

ON SOME EXTREMALITIES IN THE APPROXIMATE INTEGRATION

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(Communicated by Zs. Páles)

Abstract. Some extremalities for quadrature operators are proved for convex functions of higher order. Such results are known in the numerical analysis, however they are often proved under suitable differentiability assumptions. In our considerations we do not use any other assumptions apart from higher order convexity itself. The obtained inequalities refine the inequalities of Hadamard type. They are applied to give error bounds of quadrature operators under the assumptions weaker from the commonly used.

1. Introduction

In the theory of convex functions the Hermite–Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \quad (1)$$

which holds for convex functions (and, in fact, characterizes them), plays a very important role. The first inequality follows by the existence of a support line for f at the midpoint, while the second one can be obtained using the fundamental property of convexity stating that a graph of a convex function f lies on $[a, b]$ below the chord joining the points $(a, f(a))$, $(b, f(b))$. We have also the following

OBSERVATION 1. *If a real function f is convex on an interval $[a, b]$ then*

$$f\left(\frac{a+b}{2}\right) \leq \sum_{i=1}^N \lambda_i f(\xi_i) \leq \frac{f(a)+f(b)}{2} \quad (2)$$

for any $N \in \mathbb{N}$, $\xi_1, \dots, \xi_N \in [a, b]$ and $\lambda_1, \dots, \lambda_N \geq 0$ with $\sum_{i=1}^N \lambda_i = 1$ such that $\sum_{i=1}^N \lambda_i \xi_i = \frac{a+b}{2}$.

Proof. The first inequality is an immediate consequence of convexity, the second one we prove similarly to the second inequality of (1). □

The term on the left hand side of (2) is connected with the midpoint rule of the approximate integration, while the term on the right hand side is connected with the

Mathematics subject classification (2010): 26D15, 41A55, 41A80, 26A51, 65D30, 65D32.

Keywords and phrases: Approximate integration, error bounds, extremalities, Hadamard–type inequalities, higher order convexity, quadrature rules, support theorems.

trapezoidal rule. Then the inequality (2) can be regarded as an example of an extremality for quadrature operators. Many extremalities are known in the numerical analysis (cf. [4], cf. also [3] and the references therein). The numerical analysts prove them using the suitable differentiability assumptions. As we will show in this paper, for convex functions of higher order some extremalities can be obtained without assumptions of this kind, using only the higher order convexity itself. The support-type properties play here the crucial role. A general theorem of this nature was recently proved by the author in [19]. The obtained extremalities are useful in proving error bounds of quadrature operators under regularity assumptions weaker from the commonly used. The results of this sort are also known, however our method seems to be quite easy. But the price we must pay is high: the obtained error bounds are far to be optimal (cf. [3]). Some results concerning the inequalities between the quadrature operators and error bounds of quadrature rules, which are partial cases of our results, can be found in author's earlier papers [16, 17, 18]. The paper [15] contains the extension of our results concerning convex functions to the functions of several variables.

For $n \in \mathbb{N}$ denote by Π_n the space of all polynomials of degree at most n . Recall that a linear functional \mathcal{I} defined on a linear space X of (not necessarily all) functions mapping some nonempty set into \mathbb{R} is called *positive* if

$$f \leq g \implies \mathcal{I}(f) \leq \mathcal{I}(g)$$

for any $f, g \in X$. An important class of positive linear operators form the conical combinations of the involved function at appropriately chosen points of a domain. Obviously, if a domain is a real interval, then quadrature operators with nonnegative coefficients are linear and positive.

Dealing with a problem of approximate computation of the integral over an interval $[a, b]$ it is enough to change the variable and to compute it over a fixed interval. The interval $[-1, 1]$ is frequently used. For a Riemann integrable function $f : [-1, 1] \rightarrow \mathbb{R}$ let

$$\mathcal{I}(f) := \int_{-1}^1 f(x) dx.$$

DEFINITION 2. Let \mathcal{I} be a linear functional defined on a linear space of (not necessarily all) Riemann integrable functions mapping $[-1, 1]$ into \mathbb{R} containing Π_n . We say that \mathcal{I} is *exact on Π_n* if $\mathcal{I}(p) = \int p$ for all $p \in \Pi_n$.

