpha$.

Let us define the function on the sphere $V : S \to R$ given in sherical coordinates as $V(\theta, \varphi) = v(\cos\varphi)$, where $v(t)$ is the continuous mapping, $v : I \to R$, described in former sections. It is clear that $\forall f \in \mathscr{L}^2(S)$,

$$\|Vf\|_{\mathscr{L}^2(S)} \leqslant \|V\|_{\mathscr{C}(S)} \|f\|_{\mathscr{L}^2(S)} = \|v\|_\infty \|f\|_{\mathscr{L}^2(S)}. \tag{44}$$

LEMMA 3.8.

$$\|\mathscr{S}^\alpha(U_n^m) - U_n^m\|_{\mathscr{L}^2(S)} \leqslant |\alpha|_\infty \|\mathscr{S}^\alpha(U_n^m) - VU_n^m\|_{\mathscr{L}^2(S)},$$

$$\|\mathscr{S}^\alpha(V_n^m) - V_n^m\|_{\mathscr{L}^2(S)} \leqslant |\alpha|_\infty \|\mathscr{S}^\alpha(V_n^m) - VV_n^m\|_{\mathscr{L}^2(S)}.$$

Proof. The theorem 3.6 implies that, for instance,

$$\mathscr{S}^\alpha(U_n^m)(\theta, \varphi) = (U_n^m)^\alpha(\theta, \varphi) = (P_n^m)^\alpha(\cos\varphi)\cos(m\theta)$$

then

$$\begin{aligned}\|\mathscr{S}^{\alpha}(U_n^m)-U_n^m\|^2_{\mathscr{L}^2(S)} &= \int_0^{2\pi}\int_0^{\pi}|((P_n^m)^{\alpha}-P_n^m)(\cos\varphi)|^2\cos^2 m\theta\sin\varphi d\varphi d\theta\\ &= \pi\int_{-1}^{1}|((P_n^m)^{\alpha}-P_n^m)(t)|^2dt\\ &= \pi\|(P_n^m)^{\alpha}-P_n^m\|^2_{\mathscr{L}^2}\\ &\leqslant \pi|\alpha|^2_{\infty}\|(P_n^m)^{\alpha}-vP_n^m\|^2_{\mathscr{L}^2}\end{aligned}$$

(using (20)). On the other hand,

$$\begin{aligned}\|\mathscr{S}^{\alpha}(U_n^m)-VU_n^m\|^2_{\mathscr{L}^2(S)} &= \int_0^{2\pi}\int_0^{\pi}|((P_n^m)^{\alpha}-vP_n^m)(\cos\varphi)|^2\cos^2 m\theta\sin\varphi d\varphi d\theta\\ &= \pi\int_{-1}^{1}|((P_n^m)^{\alpha}-vP_n^m)(t)|^2dt = \pi\|(P_n^m)^{\alpha}-vP_n^m\|^2_{\mathscr{L}^2}.\end{aligned}$$

These expressions imply that

$$\|\mathscr{S}^{\alpha}(U_n^m)-U_n^m\|^2_{\mathscr{L}^2(S)} \leqslant |\alpha|^2_{\infty}\|\mathscr{S}^{\alpha}(U_n^m)-VU_n^m\|^2_{\mathscr{L}^2(S)}.$$

PROPOSITION 3.9. $\forall f\in\mathscr{L}^2(S)$,

$$\|\mathscr{S}^{\alpha}(f)-f\|_{\mathscr{L}^2(S)} \leqslant |\alpha|_{\infty}\|\mathscr{S}^{\alpha}(f)-Vf\|_{\mathscr{L}^2(S)}$$

and

$$\|\mathscr{S}^{\alpha}(f)-f\|_{\mathscr{L}^2(S)} \leqslant \frac{|\alpha|_{\infty}}{1-|\alpha|_{\infty}}(1+\|v\|_{\infty})\|f\|_{\mathscr{L}^2(S)}. \tag{45}$$

Proof. Let us begin proving the inequality for a spherical harmonic $Y\in\mathscr{H}_n$. If $\{Y_{nk};k=0,\dots,2n\}$ is the basis of spherical harmonics of $\mathscr{H}_n$,

$$Y=\sum_{k=0}^{2n}c_{nk}Y_{nk}$$

The orthogonality of $\{(Y_{nk}^{\alpha}-Y_{nk})\}_{k=0}^{2n}$ implies that

$$\|\mathscr{S}^{\alpha}(Y)-Y\|^2_{\mathscr{L}^2(S)}=\sum_{k=0}^{2n}|c_{nk}|^2\|Y_{nk}^{\alpha}-Y_{nk}\|^2_{\mathscr{L}^2(S)} \leqslant |\alpha|^2_{\infty}\sum_{k=0}^{2n}|c_{nk}|^2\|Y_{nk}^{\alpha}-VY_{nk}\|^2_{\mathscr{L}^2(S)}$$

