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A CLASS OF ABSTRACT VOLTERRA EQUATIONS, VIA WEAKLY PICARD OPERATORS TECHNIQUE

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Abstract. In this paper we consider the following abstract Volterra equations:

$$x(t) = G(t, g(x)(t), x(t), x(0)) + \int_{-t}^{t} K(t, s, x(s), x(h(s))), \ \ t \in \mathbb{R}$$

and

$$x(t) = G(t, g(x)(t), x(t), x(0)) + \int_{-|t|}^{|t|} K(t, s, x(s), x(h(s)))\, ds, \ \ t \in \mathbb{R}.$$

Using the weakly Picard operator technique we establish existence, data dependence and comparison results for the solution. The derivability of the solutions with respect to a parameter is also studied.

1. Introduction

The purpose of this paper is to study the following abstract Volterra equation:

$$x(t) = G(t, g(x)(t), x(t), x(0)) + \int_{-t}^{t} K(t, s, x(s), x(h(s))), \ \ t \in \mathbb{R}. \tag{1.1}$$

This equation has, as particular cases, the following well-known equations:

EXAMPLE 1.1. ([1], [15]) Volterra's equation:

$$x(t) = G(t) + \int_{-t}^{t} K(t, s, x(s)), \ \ t \in \mathbb{R}.$$

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EXAMPLE 1.2. ([14]) The functional Volterra equation:

$$x(t) = G(t) + \int_{-t}^{t} K(t, s, x(s), x(h(s))), \ \ t \in \mathbb{R}.$$

EXAMPLE 1.3. ([3], [9]) Tonelli's equation:

$$x(t) = G(t) + \int_{-t}^{t} K(t, s, x(s)) + \int_{-t}^{t}\int_{-t}^{t} H(t, s, x(s)x(u))\,dsdu, \ \ t \in \mathbb{R}.$$

EXAMPLE 1.4. ([3]–[8]) The abstract Volterra equation:

$$x(t) = V(x)(t), \ \ t \in \mathbb{R}.$$

In order to study the equation (1.1) we shall use the weakly Picard operators technique.

2. Basic notions and results of the weakly Picard operators theory

In this paper we shall use the terminologies and notations from [10] and [11]. For the convenience of the reader we shall recall some of them.

Let $(X, \rightarrow)$ be an L-space and $A : X \rightarrow X$ an operator. We also denote by $A^0 := 1_X$, $A^1 := A$, $A^{n+1} := A^n \circ A$, $n \in \mathbb{N}$ the iterate operators of the operator A. We also have:

$$P(X) := \{Y \subseteq X \mid Y \neq \emptyset\}$$

$$F_A := \{x \in X \mid A(x) = x\}$$

$$I(A) := \{Y \in P(X) \mid A(Y) \subseteq Y\}$$

DEFINITION 2.1. $A : X \rightarrow X$ is called a Picard operator (briefly PO) if:
(i) $F_A = \{x^*\}$;
(ii) $A^n(x) \rightarrow x^*$ as $n \rightarrow \infty$, for all $x \in X$.

The operator A is Picard if and only if the discrete dynamical system generated by A has an equilibrium state which is globally asymptotically stable.

DEFINITION 2.2. $A : X \rightarrow X$ is said to be a weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n\in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.

If $A : X \rightarrow X$ is a WPO, then we define the operator $A^\infty : X \rightarrow X$ by the formula

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Obviously $A^\infty(X) = F_A$. Moreover, if A is a PO and we denote by x^* its unique fixed point, then $A^\infty(x) = x^*$, for each $x \in X$.

DEFINITION 2.3. Let $(X,\to)$ be an L-space, $c>0$ and $d: X\times X\to\mathbb{R}_+$. By definition the operator A is c-WPO with respect to the functional d iff

$$d\left(x,A^{\infty}(x)\right)\leqslant c\cdot d\left(x,A(x)\right),\quad \text{for all } x\in X.$$

We have (see [10], [11]):

THEOREM 2.1. (Characterization theorem) *Let $(X,\to)$ be an L-space and $A: X\to X$ be an operator. Then, A is a WPO if and only if there exists a partition of X, $X=\bigcup_{\lambda\in\Lambda} X_\lambda$, such that:*

(a) $X_\lambda\in I(A)$, for all $\lambda\in\Lambda$;

(b) $A\big|_{X_\lambda}: X_\lambda\to X_\lambda$ is PO, for all $\lambda\in\Lambda$.

