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SERIES INVOLVING THE LEAST AND THE GREATEST PRIME FACTOR OF A NATURAL NUMBER

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Abstract. We determine necessary and sufficient conditions satisfied by the real numbers a, b, c that the series $\sum_{n\geqslant 2} n^a p^b(n)P^c(n)$ is convergent. Here $p(n)$ and $P(n)$ are the least and respectively the greatest prime factor of an integer $n\geqslant 2$.

1. Introduction

We introduce in the first part the notations used and recall some classical results needed in the following sections. The reader is directed to references for proofs.

Let $p(n)$, $P(n)$ and p_n be, respectively, the least prime factor of $n\geqslant 2$, the greatest prime factor of $n\geqslant 2$ and the n-th prime number.

The next three results involve $p(n)$ and $P(n)$.

In [3] is shown that

$$\sum_{2\leqslant n\leqslant x}\frac{1}{nP(n)}=e^{\gamma}\log\log x+O(1),\tag{1}$$

where γ stands for Euler's constant.

According to [4] (see also [1])

$$\sum_{2\leqslant n\leqslant x}\frac{p(n)}{P(n)}=\frac{x}{\log x}+\frac{3x}{\log^2 x}+\frac{15x}{\log^3 x}+o\left(\frac{x}{\log^3 x}\right).\tag{2}$$

Moreover, in [6] is proved that

$$\sum_{2\leqslant n\leqslant x}\frac{1}{p(n)}=x\left(A+O\left((\log x)^{-1/14}\right)\right),\tag{3}$$

where A is a constant.

Similar results can be found in [5] at pages 121–126.

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We will use several times the following results about p_n:

A. We have $p_n \sim n \log n$.

B. The series $\sum_{n \geqslant 1} \frac{1}{p_n}$ is divergent.

C. We have $\prod_{i=1}^{n} \left(1 - \frac{1}{p_i}\right) = \frac{e^{-\gamma}}{\log n}\left(1 + O\left(\frac{1}{\log n}\right)\right)$.

This result is due to Mertens and can be found e.g., in [5] at page 259.

Some other classical results needed in proofs are:

D. The series $\sum_{n \geqslant 1} \frac{1}{n^{\alpha}}$ is convergent if and only if $\alpha > 1$.

E. We have $\frac{1}{1-u} > e^u$ whenever $0 < u < 1$.

OBSERVATION 1. *The series* $\sum_{n \geqslant 2} p^b(n) P^c(n)$ *is divergent for all real numbers b and c.*

For $n = 2^i$, $i \geqslant 2$, we have $p(n) = P(n) = 2$ and the term 2^{b+c} occurs infinitely many times in the series.

We consider the series $\sum_{n \geqslant 2} n^a p^b(n) P^c(n)$ and denote it by $S(a,b,c)$. Our main result states that:

THEOREM 1. *The series* $S(a,b,c)$ *is convergent if and only if we have simultaneously* $a \leqslant -1$, $a + c < -1$ *and* $a + b + c < -1$.

OBSERVATION 2. *For* $x \geqslant 2$ *we denote by* $S(x,a,b,c) = \sum_{n \leqslant x} n^a p^b(n) P^c(n)$. *Since the series* $S(a,b,c)$ *has nonnegative terms, it is convergent if and only if* $S(x,a,b,c)$ *is bounded.*

OBSERVATION 3. *The nature of the series* $S(a,b,c)$ *is preserved by permuting its terms.*

The study can be carried out by means of the sums $S_n(a,b,c)$, where $S_n(a,b,c) = \sum_{\substack{k \geqslant 2 \\ P(k) \leqslant p_n}} k^a p^b(k) P^c(k)$. Therefore the convergence of the series $S(a,b,c)$ is still equivalent to the convergence of the sequence $(S_n(a,b,c))_{n \geqslant 1}$. According with the above observations, from now on, the letter S will stand for one of $S(a,b,c)$, $S_n(a,b,c)$ or $S(x,a,b,c)$.

2. Sufficient conditions for the divergence of the series S

We give in this section sufficient conditions that a, b, c must satisfy in order that series S diverges.