2. Convex functions of higher order

Recall that the divided differences are defined as follows: $[x_1; f] := f(x_1)$ and for $k \in \mathbb{N}$

$$[x_1, \dots, x_{k+1}; f] := \frac{[x_2, \dots, x_{k+1}; f] - [x_1, \dots, x_k; f]}{x_{k+1} - x_1}.$$

DEFINITION 3. Let $I \subset \mathbb{R}$ be an interval and $n \in \mathbb{N}$. A function $f : I \rightarrow \mathbb{R}$ is *n -convex* if $[x_1, \dots, x_{n+2}; f] \geq 0$ for any distinct $x_1, \dots, x_{n+2} \in I$.

There is an easy to imagine geometrical equivalent condition of n -convexity (for the proof cf. e.g. [9, 11]).

PROPOSITION 4. *A function f is n -convex if and only if for any $n + 1$ distinct points $x_1, \dots, x_{n+1} \in I$, the graph of the (unique) polynomial $p \in \Pi_n$ interpolating f at these points, passing through each point $(x_i, f(x_i))$, $i = 1, \dots, n + 1$, changes the side of the graph of f (always $p \geq f$ on $[x_n, x_{n+1}]$).*

Then trivially 1-convexity reduces to the classical convexity.

Convex functions of higher order are very well known and investigated (see e.g. [5, 9, 10, 11, 13]). But up to now there is no common terminology, which sometimes may be confusing.

The first person who dealt with the topic in question was Hopf. He considered in his dissertation [7] from 1926 the functions with nonnegative divided differences without naming them at all. The notion of higher order convexity was introduced by Popoviciu in his famous dissertation [11] from 1934 exactly in the sense of the above Definition 3. The Kuczma's monograph [9] devoted to functional equations and inequalities in several variables as well as the classical Roberts and Varberg's book on convex functions [13] use the same terminology (according to which an ordinary convex function is 1-convex). During the years another way of naming convex functions of higher order became popular. Some authors (cf. e.g. [5, 10]) call a function f to be n -convex if $[x_1, \dots, x_{n+1}; f] \geq 0$ (then a convex function is 2-convex). Now these two terminologies appear simultaneously in the literature. The first one is concentrated on the maximal degree of the interpolating polynomial, while the accent of the second one is put on the dimension of the space of polynomials of degree not exceeding some natural number. Then the second terminology is more coherent with convexity with respect to Chebyshev systems (cf. [8]). Both conventions have some advantages and disadvantages and it is not the author's intention to judge neither which one is better nor which one is classical.

Having in mind the above remarks let us declare that in this paper we understand the higher order convexity in the sense of Definition 3.

Convex functions of higher order have many regularity properties. For details see [9, 11, 13]. The paper [10] contains a brief survey of the topic given in one place. Below we list the properties which either we use in the paper or we discuss below.

THEOREM 5. *If $f : [a, b] \rightarrow \mathbb{R}$ is n -convex then f is continuous on (a, b) and bounded on $[a, b]$.*

COROLLARY 6. *If $f : [a, b] \rightarrow \mathbb{R}$ is n -convex then f is Riemann integrable.*

THEOREM 7. *The real function f defined on an open interval I is n -convex if and only if $f^{(n-1)}$ is convex on I .*

COROLLARY 8. *If the real function f defined on an open interval I is n -convex then $f_-^{(n)}$, $f_+^{(n)}$ exist on I and $f^{(n)}$ exists almost everywhere on I .*

Notice that there are n -convex functions which are not n times differentiable (e.g. $f(x) = |x|$ for $n = 1$). Sometimes what is proved under differentiability assumptions,

holds in fact for any n -convex function without further assumptions. We return to this matter in Section 3. However, the following result requiring the differentiability assumption seems to be important (cf. [9, 11, 13], for a quick reference cf. also [17, Theorems A and B] and [19, Theorem D]).

THEOREM 9. *Assume that $f : [a, b] \rightarrow \mathbb{R}$ is $(n+1)$ -times differentiable on (a, b) and continuous on $[a, b]$. Then f is n -convex if and only if $f^{(n+1)}(x) \geq 0$, $x \in (a, b)$.*

It is well known that a convex function defined on a real interval admits an affine support at every interior point of a domain. In the paper [19] we have proved a general support-type result for convex functions of higher order. Four special cases ([19, Corollaries 8–11]) play the crucial role in the proofs presented in this paper.