THEOREM 2.2. (Abstract Gronwall lemma) *Let $(X,\to,\leqslant)$ be an ordered L-space and $A: X\to X$ be n operator. We suppose that:*

(i) A is a PO;

(ii) A is increasing.

If we denote by x_A^* the unique fixed point of A, then:

(a) $x\leqslant A(x)\Longrightarrow x\leqslant x_A^*$;

(b) $x\geqslant A(x)\Longrightarrow x\geqslant x_A^*$.

THEOREM 2.3. (Comparison theorem) *Let $(X,\to,\leqslant)$ be an ordered L-space and $A,B,C: X\to X$ be three operators such that:*

(i) $A\leqslant B\leqslant C$;

(ii) A,B,C are WPOs;

(iii) the operator B is increasing.

Then

$$x\leqslant y\leqslant z\Longrightarrow A^{\infty}(x)\leqslant B^{\infty}(y)\leqslant C^{\infty}(z).$$

We also need the following results:

THEOREM 2.4. (Data dependence) *Let $\left(X,(d_\alpha)_{\alpha\in\mathscr{A}}\right)$ be a gauge space and $A,B: X\to X$ be two $c_\alpha-$WPOs. We suppose that, for each $\alpha\in\mathscr{A}$ there exists $\eta_\alpha>0$, such that*

$$d_\alpha\left(A(x),B(x)\right)\leqslant\eta_\alpha,\quad \text{for all } x\in X.$$

Then

$$H_{d_\alpha}\left(F_A,F_B\right)\leqslant c_\alpha\cdot\eta_\alpha,\ \text{for all } \alpha\in\mathscr{A}.$$

Proof. The conclusion follows by the following relations:

$$H_\alpha(F_A,F_B) \leqslant \max\left\{\sup_{x\in F_B} d_\alpha(x,A^\infty(x)), \sup_{x\in F_A} d_\alpha(x,B^\infty(x))\right\}$$

and

$$d_\alpha(x,A^\infty(x)) \leqslant c_\alpha d_\alpha(x,A(x)) = c_\alpha d_\alpha(B(x),A(x)) \leqslant c_\alpha \eta_\alpha,\ x\in F_B,$$

$$d_\alpha(x,B^\infty(x)) \leqslant c_\alpha d_\alpha(x,B(x)) = c_\alpha d_\alpha(A(x),B(x)) \leqslant c_\alpha \eta_\alpha,\ x\in F_A. \qquad \square$$

REMARK 2.5. For some similar results and applications see [2] and [12], [5], [6].

THEOREM 2.6. (Fibre contraction principle [14]) *Let $(X,\to)$ be an L-space and $\left(Y,(d_\alpha)_{\alpha\in\mathscr{A}}\right)$ be a sequentially complete Hausdorff gauge space. Let $B: X\to X$ and $C: X\times Y\to Y$ be two operators. We suppose that:*

(i) B is a PO;

(ii) for every $\alpha\in\mathscr{A}$ there exists $l_\alpha\in[0;1[$ such that $C(x,\cdot): Y\to Y$, $x\in X$, is l_α-contraction;

(iii) if $(x^*,y^*)\in F_A$, where $A: X\times Y\to X\times Y$, $A(x,y)=(B(x),C(x,y))$, then $C(\cdot,y^*)$ is continuous in x^*.

Then A is a PO.

3. The solution set of the equation (1.1)

Let $(\mathbb{B},+,\mathbb{R}.|\cdot|)$ be a Banach space. Let us consider the equation (1.1) and suppose that:

(C_1) $G\in C\left(\mathbb{R}\times\mathbb{B}^3,\mathbb{B}\right)$ and $K\in C\left(\mathbb{R}^2\times\mathbb{B}^2,\mathbb{B}\right)$;

(C_2) $g: C(\mathbb{R},\mathbb{B})\to C(\mathbb{R},\mathbb{B})$ is an abstract Volterra operator and there exists $l_g>0$ such that

$$|g(x)(t)-g(y)(t)| \leqslant l_g|x(t)-y(t)|$$

for all $x,y\in C(\mathbb{R},\mathbb{B})$, $t\in\mathbb{R}$;

(C_3) $h\in C(\mathbb{R},\mathbb{R})$ and $|h(t)|\leqslant|t|$ for all $t\in\mathbb{R}$;