(applying Lemma 3.8). The orthogonality of the elements $\cos k\theta,\cos r\theta$ or $\cos k\theta,\sin r\theta$ implies that of $Y_{nk}^{\alpha}-VY_{nk},Y_{nr}^{\alpha}-VY_{nr}$ for $k\neq r$. In this case,

$$\|\mathscr{S}^{\alpha}(Y)-Y\|^2_{\mathscr{L}^2(S)} \leqslant |\alpha|^2_{\infty}\|\mathscr{S}^{\alpha}(Y)-VY\|^2_{\mathscr{L}^2(S)}. \tag{46}$$

For a general $f\in\mathscr{L}^2(S)$, let us consider a sequence Y_m of spherical harmonics such that $\lim Y_m=f$ in the $\mathscr{L}^2$-norm (see the beginning of the proof of Theorem 3.6) and thus $\lim\|Y_m-f\|_{\mathscr{L}^2(S)}=0$. Then $\lim VY_m=Vf$.

The continuity of $\mathscr{S}^\alpha$ and that of the norm imply that

$$\|\mathscr{S}^\alpha(f)-f\|^2_{\mathscr{L}^2(S)} = \lim \|\mathscr{S}^\alpha(Y_m)-Y_m\|^2_{\mathscr{L}^2(S)}.$$

Applying this equality and (46) for $Y = Y_m$,

$$\|\mathscr{S}^\alpha(f)-f\|^2_{\mathscr{L}^2(S)} = \lim \|\mathscr{S}^\alpha(Y_m)-Y_m\|^2_{\mathscr{L}^2(S)} \leqslant |\alpha|^2_\infty \lim \|\mathscr{S}^\alpha(Y_m)-VY_m\|^2_{\mathscr{L}^2(S)}$$

and we obtain the first inequality of the Proposition:

$$\|\mathscr{S}^\alpha(f)-f\|_{\mathscr{L}^2(S)} \leqslant |\alpha|_\infty \|\mathscr{S}^\alpha(f)-Vf\|_{\mathscr{L}^2(S)}.$$

Moreover,

$$\begin{aligned}\|\mathscr{S}^\alpha(f)-f\|_{\mathscr{L}^2(S)} &\leqslant |\alpha|_\infty \|\mathscr{S}^\alpha(f)-Vf\|_{\mathscr{L}^2(S)} \\ &\leqslant |\alpha|_\infty (\|\mathscr{S}^\alpha(f)-f\|_{\mathscr{L}^2(S)} + \|f-Vf\|_{\mathscr{L}^2(S)}),\end{aligned}$$

therefore, using (44),

$$\|\mathscr{S}^\alpha(f)-f\|_{\mathscr{L}^2(S)} \leqslant |\alpha|_\infty \|\mathscr{S}^\alpha(f)-f\|_{\mathscr{L}^2(S)} + |\alpha|_\infty (1+\|v\|_\infty)\|f\|_{\mathscr{L}^2(S)},$$

and the second inequality is obtained.

THEOREM 3.10. *If $|\alpha|_\infty < \|v\|_\infty^{-1}$, the range of $\mathscr{S}^\alpha$ is closed.*

Proof. By hypothesis

$$|\alpha|_\infty < \|v\|_\infty^{-1}$$

and

$$1-|\alpha|_\infty \|v\|_\infty > 0. \tag{47}$$

According to the first result of the previous Proposition, $\forall f \in \mathscr{L}^2(S)$,

$$\|f\|_{\mathscr{L}^2(S)} - \|\mathscr{S}^\alpha(f)\|_{\mathscr{L}^2(S)} \leqslant |\alpha_\infty| \|\mathscr{S}^\alpha(f)-Vf\|_{\mathscr{L}^2(S)} \tag{48}$$

moreover (44),

$$\|\mathscr{S}^\alpha(f)-Vf\|_{\mathscr{L}^2(S)} \leqslant \|\mathscr{S}^\alpha(f)\|_{\mathscr{L}^2(S)} + \|v\|_\infty \|f\|_{\mathscr{L}^2(S)}. \tag{49}$$