THEOREM 10. *If $f : [a, b] \rightarrow \mathbb{R}$ is $(2n-1)$ -convex and $x_1, \dots, x_n \in (a, b)$, then there exists a polynomial $p \in \Pi_{2n-1}$ such that $p(x_i) = f(x_i)$, $i = 1, \dots, n$, and $p \leq f$ on $[a, b]$.*

THEOREM 11. *If $f : [a, b] \rightarrow \mathbb{R}$ is $(2n-1)$ -convex and $x_1 = a$, $x_2, \dots, x_n \in (a, b)$, $x_{n+1} = b$, then there exists a polynomial $p \in \Pi_{2n-1}$ such that $p(x_i) = f(x_i)$, $i = 1, \dots, n+1$, and $p \geq f$ on $[a, b]$.*

THEOREM 12. *If $f : [a, b] \rightarrow \mathbb{R}$ is $2n$ -convex, $x_1 = a$, $x_2, \dots, x_{n+1} \in (a, b)$, then there exists a polynomial $p \in \Pi_{2n}$ such that $p(x_i) = f(x_i)$, $i = 1, \dots, n+1$, and $p \leq f$ on $[a, b]$.*

THEOREM 13. *If $f : [a, b] \rightarrow \mathbb{R}$ is $2n$ -convex, $x_1, \dots, x_n \in (a, b)$ and $x_{n+1} = b$, then there exists a polynomial $p \in \Pi_{2n}$ such that $p(x_i) = f(x_i)$, $i = 1, \dots, n+1$, and $p \geq f$ on $[a, b]$.*

3. Extremalities for quadrature operators

What we recall below is very well known from the numerical analysis (cf. e.g. [1, 6, 12, 14, 20, 21, 22]). Let P_n be the n -th degree member of the sequence of Legendre polynomials.

Gauss–Legendre quadratures

For $f : [-1, 1] \rightarrow \mathbb{R}$ and $n \in \mathbb{N}$ let

$$\mathcal{G}_n(f) := \sum_{i=1}^n w_i f(x_i),$$

where x_1, \dots, x_n are the roots of P_n (which are real, distinct and belong to $(-1, 1)$) and

$$w_i = \frac{2(1-x_i^2)}{(n+1)^2 P_{n+1}^2(x_i)}, \quad i = 1, \dots, n.$$

Then \mathcal{G}_n is exact on Π_{2n-1} . If $f \in \mathcal{C}^{2n}([-1, 1])$ then

$$\mathcal{I}(f) = \mathcal{G}_n(f) + \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi) \quad (3)$$

for some $\xi \in (-1, 1)$.

Lobatto quadratures

For $f : [-1, 1] \rightarrow \mathbb{R}$ let $\mathcal{L}_2(f) := f(-1) + f(1)$. For $n \in \mathbb{N}$, $n \geq 3$, let

$$\mathcal{L}_n(f) := w_1 f(-1) + w_n f(1) + \sum_{i=2}^{n-1} w_i f(x_i),$$

where x_2, \dots, x_{n-1} are the roots of P'_{n-1} (which are also real, distinct and belong to $(-1, 1)$) and

$$w_1 = w_n = \frac{2}{n(n-1)}, \quad w_i = \frac{2}{n(n-1)P_{n-1}^2(x_i)}, \quad i = 2, \dots, n-1.$$

Then \mathcal{L}_n is exact on Π_{2n-3} . If $f \in \mathcal{C}^{2n-2}([-1, 1])$ then

$$\mathcal{I}(f) = \mathcal{L}_n(f) - \frac{n(n-1)^3 2^{2n-1} [(n-2)!]^4}{(2n-1)[(2n-2)!]^3} f^{(2n-2)}(\xi) \quad (4)$$

for some $\xi \in (-1, 1)$.