(C_4) there exist $l_{1G}>0$, $l_{2G}>0$ such that

$$|G(t,u_1,v_1,w)-G(t,u_2,v_2,w)| \leqslant l_{1G}|u_1-u_2|+l_{2G}|v_1-v_2|,$$

for all $t\in\mathbb{R}$, $u_i,v_i,w\in\mathbb{B}$ and $i\in\{1,2\}$;

(C_5) $l_{1G}\cdot l_g+l_{2G}<1$;

(C_6) there exists $l_K > 0$ such that:

$$|K(t,s,u_1,v_1) - K(t,s,u_2,v_2)| \leqslant l_K \max\{|u_1 - u_2|, |v_1 - v_2|\},$$

for all $t \in \mathbb{R}$, $u_i, v_i \in \mathbb{B}$, $i \in \{1,2\}$;

(C_7) there exists $e \in \mathbb{B}$ such that

$$g(x)(0) = e, \quad \text{for all } x \in C(\mathbb{R},\mathbb{B}).$$

With respect to the equation (1.1) we consider the equation (in $\beta \in \mathbb{B}$),

$$\beta = G(0,e,\beta,\beta) \tag{3.1}$$

Let S_G be the solution set of the equation (3.1).

In what follows we consider the gauge space $X := \left(C(\mathbb{R},\mathbb{B}), (d_n)_{n\in\mathbf{N}}\right)$, where

$$d_n(x,y) = \max_{-n \leqslant t \leqslant n} \left(|x(t) - y(t)| \cdot e^{-\tau|t|}\right), \quad \tau > 0.$$

Let $B : X \to X$ be defined by

$$B(x)(t) = G(t, g(x)(t), x(t), x(0)) + \int_{-t}^{t} K(t,s,x(s),x(h(s))), \quad t \in \mathbb{R}.$$

It is obvious that the solution set of the equation (1.1) coincides with F_B.

Let $X_\beta = \{x \in X \mid x(0) = \beta\}$. Notice that $X = \bigcup\limits_{\beta\in\mathbb{B}} X_\beta$ is a partition of X. We have:

LEMMA 3.1.

(i) If $x \in F_B$, then $x(0) \in S_G$;

(ii) $F_B \cap X_\beta \neq \emptyset \Longleftrightarrow \beta \in S_G$.

LEMMA 3.2. *If $\beta \in S_G$, then $X_\beta \in I(B)$.*

Our first main result is the following.

THEOREM 3.1. *If the conditions (C_1)–(C_7) are satisfied, then*

$$B\Big|_{\bigcup\limits_{\beta\in S_G} X_\beta} : \bigcup_{\beta\in S_G} X_\beta \to \bigcup_{\beta\in S_G} X_\beta$$

is a WPO and $Card(F_B) = Card(S_G)$.

Proof. Let $\beta \in S_G$. We denote by $B_\beta := B\Big|_{X_\beta} : X_\beta \to X_\beta$. From (C_1)–(C_7) we have

$$\begin{aligned}\left|B_\beta(x)(t) - B_\beta(y)(t)\right| &\leqslant (l_{1G}\cdot l_g + l_{2G})\, d_n(x,y)\, e^{\tau|t|} + l_K d_n(x,y)\left|\int_{-t}^{t} e^{\tau|s|}ds\right| \\ &\leqslant (l_{1G}\cdot l_g + l_{2G})\, d_n(x,y)\, e^{\tau|t|} + l_K d_n(x,y)\int_{-|t|}^{|t|} e^{\tau|s|}ds \\ &\leqslant (l_{1G}\cdot l_g + l_{2G})\, d_n(x,y)\, e^{\tau|t|} + \tfrac{2l_K}{\tau} d_n(x,y)\, e^{\tau|t|},\ \ t\in[-n;n].\end{aligned}$$

and therefore

$$d_n\left(B_\beta(x), B_\beta(y)\right) \leqslant \left(l_{1G}\cdot l_g + l_{2G} + \frac{2l_K}{\tau}\right) d_n(x,y)$$

for any $x,y \in X_\beta$.

