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SPATIAL DECAY AND BLOW–UP FOR SOLUTIONS TO SOME PARABOLIC EQUATIONS IN THE HALF CYLINDER

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Abstract. In this paper,the spatial behaviors of a nonlinear and a quasi-linear parabolic equations with nonlinear boundary conditions are studied on a half cylinder.Under suitable conditions, we get a various, but closely related forms of Phragmén-Lindelöf principle, and we have proved the smooth solution either fails to exist globally, or when it does exist globally,it must tend asymptotically to zero with increasing long distance along the cylinder from the base.

1. Introduction

Saint-Venant in [25] formulated and conjectured a famous mathematical and mechanical principle which came to be known in subsequent literatures as $Saint-Venant's$ principle and led to an extensive investigation in the framework of applied mathematics. A review of recent work on Saint-Venant's principle is given in the work of Horgan and Knowles [11], and has been periodically updated by Horgan [9], [10]. Early work on Saint-Venant's principle primarily focused on the spatial behavior of elliptic equations, and one could refer to Flavin and Knops [7], Horgan and Payne [12], [13].

In recent years, a number of papers have dealt with the spatial decay of solutions of linear and nonlinear parabolic initial-boundary problems defined in a semi-infinite strips or cylinders. This work began perhaps with Boley [3], [4], who investigated the spatial decay of heat equation. Other subsequent contributions mainly on the spatial behaviors of parabolic equations. These papers include Bofill and Quintanilla [5], Horgan and Payne et. al. [14], Lin and Payne [15], [16], Liu and Lin [17], Payne and Phillipin [18], Payne and Schaefer et. al. [19] and Quintanilla [23]. Most of this work has dealt with the solutions of parabolic problems in a semi-infinite cylinder with homogeneous initial data and homogeneous Dirichlet data on the lateral surface of the cylinder for positive time.

It is known to us all that, the "blow-up", or "non-existence" of the solution is an important aspect in the study of partial differential equation, which had been extensively studied in the literature. The contributions concerning this question were explained in the work of Ames [1], the books of Ames and Straughan [2], Evans [6], Flavin and Rionero [8], and the book of Straughan [26]. Recently, papers deal with the blow-up

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phenomenon is abound in literature. For example, Payne and Song [22], Payne and Schaefer [20], [21], and Quintanilla [24], but most of them are concerned with the time variable. Little attention has been given to the spatial blow-up of solutions in general.

In the present paper, the spatial behaviors of a nonlinear and a quasi-linear parabolic equations with nonlinear boundary conditions are studied on a half cylinder. Under suitable conditions, we get a various, but closely related forms of Phragmén-Lindelöf principle, and we have proved the smooth solution either fails to exist globally, or when it does exist globally, it must tend asymptotically to zero with increasing long distance along the cylinder from the base.

Let R be the cylinder $(0,\infty)\times D$, where D is a two dimensional bounded domain such that the boundary ∂D is smooth enough to apply the divergence theorem. Let $D(z)$ denote the cross section of those points in R such that $x_1 = z$, $R(z)$ denotes the points of R such that $x_1 > z$ and $\Sigma(z)$ denotes $(z,\infty)\times\partial D$. The equations we study here are determined on the semi-infinite cylinder R.

The plan of the paper is the following: Section 2 is devoted to the study of the nonlinear parabolic equation. We give sufficient conditions to assess that the solutions either don't exist for all spatial values or else they decay algebraically. In Section 3, we give sufficient conditions to obtain a similar result for the quasi-linear parabolic equation.

In this article, the usual summation convention is employed with repeated Latin subscripts summed from 1 to 3, and repeated Greek subscripts from 2 to 3. The comma is used to indicate partial differentiation, and the differentiation with respect to the direction x_k is denoted by, k.