THEOREM 2. *The series* S *is divergent in each of the cases*

i) $a + b + c \geqslant -1$;

ii) $a + c \geqslant -1$;

iii) $a > -1$.

Proof. We have

$$S(x,a,b,c) = \sum_{2\leqslant n\leqslant x} n^a p^b(n)P^c(n) \geqslant \sum_{p_n\leqslant x} p_n^a p^b(n)P^c(p_n) = \sum_{2\leqslant p_n\leqslant x} p_n^{a+b+c}.$$

By **C.**, it follows that the series S is divergent for $a+b+c \geqslant -1$.

We have also

$$S(x) > \sum_{2p_n\leqslant x} (2p_n)^a p(2p_n)^b P(2p_n)^c = 2^{a+b} \sum_{p_n\leqslant x/2} p_n^{a+c}.$$

If $a+c \geqslant -1$, then $p_n^{a+c} \geqslant 1/p_n$. As a consequence of **B.**, the series S is divergent.

Foe iii), we distinguish two cases.

When $0 \leqslant a$ we have $n^a p^b(n)P^c(n) \geqslant p^b(n)P^c(n)$. Now from Observation 1 the series $\sum\limits_{n\geqslant 2} p^b(n)P^c(n)$ is divergent. This implies that the series $S(a,b,c)$ is divergent as well.

When $-1 < a < 0$ we have

$$\begin{aligned} S_n(a,b,c) &> \sum_{\substack{h\geqslant 1\\ P(2h)\leqslant p_n}} (2h)^a p^b(2h)P^c(2h) = 2^{a+b} \sum_{\substack{2\leqslant h\\ P(h)\leqslant p_n}} h^a P^c(h) \\ &= 2^{a+b} \sum_{1\leqslant i\leqslant n} p_i^{a+c} \prod_{j\leqslant i} (1+p_j^a+p_j^{2a}+\ldots) \\ &= 2^{a+b} \sum_{i\leqslant n} p_i^{a+c} \frac{1}{\prod\limits_{j\leqslant i}(1-p_j^a)} \end{aligned}$$

In view of **E.**, the following inequality holds

$$S_n(a,b,c) \geqslant 2^{a+b} \sum_{i\leqslant n} p_i^{a+c} e^{\sum\limits_{j\leqslant i} p_j^a} \tag{4}$$

For $a > -1$, it easily follows that for $i \geqslant 2$

$$\sum_{j=1}^{i} (j\log j)^a \sim \frac{1}{a+1} i^{a+1} \log^a i. \tag{5}$$

From **A.** and (5) we get $\sum\limits_{j\leqslant i} p_j^a \sim \frac{1}{a+1} i^{a+1} \log^a i$. This implies the existence of a constant $C_1 > 0$ such that $\sum\limits_{j\leqslant i} p_j^a > C_1 i^{a+1} \log^a i$. Thus

$$\log\left(p_i^{a+c} e^{\sum\limits_{j\leqslant i} p_j^a}\right) > (a+c)\log p_i + C_1 i^{a+1} \log^a i \tag{6}$$

Define $u_i = (a+c)\log p_i + C_1 i^{a+1}\log^a i = i^{a+1}\log^a i\left(\frac{(a+c)\log p_i}{i^{a+1}\log^a i} + C_1\right)$, for $i \geqslant 2$. Using that $a+1>0$ and **A.** it follows that $\lim\limits_{i\to\infty} u_i = \infty$. Now (4) and (6) imply that the series $S(a,b,c)$ diverges.