Radau quadratures

For $f : [-1, 1] \rightarrow \mathbb{R}$ and $n \in \mathbb{N}$, $n \geq 2$, let

$$\mathcal{R}_n^1(f) := w_1 f(-1) + \sum_{i=2}^n w_i f(x_i),$$

where x_2, \dots, x_n are the roots of the polynomial

$$Q_{n-1}(x) = \frac{P_{n-1}(x) + P_n(x)}{x+1}$$

(again real, distinct and belonging to $(-1, 1)$) and

$$w_1 = \frac{2}{n^2}, \quad w_i = \frac{1}{(1-x_i)[P'_{n-1}(x_i)]^2}, \quad i = 2, \dots, n.$$

Then \mathcal{R}_n^1 is exact on Π_{2n-2} . If $f \in \mathcal{C}^{2n-1}([-1, 1])$ then

$$\mathcal{I}(f) = \mathcal{R}_n^1(f) + \frac{2^{2n-1} n [(n-1)!]^4}{[(2n-1)!]^3} f^{(2n-1)}(\xi) \quad (5)$$

for some $\xi \in (-1, 1)$.

In [19] we also considered the operator

$$\mathcal{R}_n^r(f) := \mathcal{R}_n^l(f(-\cdot)).$$

It was, in fact, defined in terms of orthogonal polynomials. However, these two definitions coincide. This is not difficult to check. We would not like to go into details since this is not the goal of the paper. Let us only mention that (changing the way of naming and numbering the abscissas and weights) we have

$$\mathcal{R}_n^r(f) = \sum_{i=1}^n w_i f(y_i) + w_n f(1)$$

and \mathcal{R}_n^r is exact on Π_{2n-2} . The error term of \mathcal{R}_n^r is similar to (5), precisely, if $f \in \mathcal{C}^{2n-1}([-1, 1])$ then

$$\mathcal{I}(f) = \mathcal{R}_n^r(f) - \frac{2^{2n-1} n [(n-1)!]^4}{[(2n-1)!]^3} f^{(2n-1)}(\eta) \quad (6)$$

for some $\eta \in (-1, 1)$.

As we can see, all the weights of the above quadratures are positive, so these operators are positive. This is also the case for many other quadratures. However, there are the quadratures with negative coefficients (e.g. among the Newton–Cotes formulas).

Now we can prove the main results of this section.

THEOREM 14. *Fix $n \in \mathbb{N}$. Let \mathcal{I} be the positive linear operator defined (at least) on a linear subspace of $\mathbb{R}^{[-1,1]}$ generated by a cone of $(2n-1)$ -convex functions. Assume that \mathcal{I} is exact on Π_{2n-1} . If a function $f : [-1, 1] \rightarrow \mathbb{R}$ is $(2n-1)$ -convex then*

$$\mathcal{G}_n(f) \leq \mathcal{I}(f) \leq \mathcal{L}_{n+1}(f). \quad (7)$$

Proof. By Theorems 10 and 11 there exist two polynomials $p, q \in \Pi_{2n-1}$ interpolating f at the abscissas of the operators $\mathcal{G}_n, \mathcal{L}_{n+1}$, respectively, such that $p \leq f \leq q$ on $[-1, 1]$. Since $\mathcal{G}_n = \mathcal{L}_{n+1} = \mathcal{I}$ on Π_{2n-1} , we get

$$\mathcal{G}_n(f) = \mathcal{G}_n(p) = \mathcal{I}(p) = \mathcal{I}(f) \leq \mathcal{I}(q) = \mathcal{L}_{n+1}(q) = \mathcal{L}_{n+1}(f).$$

□

THEOREM 15. *Fix $n \in \mathbb{N}$. Let \mathcal{I} be the positive linear operator defined (at least) on a linear subspace of $\mathbb{R}^{[-1,1]}$ generated by a cone of $2n$ -convex functions. Assume that \mathcal{I} is exact on Π_{2n} . If a function $f : [-1, 1] \rightarrow \mathbb{R}$ is $2n$ -convex then*

$$\mathcal{R}_{n+1}^l(f) \leq \mathcal{I}(f) \leq \mathcal{R}_{n+1}^r(f). \quad (8)$$

Proof. Use Theorems 12 and 13 and the abscissas of the operators $\mathcal{R}_{n+1}^l, \mathcal{R}_{n+1}^r$, respectively, and argue similarly as in the proof of Theorem 14. □

We would like to emphasize two particular cases of the above results. The first one concerns the inequalities of Hadamard type. The assertions of the Corollary below

were proved in [19, Propositions 12 and 13] (cf. also the earlier paper [1], where these results were obtained by another method).