2. Nonlinear parabolic equation

The first problem we consider is determined by the equation:

$$s(x,u)u_{,t} = (\rho(x,u,q^2)u_{,i})_{,i} - E(u), q^2 = |\nabla u|^2, \tag{2.1}$$

and the boundary condition

$$\rho\frac{\partial u}{\partial n} + f(u) = 0, \partial D\times(0,\infty), \tag{2.2}$$

where n_α are the components of the unit outward normal on ∂D, and f(u) is assumed to satisfy

$$f(u) \geqslant 0. \tag{2.3}$$

It is worth noting that the homogeneous Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ is included in condition (2.2), because it corresponds to the case f(u)=0. To have the problem determined, we need to impose boundary conditions on the finite end of the cylinder. However, we don't mention the explicit boundary condition on this part of the boundary because this is not relevant in our analysis. The solution u satisfies the initial condition:

$$u(x,0) = 0 \quad x\in R. \tag{2.4}$$

For simplicity, we deal with the function ρ that verifies the following assumption: There is a positive constant ρ_M such that

$$0 \leqslant \rho \leqslant \rho_M. \tag{2.5}$$

And we consider the class for which

$$E(u) > 0, \tag{2.6}$$

and

$$\int_0^\infty E^{-\beta}(\eta)d\eta < M. \tag{2.7}$$

where M is a positive constant.
Simple example of such a nonlinearity is $E(u) = (1+u^2)^2$ satisfies the conditions (2.6) and (2.7), for which $\beta = \frac{1}{2}$.

Now, we assume there exists a function $S(x,u) \geqslant 0$ which satisfies the conditions:

$$s(x,u)g(u) = \frac{\partial S(x,u)}{\partial u}, \tag{2.8}$$

$$S(x,0) = 0. \tag{2.9}$$

where g(u) is a positive function which will be defined later.

In this section, we consider the function:

$$\Phi(z,t) = -\int_0^t \int_{D(z)} \exp(-\omega s)\rho g(u)u_{,1}dads, \tag{2.10}$$

where $\omega > 0$ is a constant.

Using the divergence theorem and the boundary conditions, we see that

$$\begin{aligned}\Phi(z,t) &= \Phi(z_0,t) - \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-\omega s)(\rho g(u)u_{,i})_{,i}dvds \\ &\quad - \int_0^t \int_{z_0}^z \int_{\partial D(z)} \exp(-\omega s)g(u)f(u)dads \qquad (2.11)\\ &= \Phi(z_0,t) - \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-\omega s)(\rho u_{,i})_{,i}g(u)dvds \\ &\quad - \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-\omega s)\rho u_{,i}g'(u)u_{,i}dvds \\ &\quad - \int_0^t \int_{z_0}^z \int_{\partial D(z)} \exp(-\omega s)g(u)f(u)dads. \qquad (2.12)\end{aligned}$$

In view of (2.1), we obtain

$$\begin{aligned}\Phi(z,t) &= \Phi(z_0,t) - \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-\omega s)g(u)u_{,s}s(x,u)dvds \\ &\quad - \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-\omega s)g(u)E(u)dvds - \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-\omega s)\rho g'(u)q^2dvds \\ &\quad - \int_0^t \int_{z_0}^z \int_{\partial D(z)} \exp(-\omega s)g(u)f(u)dads. \qquad (2.13)\end{aligned}$$

Making use of (2.8) and (2.9), we get

$$\begin{aligned}\Phi(z,t) = &\ \Phi(z_0,t) - \omega \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-\omega s) S(x,u) dv ds \\ &- \int_{z_0}^z \int_{D(z)} \exp(-\omega s) S(x,u) dv|_{s=t} - \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-\omega s) \rho g'(u) q^2 dv ds \\ &- \int_0^t \int_{z_0}^z \int_{\partial D(z)} \exp(-\omega s) g(u) f(u) da ds \\ &- \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-\omega s) g(u) E(u) dv ds, \end{aligned} \tag{2.14}$$

for every $z \geqslant z_0$. In case that $\Phi(z,t) \to 0$ as $z \to \infty$, from (2.14) we have

$$\begin{aligned}\Phi(z,t) = &\ \omega \int_0^t \int_{R(z)} \exp(-\omega s) S(x,u) dv ds \\ &+ \int_{R(z)} \exp(-\omega s) S(x,u) dv|_{s=t} + \int_0^t \int_{R(z)} \exp(-\omega s) \rho g'(u) q^2 dv ds \\ &+ \int_0^t \int_{\Sigma(z)} \exp(-\omega s) g(u) f(u) da ds + \int_0^t \int_{R(z)} \exp(-\omega s) g(u) E(u) dv ds. \end{aligned} \tag{2.15}$$

From (2.14), we see that

$$\begin{aligned}\frac{\partial \Phi(z,t)}{\partial z} = &-\omega \int_0^t \int_{D(z)} \exp(-\omega s) S(x,u) da ds \\ &- \int_{D(z)} \exp(-\omega s) S(x,u) da|_{s=t} - \int_0^t \int_{D(z)} \exp(-\omega s) \rho g'(u) q^2 da ds \\ &- \int_0^t \int_{\partial D(z)} \exp(-\omega s) g(u) f(u) dl ds - \int_0^t \int_{D(z)} \exp(-\omega s) g(u) E(u) da ds. \end{aligned} \tag{2.16}$$