From (48), and applying (49),

$$\|\mathscr{S}^\alpha(f)\|_{\mathscr{L}^2(S)} \geqslant \|f\|_{\mathscr{L}^2(S)} - |\alpha|_\infty \|\mathscr{S}^\alpha(f)-Vf\|_{\mathscr{L}^2(S)},$$

$$\|\mathscr{S}^\alpha(f)\|_{\mathscr{L}^2(S)} \geqslant \|f\|_{\mathscr{L}^2(S)} - |\alpha|_\infty \|\mathscr{S}^\alpha(f)\|_{\mathscr{L}^2(S)} - |\alpha|_\infty \|v\|_\infty \|f\|_{\mathscr{L}^2(S)},$$

and

$$\frac{(1-|\alpha|_\infty \|v\|_\infty)}{1+|\alpha|_\infty} \|f\|_{\mathscr{L}^2(S)} \leqslant \|\mathscr{S}^\alpha(f)\|_{\mathscr{L}^2(S)}. \tag{50}$$

In order to prove the closure of the range of $\mathcal{S}^\alpha$ we consider a sequence $\mathcal{S}^\alpha(f_n)$ convergent, $\mathcal{S}^\alpha(f_n) \to g$. The original $\{f_n\} \subset \mathcal{L}^2(S)$ is Cauchy because, due to the linearity of $\mathcal{S}^\alpha$, (47) and (50),

$$\|f_n - f_m\|_{\mathcal{L}^2(S)} \leqslant \frac{1+|\alpha|_\infty}{(1-|\alpha|_\infty\|v\|_\infty)} \|\mathcal{S}^\alpha(f_n) - \mathcal{S}^\alpha(f_m)\|_{\mathcal{L}^2(S)} \tag{51}$$

thus, the Cauchy property of $\mathcal{S}^\alpha(f_n)$ is inherited by f_n. As a consequence f_n is convergent. Let us consider, $f = \lim f_n$. The continuity of $\mathcal{S}^\alpha$ provides $\mathcal{S}^\alpha(f) = \lim \mathcal{S}^\alpha(f_n) = g$ and g belongs to the range of $\mathcal{S}^\alpha$.

COROLLARY 3.11. *If* $|\alpha|_\infty < \|v\|_\infty^{-1}$,

$$\mathcal{L}^2(S) = \operatorname{rg}\mathcal{S}^\alpha \bigoplus \ker(\mathcal{S}^\alpha)^*, \tag{52}$$

where $(\mathcal{S}^\alpha)^*$ *is the adjoint operator of* $\mathcal{S}^\alpha$.

Proof. For a bounded and linear operator of a Hilbert space, the following orthogonal decomposition is satisfied,

$$\mathcal{L}^2(S) = \overline{\operatorname{rg}\mathcal{S}^\alpha} \bigoplus \ker(\mathcal{S}^\alpha)^*. \tag{53}$$

In this case the range of the operator is closed and the result is obtained.

COROLLARY 3.12. *If* $|\alpha|_\infty < \|v\|_\infty^{-1}$, *there exists a solution in* f *for the equation* $\mathcal{S}^\alpha(f) = \widetilde{f}$ *if and only if* $\widetilde{f}$ *is orthogonal to* $\ker(\mathcal{S}^\alpha)^*$.

Proof. It is a trivial consequence of the previous corollary.

PROPOSITION 3.13. *If* $|\alpha|_\infty < \|v\|_\infty^{-1}$, $\mathcal{S}^\alpha$ *is injective.*

Proof. It is a consequence of the inequality (50), taking $\mathcal{S}^\alpha(f) = 0$.

PROPOSITION 3.14. *If* $|\alpha|_\infty < \|v\|_\infty^{-1}$, $\operatorname{rg}(\mathcal{S}^\alpha)^*$ *is dense in* $\mathcal{L}^2(S)$.

Proof. In general,

$$\mathcal{L}^2(S) = \overline{\operatorname{rg}(\mathcal{S}^\alpha)^*} \bigoplus \ker(\mathcal{S}^\alpha). \tag{54}$$

In this case $\mathcal{S}^\alpha$ is injective, $\ker(\mathcal{S}^\alpha) = \{0\}$ and $\operatorname{rg}(\mathcal{S}^\alpha)^*$ is dense in $\mathcal{L}^2(S)$.

