Mathematical
Inequalities
& Applications
Volume 13, Number 1 (2010), 157–163

SOME INEQUALITIES ON SUBLINEAR FUNCTIONALS RELATED TO THE INVARIANT MEAN FOR DOUBLE SEQUENCES

M. MURSALEEN AND S. A. MOHIUDDINE

(*Communicated by J. Pečarić*)

Abstract. In this paper we define invariant mean for double sequences and construct a sublinear functional which dominates and generates invariant mean.

1. Introduction

A double sequence $x=(x_{jk})$ is said to be *convergent* in the Pringsheim sense (or P-*convergent*) if for given $\varepsilon>0$ there exists an integer N such that $|x_{jk}-\ell|<\varepsilon$ whenever $j,k>N$. We shall write this as

$$\lim_{j,k\to\infty} x_{jk}=\ell,$$

where j and k tending to infinity independent of each other (cf[15]). We denote by c_2, the space of P-convergent sequences. Throughout this paper limit of a double sequence means limit in the Pringsheim sense.

A double sequence x is *bounded* if

$$\| x \|=\sup_{j,k\geqslant 0} |x_{jk}|<\infty.$$

Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. By c_2^∞, we denote the space of double sequences which are bounded convergent, and by ℓ_2^∞ the space of bounded double sequences. Note that $c_2^\infty\subset\ell_2^\infty$.

In this paper, firstly we define the concept of invariant mean for double sequences.

DEFINITION. Let σ be a one-to-one mapping from the set $\mathbb{N}$ of natural numbers into itself. A continuous linear functional φ_2 on ℓ_2^∞ is said to be an *invariant mean* or a σ-*mean* if and only if

(i) $\varphi_2(x)\geqslant 0$ if $x\geqslant 0$ (i.e. $x_{jk}\geqslant 0$ for all j,k);

(ii) $\varphi_2(E)=1$, where $E=(e_{jk})$, $e_{jk}=1$ for all j,k;

(iii) $\varphi_2(x)=\varphi_2((x_{\sigma(j),\sigma(k)}))=\varphi_2((x_{\sigma(j),k}))=\varphi_2((x_{j,\sigma(k)}))$.

Mathematics subject classification (2010): 40C05, 40H05.
Keywords and phrases: Double sequences, P-convergence, invariant mean, σ-convergence.

Paper MIA-13-12

Throughout this paper we consider the mapping σ which has no finite orbits, that is, $\sigma^p(k) \neq k$ for all integer $k \geqslant 0$ and $p \geqslant 1$, where $\sigma^p(k)$ denotes the pth iterate of σ at k. Note that, as in case of single sequences [12], a σ-mean extends the limit functional on c_2^∞ in the sense that $\varphi_2(x) = \lim x$ for all $x \in c_2^\infty$.

The space V_2^σ of σ-convergent double sequences was introduced in [2] and further studied by Mursaleen and Mohiuddine [10]. That is, a double sequence $x = (x_{jk})$ of real numbers is said to be σ-*convergent* to a number L if and only if $x \in V_2^\sigma$, where

$$V_2^\sigma = \{x \in \ell_2^\infty : \lim_{p,q\to\infty} \tau_{pqst}(x) = L \text{ uniformly in } s,t; L = \sigma\text{-}\lim x\} \tag{1.1}$$

$$\tau_{pqst}(x) = \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)}$$

and $\tau_{-1,q,s,t} = \tau_{p,-1,s,t} = \tau_{-1,-1,s,t} = 0$.
For $\sigma(n) = n+1$, the set V_2^σ is reduced to the set f_2 of almost convergent double sequences [7]. The concept of almost convergence for single sequences was introduced by Lorentz [6]. Note that $c_2^\infty \subset V_2^\sigma \subset \ell_2^\infty$.

For matrix transformations of double sequences and related methods, we refer to Altay-Başar [1], Gökhan-Çolak [3, 4], Hamilton [5], Patterson [14], Mursaleen [13], Mursaleen-Edely [8], Mursaleen-Mohiuddine [10, 11], Mursaleen-Savas [9], Robinson [16], and Zeltser [17].