Now, we define

$$g(u) = \left[\int_u^\infty E^{-\beta}(\eta) d\eta \right]^{\frac{-1}{1-\beta}}, \tag{2.17}$$

where β is a constant such that $0 < \beta < 1$. Thus we have immediately that

$$g'(u) = g^{2-\beta}(u) E^{-\beta}(u)/(1-\beta) \geqslant 0, \tag{2.18}$$

and hence from (2.10), we have

$$\begin{aligned}|\Phi(z,t)| \leqslant \rho_M^{\frac{1}{2}} \Bigg[&\int_0^t \int_{D(z)} \exp(-\omega s) \rho u_{,i} u_{,i} E^{-\beta}(u) g^{2-\beta}(u) da ds \\ &\times \int_0^t \int_{D(z)} \exp(-\omega s) E^{\beta}(u) g^{\beta}(u) da ds \Bigg]^{\frac{1}{2}}. \end{aligned} \tag{2.19}$$

Applying the Schwarz and *Hölder* inequalities to (2.19), and using (2.18), we obtain

$$\begin{aligned} |\Phi(z,t)| &\leqslant k_1(t)(1-\beta)\left[\int_0^t\int_{D(z)}\exp(-\omega s)\rho u_{,i}u_{,i}g'(u)dads \right. \\ &\quad \left. \times\left(\int_0^t\int_{D(z)}\exp(-\omega s)E(u)g(u)dads\right)^{\beta}\right]^{\frac{1}{2}} \\ &= k_1(t)Q_1^{\frac{1}{2}}Q_2^{\frac{\beta}{2}}, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} k_1(t) &= \rho_M^{\frac{1}{2}}\left(\int_0^t\int_{D(z)}\exp(-\omega s)dads\right)^{\frac{1-\beta}{2}} \\ &= \rho_M^{\frac{1}{2}}\left(|D(z)|\left(\frac{1-\exp(-\omega t)}{w}\right)\right)^{\frac{1-\beta}{2}}, \end{aligned} \tag{2.21}$$

and $|D|$ denotes the measre of D.

$$Q_1 = (1-\beta)\int_0^t\int_{D(z)}\exp(-\omega s)\rho u_{,i}u_{,i}g'(u)dads,$$

$$Q_2 = \int_0^t\int_{D(z)}\exp(-\omega s)E(u)g(u)dads.$$

By virtue of Young's inequality in the form

$$a^q b^{1-q} \leqslant qa+(1-q)b, \quad 0<q<1,\ a>0,\ b>0. \tag{2.22}$$

We may conclude from (2.20) that

$$\begin{aligned} |\Phi(z,t)| &\leqslant k_1(t)\left[Q_1^{\frac{1}{1+\beta}}Q_2^{\frac{\beta}{1+\beta}}\right]^{\frac{1+\beta}{2}} \\ &\leqslant \frac{k_1(t)}{(1+\beta)^{\frac{1+\beta}{2}}}\left[\gamma_1^{-1}Q_1+\beta\gamma_1^{\frac{1}{\beta}}Q_2\right]^{\frac{1+\beta}{2}}, \end{aligned} \tag{2.23}$$

where γ_1 is an arbitrary positive constant which we select to be given by

$$\gamma_1 = \left[\frac{\beta}{1-\beta}\right]^{-\frac{\beta}{1+\beta}},$$

thus, we have

$$|\Phi(z,t)| \leqslant k_2(t)\left[-\frac{\partial\Phi}{\partial z}\right]^{\frac{1+\beta}{2}}, \tag{2.24}$$

where

$$k_2(t) = \frac{k_1(t)}{(1+\beta)^{\frac{1+\beta}{2}}}(1-\beta)^{\frac{1}{1+\beta}}\beta^{\frac{\beta}{1+\beta}}.$$

We should prove the following theorem basing on the inequality (2.24).

THEOREM 2.1. *Let u be a solution of the initial boundary value problem determined by the equation (2.1) where ρ satisfies (2.5) and E satisfies conditions (2.6), (2.7), and the initial boundary conditions (2.2), (2.3), (2.4) are also satisfied. Then, either the solution ceases to exist for a finite value of the spatial variable z, or the solution must tend asymptotically to zero as z tends to infinity.*

The proof of Theorem 2. 1 is based upon the following two propositions.