3. The case $a=-1$

This section is studying the convergence of the series S when $a=-1$. We need the following

LEMMA 1. *If* $\varepsilon>0$ *then the series* $\sum_{n\geqslant 2}\frac{1}{nP^{\varepsilon}(n)}$ *is convergent.*

Proof. We have

$$\sum_{n\geqslant 2}\frac{1}{nP^{\varepsilon}(n)}=\sum_{i\geqslant 1}\frac{1}{p_i^{1+\varepsilon}}\prod_{j\leqslant i}(1+p_j^{-1}+p_j^{-2}+\ldots)=\sum_{i\geqslant 1}\frac{1}{p_i^{1+\varepsilon}\prod\limits_{j\leqslant i}\left(1-\frac{1}{p_j}\right)}.$$

The result stated in **C.** gives $\prod\limits_{j\leqslant i}\left(1-\frac{1}{p_j}\right)>\frac{C_2}{\log i}$. This inequality, together with **A.**, imply the existence of $C_3>0$ such that $p_i>C_3 i\log i$. Therefore,

$$\frac{1}{p_i^{1+\varepsilon}\prod\limits_{j\leqslant i}\left(1-\frac{1}{p_j}\right)}<\frac{\log i}{C_2C_3^{1+\varepsilon}i^{1+\varepsilon}\log^{1+\varepsilon} i}<\frac{1}{C_2C_3^{1+\varepsilon}i^{1+\varepsilon}}.$$

In view of **D.**, we deduce that the series $\sum\limits_{i\geqslant 1}\frac{1}{p_i^{1+\varepsilon}\prod\limits_{j\leqslant i}\left(1-\frac{1}{p_j}\right)}$ is convergent.

We are now ready to prove

THEOREM 3. *The series* $\sum\limits_{n\geqslant 2}\frac{p^b(n)P^c(n)}{n}$ *is convergent if and only if we have both* $c<0$ *and* $b+c<0$.

Proof. For $c\geqslant 0$ we have $S(x)>\sum\limits_{2p_i\leqslant x}\frac{p^b(2p_i)P^c(2p_i)}{2p_i}=2^{b-1}\sum\limits_{2p_i\leqslant x}p_i^{c-1}$, hence the series is divergent since $p_i^{c-1}\geqslant\frac{1}{p_i}$ and the series $\sum\limits_{i\geqslant 1}\frac{1}{p_i}$ is divergent.

If $b+c\geqslant 0$, then

$$S(x)>\sum_{p_i\leqslant x}\frac{p^b(p_i)P^c(p_i)}{p_i}=\sum_{p_i\leqslant x}p_i^{b+c-1}\geqslant\sum_{p_i\leqslant x}\frac{1}{p_i}$$

hence the series is divergent.

It remains to study the case when both c and $b+c$ are negative numbers.

Denote $b+c=-\varepsilon$ with $\varepsilon>0$.

If $b\geqslant 0$ then $p^b(n)\leqslant P^b(n)$. By Lemma 1 we have

$$S(x)\leqslant\sum_{n\leqslant x}\frac{P^{b+c}(n)}{n}=\sum_{n\leqslant x}\frac{1}{nP^{\varepsilon}(n)}<\infty.$$

If $b<0$ then $p^b(n)<1$ and by Lemma 1 we get $S(x)<\sum\limits_{n\leqslant x}\frac{1}{nP^{-c}(n)}<\infty$, since $-c>0$.

4. Proof of Theorem 1

Using the results of the previous two sections we can complete now the proof of the main result.

In view of Theorem 2 and Theorem 3, it remains only to show that, if $a < -1$, $a+c < -1$ and $a+b+c < -1$, then the series S is convergent.

Denote $a = -1-t$, $a+c = -1-y$, $a+b+c = -1-z$ with $t,y,z > 0$. It follows that $n^a p^b(n)P^c(n) = p^{y-z}(n)P^{t-y}(n)/n^{1+t}$.

If $y \leqslant z$, then

$$n^a p^b(n)P^c(n) = \left(\frac{P(n)}{n}\right)^t \cdot \frac{1}{nP^y(n)} \cdot p^{y-z}(n) \leqslant \frac{1}{nP^y(n)}.$$

In the case $y > z$ we have $p^{y-z}(n) \leqslant P^{y-z}(n)$, which gives

$$n^a p^b(n)P^c(n) \leqslant \frac{P^{t-z}(n)}{n^{1+t}} = \left(\frac{P(n)}{n}\right)^t \cdot \frac{1}{nP^z(n)} \leqslant \frac{1}{nP^z(n)}.$$

In both cases, the convergence of the series S follows now by Lemma 1. □

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