2. Sublinear Functionals that Generate Invariant Means

A sublinear functional P on ℓ_2^∞ generates invariant means if $\varphi_2 \in (\ell_2^\infty)'$ and $\varphi_2 < P$ implies φ_2 is an invariant mean. Here $\varphi_2 < P$ means $\varphi_2(x) \leqslant P(x)$ for all $x = (x_{jk}) \in \ell_2^\infty$ and $(\ell_2^\infty)'$ is the continuous dual of ℓ_2^∞, that is, $(\ell_2^\infty)'$ is the set of all continuous linear functionals defined on ℓ_2^∞. We define a subset of ℓ_2^∞ as

$$(V_2^\sigma)_0 = \{x = (x_{jk}) \in \ell_2^\infty : \lim_{p,q} \tau_{pqst}(x) = 0 \text{ uniformly in } s,t\}. \tag{2.1}$$

It is trivial that $x = (x_{jk}) \in \ell_2^\infty$ implies $x_{\sigma(j),\sigma(k)} - x \in (V_2^\sigma)_0$ because for any σ-mean φ_2,

$$\varphi_2(x_{\sigma(j),\sigma(k)} - x) = \varphi_2(x_{\sigma(j),\sigma(k)}) - \varphi_2(x) = 0.$$

From (1.1), it is clear that

$$\varphi_2(x) = \lim_{p,q} \tau_{pqst}(x),$$

uniformly in s,t. Now we define $V : \ell_2^\infty \to \mathbb{R}$ such that

$$V(x) = \inf_{p=(p_{jk})\in(V_2^\sigma)_0} \limsup_{j,k} (x_{jk} + p_{jk}), \tag{2.2}$$

V is well defined if and only if $V(p) \geqslant 0$ for all $p = (p_{jk}) \in (V_2^\sigma)_0$.

PROPOSITION 2.1. *V is sublinear on ℓ_2^∞.*

Proof. For $x, y \in \ell_2^\infty$,

$$V(x+y) = \inf_{p=(p_{jk})\in(V_2^\sigma)_0} \limsup_{j,k}(x_{jk}+y_{jk}+p_{jk}).$$

Hence

$$V(x+y) \leqslant \limsup_{j,k}(x_{jk}+y_{jk}+2p_{jk}),$$
$$\leqslant \limsup_{j,k}(x_{jk}+p_{jk}) + \limsup_{jk}(y_{jk}+p_{jk}),$$

taking infimum over $p=(p_{jk})\in(V_2^\sigma)_0$, we have

$$V(x+y) \leqslant V(x)+V(y).$$

For any $\alpha \geqslant 0$,

$$V(\alpha x) = \inf_{p=(p_{jk})\in(V_2^\sigma)_0} \limsup_{j,k}(\alpha x_{jk}+p_{jk}) = \alpha \inf_{p'_{jk}\in(V_2^\sigma)_0} \limsup_{j,k}(x_{jk}+p'_{jk}) = \alpha V(x),$$

where $p'_{jk} = p_{jk}/\alpha$.

Hence V is a sublinear on ℓ_2^∞. $\square$

PROPOSITION 2.2. *If $p=(p_{jk})\in(V_2^\sigma)_0$ then $V(p)=0$.*

Proof. We have

$$V(p) = \inf_{p=(p_{jk})\in(V_2^\sigma)_0} \limsup_{j,k}(2p_{jk}),$$
$$\leqslant 0, \text{ since } \{0\}\in(V_2^\sigma)_0.$$

Since $-p\in Z_2$, we have $V(-p)\leqslant 0$. But V being sublinear, $V(p)\geqslant -V(-p)$. Hence $V(p)=0$.

This completes the proof. $\square$

THEOREM 2.3. *V generates σ-means.*

Proof. Let $\varphi_2\in(\ell_2^\infty)'$ and $\varphi_2 < V$. We have to show that φ_2 is a σ-mean. From (2.2), we get $V(x)\leqslant \limsup x_{jk}$. Hence for $x=(x_{jk})\geqslant 0$, $V(x)\geqslant 0$ as V is sublinear. As $\varphi_2(x)\leqslant V(x)$ for all $x=(x_{jk})\in\ell_2^\infty$, we get

$$\varphi_2(x)\geqslant 0 \text{ for all } x=(x_{jk})\geqslant 0. \tag{2.3.1}$$

Now, $\varphi_2(E) \leqslant V(E)$ where $E = (e_{jk}) = 1$ for all j, k; and

$$V(E) = \inf_{p=(p_{jk})\in(V_2^\sigma)_0} \limsup_{j,k}(1+p_{jk}),$$
$$\leqslant \limsup_{j,k} 1 = 1.$$