PROPOSITION 2.2. *If we assume that there exists a $z_0 \geqslant 0$, such that $\Phi(z_0,t) < 0$, then the solution ceases to exist for a finite value of z.*

Proof. We assume the contrary, and that a solution exists for all z, we then prove a contradiction.

When we assume that $\Phi(z_0,t) < 0$, we have $\Phi(z,t) < 0$ for all $z \geqslant z_0$.

Hence, from (2.24), we conclude that

$$-\frac{\partial\Phi(z,t)}{\partial z} \geqslant \left(-\frac{\Phi(z,t)}{k_2(t)}\right)^{\frac{2}{1+\beta}}, \tag{2.25}$$

which integrating yields

$$(-\Phi(z,t))^{\frac{-(1-\beta)}{1+\beta}} \leqslant (-\Phi(z_0,t))^{-\frac{1-\beta}{1+\beta}} - \frac{1-\beta}{1+\beta}k_2(t)^{-\frac{2}{1+\beta}}(z-z_0). \tag{2.26}$$

From (2.26), if $z \longrightarrow \infty$, then $-\Phi(z,t) < 0$, this is a contradiction to $\Phi(z,t) < 0$ for all $z \geqslant z_0$. So the solutions cease to exist for a finite value of z. Thus, we have proved proposition 2. 2. □

PROPOSITION 2.3. *On the other hand, if we assume that $\Phi(z,t) \geqslant 0$ for all z, then, the solution must tend asymptotically to zero as z tends to infinity.*

Proof. If we assume that $\Phi(z,t) \geqslant 0$ for all z, then from (2.24), we have

$$\Phi(z,t) \leqslant k_2(t)\left(-\frac{\partial\Phi(z,t)}{\partial z}\right)^{\frac{1+\beta}{2}}. \tag{2.27}$$

So, we have obtain the inequality

$$k_2(t)^{-\frac{2}{1+\beta}} \leqslant -(\Phi(z,t))^{-\frac{2}{1+\beta}}\left(\frac{\partial\Phi(z,t)}{\partial z}\right), \tag{2.28}$$

and integrating, we obtain the inequality

$$\Phi(z,t) \leqslant \left[\frac{1-\beta}{1+\beta}k_2(t)^{-\frac{2}{1+\beta}}(z-z_0)+\Phi(z_0,t)^{\frac{\beta-1}{1+\beta}}\right]^{-\frac{1+\beta}{1-\beta}}. \tag{2.29}$$

That gives a description of the spatial decay of solutions whenever they exist for all $z \geqslant 0$. Then we have proved Proposition (2.3). □

Combining Proposition 2. 2 and Proposition 2. 3, we have proved Theorem 2. 1. We will discuss the quasi-linear equation in a different method in the next section.

3. Quasi-linear parabolic equation

Now, we consider the quasi-linear parabolic equation

$$u_{,t} - (a_{ij}(x,u,|\nabla u|^2)u_{,i})_{,j} + g(u) = 0, \tag{3.1}$$

in $R \times (0,\infty)$ and satisfies the initial condition

$$u(x,0) = 0 \quad in \quad R, \tag{3.2}$$

and the boundary condition

$$a_{i\alpha}u_{,i}n_\alpha + f(u) = 0 \quad on \quad \partial D \times (0,\infty), \tag{3.3}$$

and

$$uf(u) \geqslant 0. \tag{3.4}$$

We also assume

$$ug(u) \geqslant \gamma|u|^{2\alpha}, \tag{3.5}$$

where γ and α are all constants with $\gamma > 0, \alpha > 1$. And a_{ij} are symmetric and satisfy the ellipticity conditions

$$a_0\xi_i\xi_i \leqslant a_{ij}\xi_i\xi_j \leqslant a_1\xi_i\xi_i, \tag{3.6}$$

where a_0 and a_1 are positive constants, for all $\xi \in R^3$.

In this section, we consider the function

$$H(z,t) = -\int_0^t \int_{D(z)} \exp(-ws)a_{i1}uu_{,i}dads, \tag{3.7}$$

where $w > 0$ is also a constant.