For a suitable choice of τ, the operator $B\Big|_{X_\beta}$ is a contraction with respect to $(d_n)_{n\in\mathbf{N}}$. Since, for $\beta \in S_G$, the operator B_β is PO and from Lemma 3.1 we have that $CardF_B = CardS_G$. Moreover, from the characterization theorem of WPOs we get that B is a WPO. □

4. Data dependence

Consider the following integral equations:

$$x(t) = G_i(t, g(x)(t), x(t), x(0)) + \int_{-t}^{t} K_i(t,s,x(s),x(h(s))),\ \ t\in\mathbb{R},\ \ i\in\{1,2\}.$$

We have:

THEOREM 4.1. *Consider G_i, K_i, $i\in\{1,2\}$ satisfying the conditions (C_1)–(C_7). In addition, we suppose:*

(i) there exists $\eta_1 > 0$ such that

$$|G_1(t,u,v,z) - G_2(t,u,v,z)| \leqslant \eta_1$$

for all $t\in\mathbb{R}$, $u,v,z\in\mathbb{B}$;

(ii) there exists $\eta_2 > 0$ such that

$$|K_1(t,s,u,v) - K_2(t,s,u,v)| \leqslant \eta_2$$

for all $t,s\in\mathbb{R}$, $u,v\in\mathbb{B}$;

We denote by

$$B_i : X \to X$$
$$B_i(x)(t) = G_i(t, g(x)(t), x(t), x(0)) + \int_{-t}^{t} K_i(t, s, x(s), x(h(s))),\ i \in \{1,2\}$$

the operators generated by the above integral equations.

Then we have:

$$H_{d_n}(F_{B_1}, F_{B_2}) \leqslant (\eta_1 + 2n\eta_2) \max\left\{\frac{1}{1-a_1}, \frac{1}{1-a_2}\right\},$$

where $a_i = \left(l_{1G_i} \cdot l_g + l_{2G_i} + \frac{2l_{K_i}}{\tau}\right)$, *for* $i \in \{1,2\}$.

Proof. From Theorem 3.1 we have that

$$B_i\Big|_{\bigcup\limits_{\beta \in S_{G_i}} X_\beta} : \bigcup_{\beta \in S_{G_i}} X_\beta \to \bigcup_{\beta \in S_{G_i}} X_\beta,\ i \in \{1,2\}$$

are WPOs. Moreover, $B_i\big|_{X_\beta}$ is a contraction, with constant $a_i = \left(l_{1G_i} \cdot l_g + l_{2G_i} + \frac{2l_{K_i}}{\tau}\right)$, $i \in \{1,2\}$, with respect to $(d_n)_{n\in\mathbf{N}}$ for a suitable choice of τ. Therefore $B_i\Big|_{\bigcup\limits_{\beta \in S_{G_i}} X_\beta}$ is c_i-WPO, where $c_i = \frac{1}{1-a_i}$. Also, we have:

$$\begin{aligned}|B_1(x)(t) - B_2(x)(t)| &\leqslant |G_1(t, g(x)(t), x(t), x(0)) - G_1(t, g(x)(t), x(t), x(0))| \\ &\quad + \left|\int_{-t}^{t} |K_1(t, s, x(s), x(h(s))) - K_2(t, s, x(s), x(h(s)))|\, ds\right| \\ &\leqslant \eta_1 + 2\eta_2 |t| \leqslant \eta_1 + 2\eta_2 n\end{aligned}$$

for every $t \in [-n; n]$.

Thus,

$$d_n(B_1(x), B_2(x)) \leqslant \eta_1 + 2\eta_2 n.$$

The conclusion follows from Theorem 2.4. □

Further on, we consider the case $S_G = \{\beta^*\}$ and the integral equation depending on a parameter λ of the form:

$$x(t,\lambda) = G(t, x(t,\lambda), \beta^*, \lambda) + \int_{-t}^{t} K(t, s, x(s,\lambda), x(h(s),\lambda), \lambda)\, ds, \qquad (4.1)$$

for any $t \in \mathbb{R}$, $\lambda \in J$. In this case $\beta^* \in \mathbb{B}$ satisfies $\beta^* = G(0, \beta^*, \beta^*, \lambda)$ and $x(0,\lambda) = \beta^*$ for all $\lambda \in J$.