COROLLARY 16. Fix $n \in \mathbb{N}$. If $f : [-1, 1] \rightarrow \mathbb{R}$ is $(2n - 1)$ -convex then $\mathcal{G}_n(f) \leq \mathcal{I}(f) \leq \mathcal{L}_{n+1}(f)$. If $f : [-1, 1] \rightarrow \mathbb{R}$ is $2n$ -convex then $\mathcal{R}_{n+1}^l(f) \leq \mathcal{I}(f) \leq \mathcal{R}_{n+1}^r(f)$.

Proof. Use Theorems 14 and 15 for $\mathcal{T} = \mathcal{I}$. □

The second important case is connected with quadrature operators.

COROLLARY 17. Fix $n, N \in \mathbb{N}$, $\xi_1, \dots, \xi_N \in [-1, 1]$ and $\lambda_1, \dots, \lambda_N \geq 0$. Let

$$\mathcal{T}(f) := \sum_{i=1}^N \lambda_i f(\xi_i) \quad \text{for } f : [-1, 1] \rightarrow \mathbb{R}.$$

- (i) If \mathcal{T} is exact on Π_{2n-1} , then $\mathcal{G}_n(f) \leq \mathcal{T}(f) \leq \mathcal{L}_{n+1}(f)$ for any $(2n - 1)$ -convex function $f : [-1, 1] \rightarrow \mathbb{R}$.
- (ii) If \mathcal{T} is exact on Π_{2n} , then $\mathcal{R}_{n+1}^l(f) \leq \mathcal{T}(f) \leq \mathcal{R}_{n+1}^r(f)$ for any $2n$ -convex function $f : [-1, 1] \rightarrow \mathbb{R}$.

Proof. The operator \mathcal{T} trivially fulfils the assumptions of Theorems 14 and 15, respectively. □

In the numerical analysis the inequalities the above type are called *extremalities*. The extremalities of Corollary 17 (i) were earlier proved in [4, Theorem 6] under the assumption of $2n$ -times differentiability. The proof given there is based on taking double nodes. The author independently used in [19] exactly the same idea to prove support-type results of Corollaries 8–11 (quoted here in Theorems 10–13) with no use of any differentiability assumptions. Thus, as we can see from the proof of Theorems 14 and 15, the extremalities in question are proved with no further assumptions, except higher order convexity itself. So, our results are more general than these of [4].

We underline that the inequalities of Corollary 17 do not hold for any quadrature operator \mathcal{T} . The exactness assumption (i.e. $\mathcal{T} = \mathcal{I}$ for polynomials of the appropriate degree) is essential.

EXAMPLE 18. We have

$$\begin{aligned} \mathcal{G}_2(f) &= f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right), \\ \mathcal{G}_3(f) &= \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\frac{\sqrt{15}}{5}\right) + f\left(\frac{\sqrt{15}}{5}\right)\right], \\ \mathcal{R}_3^l(f) &= \frac{2}{9}f(-1) + \frac{16 + \sqrt{6}}{18}f\left(\frac{1 - \sqrt{6}}{5}\right) + \frac{16 - \sqrt{6}}{18}f\left(\frac{1 + \sqrt{6}}{5}\right) \end{aligned}$$

(cf. e.g. [20, 22]), whence $\mathcal{G}_3(\exp) > \mathcal{G}_2(\exp)$ and $\mathcal{R}_3^l(\exp) > \mathcal{G}_2(\exp)$. The exponential function is convex of any order (cf. Theorem 9). Let $\mathcal{T} = \mathcal{G}_2$. Then the inequality of Corollary 17 (i) does not hold for $n = 3$, and that of (ii) is not true for $n = 2$. Notice that for $p(x) = x^4$ we have $\mathcal{G}_2(p) \neq \mathcal{I}(p)$, so $\mathcal{G}_2 \neq \mathcal{I}$ both on Π_5 and on Π_4 .

Now the question arises if there are other, i.e. non–quadrature, operators approximating the integral, for which Theorems 14 and 15 are applicable. The positive answer given below shows that the extremalities for quadrature operators are special cases of more general inequalities. Not the form of the operators considered (linear combination, integral and so on) is important but two things play the key role: positiveness and exactness for polynomials of the appropriate degree.