THEOREM 3.15. *If* $|\alpha|_\infty < \frac{1}{2+\|v\|_\infty}$, $\mathcal{S}^\alpha : \mathcal{L}^2(S) \to \mathcal{L}^2(S)$ *has a bounded inverse.*

Proof. According to the inequality (45) and the hypothesis given

$$\|I-\mathscr{S}^{\alpha}\|_2 \leqslant \frac{(1+\|v\|_\infty)|\alpha|_\infty}{1-|\alpha|_\infty} < 1. \tag{55}$$

Lemma 2.5 ensures that $\mathscr{S}^{\alpha} = I-(I-\mathscr{S}^{\alpha})$ has a bounded inverse. Using (50),

$$\|(\mathscr{S}^{\alpha})^{-1}\|_2 \leqslant \frac{1+|\alpha|_\infty}{1-|\alpha|_\infty\|v\|_\infty}.$$

PROPOSITION 3.16. *If* $|\alpha|_\infty < \frac{1}{2+\|v\|_\infty}$, $(\mathscr{S}^{\alpha})^*$ *is injective.*

Proof. Theorem 3.15 states that $\mathrm{rg}\,\mathscr{S}^{\alpha} = \mathscr{L}^2(S)$. Since

$$|\alpha|_\infty < \frac{1}{2+\|v\|_\infty} < \frac{1}{\|v\|_\infty}$$

according to Corollary 3.11, $\ker(\mathscr{S}^{\alpha})^* = \{0\}$ and $(\mathscr{S}^{\alpha})^*$ is injective.

DEFINITION 3.17. An operator T is Fredholm if:

- $\mathrm{rg}\,T$ is closed.
- $\ker T$ and $\ker T^*$ are finite-dimensional.

PROPOSITION 3.18. *If* $|\alpha|_\infty < \frac{1}{2+\|v\|_\infty}$, $(\mathscr{S}^{\alpha})^*$ *is Fredholm and its index is 0.*

Proof. In this case, $\mathrm{rg}\,\mathscr{S}^{\alpha}$ is closed (Theorem 3.10) and $\mathscr{S}^{\alpha}$, $(\mathscr{S}^{\alpha})^*$ are injective (Propositions 3.13 and 3.16 respectively). As a consequence, $\mathscr{S}^{\alpha}$ is Fredholm. The index of a Fredholm operator is defined as:

$$\mathrm{ind}\,\mathscr{S}^{\alpha} = \dim(\ker \mathscr{S}^{\alpha}) - \dim(\ker(\mathscr{S}^{\alpha})^*).$$

In this case the index is zero.

DEFINITION 3.19. A sequence $\{x_n\} \subset E$ is closed or fundamental if every element of E can be approximated arbitrarily closely by finite linear combinations of elements of $\{x_n\}$.

THEOREM 3.20. *If* $|\alpha|_\infty < \frac{1}{2+\|v\|_\infty}$, *the system*

$$T = \{(U_n^0)^{\alpha}, (U_n^m)^{\alpha}, (V_n^m)^{\alpha}; n = 0,1,\ldots, m = 1,2,\ldots,n\} \tag{56}$$

is fundamental in $\mathscr{L}^2(S)$.

Proof. According to Theorem 3.15 for any $g \in \mathscr{L}^2(S)$, there exists $f \in \mathscr{L}^2(S)$ such that $\mathscr{S}^\alpha(f) = g$. In that case, if the Laplace series of f is

$$f = \sum_{n=0}^{\infty} \sum_{k=0}^{2n} c_{nk} Y_{nk} \tag{57}$$

the linearity and continuity of $\mathscr{S}^\alpha$ imply that

$$g = \mathscr{S}^\alpha(f) = \sum_{n=0}^{\infty} \sum_{k=0}^{2n} c_{nk} Y_{nk}^\alpha \tag{58}$$

and g can be arbitrarily approached by finite linear combinations of elements of T.

THEOREM 3.21. *If* $|\alpha|_\infty < \frac{1}{2+\|v\|_\infty}$, *the system* T *is complete in* $\mathscr{L}^2(S)$.

Proof. By Banach's theorem ([4]), in a normed linear space a system is fundamental if and only if it is complete.

COROLLARY 3.22. *If* $|\alpha|_\infty < \frac{1}{2+\|v\|_\infty}$, $\mathscr{L}^2(S)$ *owns a Hilbert basis of* α*-fractal spherical harmonics.*

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M.A. Navascués
Dpto. de Matemática Aplicada
Centro Politécnico Superior de Ingenieros
Universidad de Zaragoza
C/ María de Luna 3
50018 Zaragoza
Spain
e-mail: manavas@unizar.es