Since V is sublinear, we get

$$\varphi_2(E) = 1. \tag{2.3.2}$$

Again,

$$V(x_{\sigma(j),\sigma(k)} - x) = \inf_{p=(p_{jk})\in(V_2^\sigma)_0} \limsup_{j,k}(x_{\sigma(j),\sigma(k)} - x_{jk} + p_{jk}),$$
$$= 0\ ; \text{ by Proposition 2.2.}$$

Hence $\varphi_2(x_{\sigma(j),\sigma(k)} - x) = 0$, so that

$$\varphi_2(x_{\sigma(j),\sigma(k)}) = \varphi_2(x). \tag{2.3.3}$$

Similarly, $\varphi_2(x) = \varphi_2(x_{\sigma(j),k}) = \varphi_2(x_{j,\sigma(k)} - x)$. From (2.3.1), (2.3.2), and (2.3.3), we see that φ_2 is a σ-mean.

This completes the proof of the theorem. □

3. Sublinear Functionals that Dominate Invariant Means

A sublinear functional P on ℓ_2^∞ dominates invariant means if every invariant mean φ_2 is less than P, that is, $\varphi_2 \in M_2^\sigma$ implies $\varphi_2 < P$, where M_2^σ denotes the set of all σ-means.

Now we show that sublinear functional V dominates invariant mean. First we prove the following lemma which will be used in our next theorem.

LEMMA 3.1. *Let σ have no finite orbits, i.e. $\sigma^p(k) \neq k$ for all integer $k \geqslant 0$ and $p \geqslant 1$. Then any invariant mean φ_2 is such that*

$$\varphi_2(x) \leqslant L_2(x) \text{ for all } x \in \ell_2^\infty,$$

where

$$L_2(x) = \limsup_{j,k} x_{jk}.$$

Proof. From the definition of $L_2(x)$, we have that for given $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$x_{jk} < L_2(x) + \varepsilon \text{ for all } j,k \geqslant N.$$

Hence

$$|x_{jk} - L_2(x)| < \varepsilon \text{ for all } j,k \geqslant N,$$

and

$$|x_{jk} - L_2(x)| > \varepsilon, \tag{3.1.1}$$

for some $j,k \leqslant N$. Since $x = (x_{jk}) \in \ell_2^\infty$,

$$|x_{jk} - L_2(x)| \leqslant M, \tag{3.1.2}$$

for a positive real number M and for all j,k. Now

$$\begin{aligned}
|\tau_{pqst} - L_2(x)| &= \left| \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)} - L_2(x) \right|, \\
&= \frac{1}{(p+1)(q+1)} \Big| x_{st} - L_2(x) + x_{s,\sigma(t)} - L_2(x) + \cdots + x_{s,\sigma^q(t)} - L_2(x) \\
&\quad + x_{\sigma(s),t} - L_2(x) + \cdots + x_{\sigma^p(s),t} - L_2(x) + x_{\sigma(s),\sigma(t)} - L_2(x) \\
&\quad + \cdots + x_{\sigma^p(s),\sigma^q(t)} - L_2(x) \Big|, \\
&\leqslant \frac{1}{(p+1)(q+1)} (|x_{st} - L_2(x)| + |x_{s,\sigma(t)} - L_2(x)| + \cdots + |x_{s,\sigma^q(t)} - L_2(x)| \\
&\quad + |x_{\sigma(s),t} - L_2(x)| + \cdots + |x_{\sigma^p(s),t} - L_2(x)| + |x_{\sigma(s),\sigma(t)} - L_2(x)| \\
&\quad + \cdots + |x_{\sigma^p(s),\sigma^q(t)} - L_2(x)|.
\end{aligned} \tag{3.1.3}$$

Taking p,q very large so that atmost N of the numbers differ from $L_2(x)$ by more than ε by relation (3.1.1). The rest $(p+1)(q+1) - N$ numbers differ from $L_2(x)$ by less than ε. Hence from (3.1.3)

$$\begin{aligned}
|\tau_{pqst} - L_2(x)| &\leqslant \frac{1}{(p+1)(q+1)} (NM + ((p+1)(q+1) - N)\varepsilon), \\
&= \frac{NM}{(p+1)(q+1)} + \frac{(p+1)(q+1) - N}{(p+1)(q+1)} \varepsilon.
\end{aligned} \tag{3.1.4}$$

If p,q are very large and independent of j,k then

$$|\tau_{pqst} - L_2(x)| \leqslant \varepsilon + \left(\varepsilon - \frac{N}{(p+1)(q+1)} \right).$$