EXAMPLE 19. For a Riemann–integrable function $f : [-1, 1] \rightarrow \mathbb{R}$ let

$$\mathcal{T}(f) := \frac{3}{11}[f(-1) + f(1)] + \frac{16}{11} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) dt.$$

Then \mathcal{T} is a positive linear operator exact on Π_3 . Using Theorem 14 for $n = 2$ we obtain $\mathcal{G}_2(f) \leq \mathcal{T}(f) \leq \mathcal{L}_3(f)$ for a 3–convex function $f : [-1, 1] \rightarrow \mathbb{R}$.

For the other non–quadrature operators approximating the integral cf. e.g. [2].

4. Error bounds of quadrature operators

In this section we show that the extremalities of Corollary 17 may be applied to obtain the error bounds of the involved quadrature operator \mathcal{T} using the regularity assumptions weaker from the commonly used. The results of this type are known in the numerical analysis. We would like to point that the inequalities of Hadamard type may be used in the approximate integration. But the results obtained by our method deliver error bounds which are far to be optimal. This is a price we have to pay for simplicity. Error bounds obtained in the numerical analysis under assumptions used by us are much better (see [3] and the references therein).

For $f \in \mathcal{C}([-1, 1])$ denote $\|f\|_\infty := \sup \{|f(x)| : x \in [-1, 1]\}$.

LEMMA 20. Fix $k \in \mathbb{N}$, $k \geq 2$. Let \mathcal{K} , \mathcal{T} be linear operators defined on a linear subspace of $\mathbb{R}^{[-1,1]}$ containing all the functions involved below with the following properties:

- (i) there exists an $\alpha > 0$ such that $\mathcal{I}(f) \leq \mathcal{K}(f) + \alpha \|f^{(k)}\|_\infty$ for all $f \in \mathcal{C}^k([-1, 1])$;
- (ii) If $f : [-1, 1] \rightarrow \mathbb{R}$ is $(k - 1)$ –convex then $\mathcal{K}(f) \leq \mathcal{T}(f)$;
- (iii) $\mathcal{T}(p) = \mathcal{I}(p)$ for $p(x) = x^k$.

Then $|\mathcal{I}(f) - \mathcal{T}(f)| \leq 2\alpha \|f^{(k)}\|_\infty$ for any $f \in \mathcal{C}^k([-1, 1])$.

Proof. By (i) and (ii) we get

$$\mathcal{I}(f) - \mathcal{T}(f) \leq \alpha \|f^{(k)}\|_\infty \tag{9}$$

for any $(k - 1)$ –convex function $f \in \mathcal{C}^k([-1, 1])$.

For an arbitrary function $f \in \mathcal{C}^k([-1, 1])$ define now $g(x) := \frac{\|f^{(k)}\|_\infty}{k!} x^k$. Then $g^{(k)} = \|f^{(k)}\|_\infty$, whence $|f^{(k)}| \leq g^{(k)}$ on $[-1, 1]$, which implies $(g - f)^{(k)} \geq 0$ and $(g + f)^{(k)} \geq 0$ on $[-1, 1]$. Therefore by Theorem 9 the functions $g - f$, $g + f$ are $(k - 1)$ –convex on $[-1, 1]$. By the triangle inequality

$$\|(g - f)^{(k)}\|_\infty \leq 2\|f^{(k)}\|_\infty \quad \text{and} \quad \|(g + f)^{(k)}\|_\infty \leq 2\|f^{(k)}\|_\infty.$$

Now we apply (9) to $g - f$ and $g + f$. Then the desired inequality follows by linearity, the assumption (iii) and the above inequalities. \square

For $n \in \mathbb{N}$ let

$$\alpha_{2n} := \frac{4^{n+1}(n!)^4}{(2n+1)[(2n)!]^3}, \quad \alpha_{2n+1} := \frac{4^{n+1}(n+1)(n!)^4}{[(2n+1)!]^3}$$

THEOREM 21. Fix $k \in \mathbb{N}$, $k \geq 2$. Let \mathcal{T} be a positive linear operator defined on a domain as in Lemma 20. If \mathcal{T} is exact on Π_k , then $|\mathcal{I}(f) - \mathcal{T}(f)| \leq \alpha_k \|f^{(k)}\|_\infty$ for any $f \in \mathcal{C}^k([-1, 1])$.