Using the divergence theorem and the boundary conditions, we see that

$$\begin{aligned}
H(z,t) &= H(z_0,t) - \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-ws)(a_{ij}uu_{,i})_{,j} dvds \\
&\quad + \int_0^t \int_{z_0}^z \int_{\partial D(z)} \exp(-ws) a_{i\alpha} uu_{,i} n_\alpha dads \\
&= H(z_0,t) - \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-ws)(u_{,s}+g)udvds \\
&\quad - \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-ws) a_{ij} u_{,j} u_{,i} dvds \\
&\quad - \int_0^t \int_{z_0}^z \int_{\partial D(z)} \exp(-ws) f(u) udads \\
&= H(z_0,t) - \frac{1}{2} \int_0^t \int_{z_0}^z \int_{D(z)} w\exp(-ws)u^2 dvds - \frac{1}{2} \int_{z_0}^z \int_{D(z)} w\exp(-ws)u^2 dv \\
&\quad - \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-ws) a_{ij} u_{,j} u_{,i} dvds - \int_0^t \int_{z_0}^z \int_{D(z)} \exp(-ws) gudvds \\
&\quad - \int_0^t \int_{z_0}^z \int_{\partial D(z)} \exp(-ws) f(u) udads. \qquad (3.8)
\end{aligned}$$

From (3.8), we get

$$\begin{aligned}
\frac{\partial H(z,t)}{\partial z} &= -\frac{1}{2} \int_0^t \int_{D(z)} w\exp(-ws)u^2 dads - \frac{1}{2} \int_{D(z)} w\exp(-ws)u^2 da \\
&\quad - \int_0^t \int_{D(z)} \exp(-ws) a_{ij} u_{,j} u_{,i} dads - \int_0^t \int_{D(z)} \exp(-ws) gudads \\
&\quad - \int_0^t \int_{\partial D(z)} \exp(-ws) f(u) udlds. \qquad (3.9)
\end{aligned}$$

Using (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned}
-\frac{\partial H(z,t)}{\partial z} &\geqslant \int_0^t \int_{D(z)} \exp(-ws) a_{ij} u_{,j} u_{,i} dads + \int_0^t \int_{D(z)} \exp(-ws) gudads \\
&\geqslant \int_0^t \int_{D(z)} \exp(-ws) \gamma |u|^{2\alpha} dads \\
&\quad + \int_0^t \int_{D(z)} \exp(-ws) a_0 |\nabla u|^2 dads. \qquad (3.10)
\end{aligned}$$

For every $z \geqslant z_0$, in case that $H(z,t) \longrightarrow 0$ as $z \longrightarrow \infty$, from (3.8), we have

$$\begin{aligned}
H(z,t) &= \frac{1}{2} \int_0^t \int_{R(z)} w\exp(-ws)u^2 dvds + \frac{1}{2} \int_{R(z)} w\exp(-ws)u^2 dv \\
&\quad + \int_0^t \int_{R(z)} \exp(-ws) a_{ij} u_{,j} u_{,i} dvds \\
&\quad + \int_0^t \int_{R(z)} \exp(-ws) gudvds + \int_0^t \int_{\Sigma(z)} \exp(-ws) f(u) udads. \qquad (3.11)
\end{aligned}$$

To obtain the basic inequality like (2.24), using Schwarz inequality, we can obtain

$$\begin{aligned}|H(z,t)| &= \left|\int_0^t \int_{D(z)} \exp(-ws) a_{i1} u u_{,i} dads\right| \\ &\leqslant a_1^{\frac{1}{2}} \left(\int_0^t \int_{D(z)} \exp(-ws) a_{ij} u_{,j} u_{,i} dads \cdot \int_0^t \int_{D(z)} \exp(-ws)|u|^2 dads\right)^{\frac{1}{2}}. \quad (3.12)\end{aligned}$$

In view of (3.6), we obtain

$$\begin{aligned}|H(z,t)| &= \left|\int_0^t \int_{D(z)} \exp(-ws) a_{i1} u u_{,i} dads\right| \\ &\leqslant a_1 \left(\int_0^t \int_{D(z)} \exp(-ws)|\nabla u|^2 dads \cdot \int_0^t \int_{D(z)} \exp(-ws)|u|^2 dads\right)^{\frac{1}{2}} \\ &\leqslant c_1(t) \left(\int_0^t \int_{D(z)} \exp(-ws)|\nabla u|^2 dads \cdot \int_0^t \int_{D(z)} \exp(-ws)|u|^{2\alpha} dads\right)^{\frac{1}{2}}, \quad (3.13)\end{aligned}$$

where

$$\begin{aligned}c_1(t) &= a_1 \left(\int_0^t \int_{D(z)} \exp(-ws) dads\right)^{\frac{\alpha-1}{2\alpha}} \\ &= a_1 \left[\frac{1}{w}(1-\exp(-wt))|D|\right]^{\frac{\alpha-1}{2\alpha}}.\end{aligned}$$