Hence

$$\tau_{pqst} \leqslant L_2(x) + 2\varepsilon.$$

Since ε was arbitrary small,

$$\tau_{pqst} \leqslant L_2(x) \text{ for all } x = (x_{jk}) \in \ell_2^\infty.$$

Letting $p,q \to \infty$, we get

$$\lim_{p,q\to\infty} \tau_{pqst} \leqslant L_2(x) \text{ for all } s,t.$$

Since φ_2 is an invariant mean, $\varphi_2(x) = \lim_{p,q\to\infty} \tau_{pqst}$ uniformly in s,t; we have

$$\varphi_2(x) \leqslant L_2(x).$$

This completes the proof of the lemma. □

THEOREM 3.2. *V dominates σ-means.*

Proof. Let φ_2 be a σ-mean. For $x,y \in \ell_2^\infty$

$$\varphi_2(x) = \varphi_2(x + y_{\sigma(j),\sigma(k)} - y).$$

Using Lemma 3.1, we have

$$\begin{aligned} &\leqslant L_2(x + y_{\sigma(j),\sigma(k)} - y), \\ &= L_2(x+\alpha), \end{aligned}$$

where $\alpha = y_{\sigma(j),\sigma(k)} - y \in (V_2^\sigma)_0$. Taking infimum over α, we get

$$\varphi_2(x) \leqslant V(x).$$

This completes the proof of the theorem. □

REFERENCES

[1] B. ALTAY AND F. BAŞAR, *Some new spaces of double sequences*, J. Math. Anal. Appl., **309** (2005), 70–90.

[2] C. ÇAKAN, B. ALTAY AND M. MURSALEEN, *The σ-convergence and σ-core of double sequences*, Appl. Math. Lett., **19** (2006), 1122–1128.

[3] A. GÖKHAN AND R. ÇOLAK, *The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$*, Appl. Math. Comput., **157** (2004), 491–501.

[4] A. GÖKHAN AND R. ÇOLAK, *The double sequence spaces $l_2^\infty(p)$*, Appl. Math. Comput., **160** (2005), 147–153.

[5] H. J. HAMILTON, *Transformations of multiple sequences*, Duke Math. J., **2** (1996), 29–60.

[6] G. G. LORENTZ, *A contribution to theory of divergent sequences*, Acta Math., **80** (1948), 167–190.

[7] F. MORICZ AND B. E. RHOADES, *Almost convergence of double sequences and strong regularity of summability matrices*, Math. Proc. Camb. Phil. Soc., **104** (1988), 283–294.

[8] M. MURSALEEN AND OSAMA H. H. EDELY, *Almost convergence and a core theorem for double sequences*, J. Math. Anal. Appl., **293** (2004), 532–540.

[9] M. MURSALEEN AND E. SAVAŞ, *Almost regular matrices for double sequences*, Studia Sci. Math. Hung., **40** (2003), 205–212.

[10] M. MURSALEEN AND S. A. MOHIUDDINE, *Double σ-multiplicative matrices*, J. Math. Anal. Appl., **327** (2007), 991–996.

[11] M. MURSALEEN AND S. A. MOHIUDDINE, *Regularly σ-conservative and σ-coercive four dimensional matrices*, Comput. Math. Appl., **56** (2008), 1580–1586.

[12] M. MURSALEEN, *On some new invariant matrix methods of summability*, Quart. J. Math. Oxford, **34** (1983), 77–86.

[13] M. MURSALEEN, *Almost strongly regular matrices and a core theorem for double sequences*, J. Math. Anal. Appl., **293** (2004), 523–531.

[14] R. F. PATTERSON, *Double sequence core theorems*, Internat. J. Math.& Math. Sci., **22** (1999), 785–793.

[15] A. PRINGSHEIM, *Zur theorie der zweifach unendlichen Zahlenfolgen*, Math. Z., **53** (1900), 289–321.

[16] G. M. ROBINSON, *Divergent double sequences and series*, Trans. Amer. Math. Soc., **28** (1926), 50–73.

[17] M. ZELTSER, *Matrix transformations of double sequences*, Acta Comment. Univ. Tartu Math., **4** (2000), 39–51.

(Received December 3, 2007)

M. Mursaleen
Department of Mathematics
Aligarh Muslim University
Aligarh-202002
India
e-mail: mursaleenm@gmail.com

S. A. Mohiuddine
Department of Mathematics
Aligarh Muslim University
Aligarh-202002
India
e-mail: mohiuddine@gmail.com