Proof. If k is even, $k = 2n$, then use Lemma 20 for $\alpha = \frac{\alpha_{2n}}{2}$ and $\mathcal{H} = \mathcal{G}_n$. The condition (i) is fulfilled by (3), (ii) holds by Theorem 14 and (iii) by the assumption.

Similarly, if k is odd, $k = 2n + 1$, then use Lemma 20 for $\alpha = \frac{\alpha_{2n+1}}{2}$ and $\mathcal{H} = \mathcal{R}_{n+1}^l$. The condition (i) is fulfilled by (5), (ii) holds by Theorem 15 and (iii) by the assumption. \square

In the assertion of Theorem 14 there is an operator \mathcal{L}_{n+1} on the right hand side of the inequality (7) and in the statement of Theorem 15 there is an operator \mathcal{R}_{n+1}^r on the right hand side of the inequality (8). We could prove the result similar to Theorem 21 involving (in the proof) these operators. However, the error bound obtained in this way will not improve that of Theorem 21. Namely, the absolute value of the constant of (4) (take $n + 1$ instead of n) is greater from the similar constant of (3). For the operators \mathcal{R}_{n+1}^l and \mathcal{R}_{n+1}^r the absolute values of both constants of (5), (6) are the same.

For the quadrature operators we immediately derive from Theorem 21 the following

COROLLARY 22. Fix $k, N \in \mathbb{N}$, $k \geq 2$, $\xi_1, \dots, \xi_N \in [-1, 1]$ and $\lambda_1, \dots, \lambda_N \geq 0$.

Let

$$\mathcal{T}(f) := \sum_{i=1}^N \lambda_i f(\xi_i) \quad \text{for } f : [-1, 1] \rightarrow \mathbb{R}.$$

If \mathcal{T} is exact on Π_k , then $|\mathcal{I}(f) - \mathcal{T}(f)| \leq \alpha_k \|f^{(k)}\|_\infty$ for any $f \in \mathcal{C}^k([-1, 1])$.

Using this result we will now give the error bounds of Gauss–Legendre, Lobatto and Radau quadratures under regularity assumptions weaker from the commonly used. Denote by $\lfloor \cdot \rfloor$ the floor function, i.e. $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$, $x \in \mathbb{R}$.

PROPOSITION 23. Let $k, N \in \mathbb{N}$, $k \geq 2$ and $N > \lfloor \frac{k}{2} \rfloor$. If $f \in \mathcal{C}^k([-1, 1])$ then $|\mathcal{I}(f) - \mathcal{G}_N(f)| \leq \alpha_k \|f^{(k)}\|_\infty$.

Proof. If $N > \lfloor \frac{k}{2} \rfloor$ then $2N - 1 \geq k$, whence \mathcal{G}_N is exact on Π_k (cf. (3)). Then the result follows immediately by Corollary 22. \square

PROPOSITION 24. Let $k, N \in \mathbb{N}$, $k \geq 2$ and $N > \lfloor \frac{k}{2} \rfloor + 1$. If $f \in \mathcal{C}^k([-1, 1])$ then $|\mathcal{I}(f) - \mathcal{L}_N(f)| \leq \alpha_k \|f^{(k)}\|_\infty$.

Proof. This is also an immediate consequence of Corollary 22 since $N > \lfloor \frac{k}{2} \rfloor + 1$ implies $2N - 3 \geq k$ and then \mathcal{L}_N is exact on Π_k (cf. (4)). \square

PROPOSITION 25. Let $k, N \in \mathbb{N}$, $k \geq 2$ and $N > \lfloor \frac{k+1}{2} \rfloor$. If $f \in \mathcal{C}^k([-1, 1])$ then $|\mathcal{I}(f) - \mathcal{R}_N^l(f)| \leq \alpha_k \|f^{(k)}\|_\infty$. The same assertion holds for the operator \mathcal{R}_N^r .

Proof. By $N > \left\lfloor \frac{k+1}{2} \right\rfloor$ we get $2N - 2 \geq k$ and we can see (cf. (5), (6)) that \mathcal{R}_N^1 and \mathcal{R}_N^r are exact on Π_k , which, together with Corollary 22, concludes the proof. \square

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(Received December 19, 2007)

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