For arbitrary positive constant υ,

$$\begin{aligned}|H(z,t)| \leqslant c_1(t) \Bigg[&\left(\upsilon^\alpha \int_0^t \int_{D(z)} \exp(-ws)|u|^{2\alpha} dads\right)^{\frac{1}{\alpha+1}} \\ &\times \left(\upsilon^{-1} \int_0^t \int_{D(z)} \exp(-ws)|\nabla u|^2 dads\right)^{\frac{\alpha}{\alpha+1}}\Bigg]^{\frac{\alpha+1}{2\alpha}}. \quad (3.14)\end{aligned}$$

Applying Young's inequality, we get

$$\begin{aligned}|H(z,t)| \leqslant c_2(t) \Bigg[&\frac{\upsilon^\alpha}{\gamma} \int_0^t \int_{D(z)} \exp(-ws)\gamma|u|^{2\alpha} dads \\ &+ \frac{\alpha\upsilon^{-1}}{a_0} \int_0^t \int_{D(z)} \exp(-ws) a_0 |\nabla u|^2 dads\Bigg]^{\frac{\alpha+1}{2\alpha}} \quad (3.15)\end{aligned}$$

where $c_2(t) = \frac{c_1(t)}{(\alpha+1)^{\frac{\alpha+1}{2\alpha}}}$.

If we choose $\upsilon = \left(\frac{\alpha\gamma}{a_0}\right)^{\frac{1}{\alpha+1}}$, we can get from (3.15) that

$$|H(z,t)| \leqslant c_3(t) \left(-\frac{\partial H(z,t)}{\partial z}\right)^{\frac{\alpha+1}{2\alpha}}, \quad (3.16)$$

where $c_3(t) = c_2(t) a_0^{-\frac{\alpha}{1+\alpha}} \gamma^{-\frac{1}{\alpha+1}} \alpha^{\frac{\alpha}{1+\alpha}}$.

When we assume that $H(z_0,t)<0$, we have that $H(z,t)<0$ for all $z \geqslant z_0$. Hence, we conclude that

$$-\frac{\partial H(z,t)}{\partial z} \geqslant \left(-\frac{H(z,t)}{c_3(t)}\right)^{\frac{2\alpha}{\alpha+1}}, \tag{3.17}$$

which integrating yields

$$(-H(z,t))^{-\frac{\alpha-1}{\alpha+1}} \leqslant (-H(z_0,t))^{-\frac{\alpha-1}{\alpha+1}} - \frac{\alpha-1}{\alpha+1} c_3(t)^{-\frac{2\alpha}{\alpha+1}}(z-z_0). \tag{3.18}$$

The inequality (3.18) shows that the solutions cease to exist for a finite value z.

On the other hand, if we assume that $H(z,t) \geqslant 0$ for all z, we obtain from (3.16),

$$c_3(t)^{-\frac{2\alpha}{1+\alpha}} \leqslant (H(z,t))^{-\frac{2\alpha}{1+\alpha}} \frac{\partial H(z,t)}{\partial z}. \tag{3.19}$$

Integrating (3.19), we obtain

$$H(z,t) \leqslant \left[H(z_0,t)^{\frac{1-\alpha}{1+\alpha}} + \frac{\alpha-1}{\alpha+1} c_3(t)^{\frac{2\alpha}{1+\alpha}}(z-z_0)\right]^{-\frac{\alpha+1}{\alpha-1}}. \tag{3.20}$$

that gives a description of the spatial decay of the solutions whenever they exist for all $z \geqslant 0$.

Above all, we have proved the following theorem:

THEOREM 3.1. *Let u be a solution of the initial boundary value problem determined by the equation (3.1), g(u) satisfies the condition (3.5), a_{ij} satisfies (3.6), and the initial-boundry conditions (3.2), (3.3), (3.4) are also satisfied. Then, either the solution ceases to exist for a finite value of the spatial variable z, or the solution must tend asymptotically to zero as z tends to infinity.*

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