THEOREM 4.2. *We suppose that:*

(i) J *is a compact interval of* $\mathbb{R}$ *and* $K \in C\left(\mathbb{R}^2 \times \mathbb{B}^2 \times J, \mathbb{B}\right)$*;*

(ii) h *satisfies condition* (C_3)*;*

(iii) there exists a unique $\beta^* \in \mathbb{B}$ *such that*

$$\beta^* = G(0, \beta^*, \beta^*, \lambda), \quad \text{for all } \lambda \in J.$$

(iv) $G(t, \cdot, \beta^*, \lambda) \in C^1(\mathbb{B}, \mathbb{B})$ *and there exists* $M_1 > 0$ *such that*

$$\|D_2 G(t, \cdot, \beta^*, \lambda)\| \leqslant M_1, \text{ for all } t \in \mathbb{R},\ \lambda \in J;$$

(v) $l_{1G} < 1$*;*

(vi) $K(t, s, \cdot, v, \lambda),\ K(t, s, u, \cdot, , \lambda) \in C^1(\mathbb{B}, \mathbb{B})$ *and there exists* $M_2 > 0$ *such that*

$$\|D_3 K(t, s, \cdot, v, \lambda)\| \leqslant M_2, \quad \|D_4 K(t, s, u, \cdot, \lambda)\| \leqslant M_2,$$

for all $t, s \in \mathbb{R},\ u, v \in \mathbb{B}$ *and* $\lambda \in J$*;*

(vii) $K(t, s, u, v, \cdot) \in C^1(J, \mathbb{B})$*, for all* $t, s \in \mathbb{R},\ u, v \in \mathbb{B}$.

In these conditions we have the following conclusions:

(a) the equation (4.1) *has a unique solution* x^* *in* $C(\mathbb{R} \times J, \mathbb{B})$*;*

(b) for all $x_0 \in C(\mathbb{R} \times J, \mathbb{B})$ *the sequence* $(x_n)_{n \in \mathbb{N}}$ *defined by:*

$$x_{n+1}(t, \lambda) = G(t, x_n(t, \lambda), \beta^*, \lambda) + \int_{-t}^{t} K(t, s, x_n(s, \lambda), x_n(h(s), \lambda), \lambda)\, ds,\ t \in \mathbb{R},\ \lambda \in J,$$

converges uniformly, on each compact of $\mathbb{R} \times J$*, to* x^**;*

(c) $x^*(t, \cdot) \in C^1(J, \mathbb{B})$*, for all* $t \in \mathbb{R}$.

Proof. Let $X = C(\mathbb{R} \times J, \mathbb{B})$ and $B : X \to X$,

$$B(x)(t, \lambda) = G(t, x(t, \lambda), \beta^*, \lambda) + \int_{-t}^{t} K(t, s, x(s), x(h(s), \lambda), \lambda)\, ds,\ t \in \mathbb{R},\ \lambda \in J.$$

The conclusions (a) and (b) follow as in the proof of Theorem 3.1. Since $S_G = \{\beta^*\}$ we have that B is a PO.

(c) We shall use the following heuristic argument. We suppose that there exists $\frac{\partial x^*}{\partial \lambda}$. Then, from (4.1), we have:

$$\begin{aligned}\frac{\partial x^*}{\partial \lambda}(t,\lambda) = {} & (D_2 G(t,x^*(s,\lambda),\beta^*,\lambda))\left(\tfrac{\partial x^*}{\partial \lambda}(t,\lambda)\right) + \tfrac{\partial G}{\partial \lambda}(t,x^*(s,\lambda),\beta^*,\lambda) \\ & + \int_{-t}^{t} (D_3 K(t,s,x^*(s,\lambda),x^*(h(s),\lambda),\lambda))\left(\tfrac{\partial x^*}{\partial \lambda}(s,\lambda)\right) ds \\ & + \int_{-t}^{t} (D_4 K(t,s,x^*(s,\lambda),x^*(h(s),\lambda),\lambda))\left(\tfrac{\partial x^*}{\partial \lambda}(h(s),\lambda)\right) ds \\ & + \int_{-t}^{t} \tfrac{\partial K}{\partial \lambda}(t,s,x^*(s,\lambda),x^*(h(s),\lambda),\lambda)\, ds\end{aligned}$$

This relation suggests to consider the following operator

$$\begin{gathered} C : X \times X \to X \\ (x,y) \mapsto C(x,y) \end{gathered}$$

where

$$\begin{aligned} C(x,y)(t,\lambda) = {} & (D_2 G(t,x(s,\lambda),\beta^*,\lambda))(y(t,\lambda)) + \tfrac{\partial G}{\partial \lambda}(t,x(s,\lambda),\beta^*,\lambda) \\ & + \int_{-t}^{t} (D_3 K(t,s,x(s,\lambda),x(h(s),\lambda),\lambda))(y(s,\lambda))\, ds \\ & + \int_{-t}^{t} (D_4 K(t,s,x(s,\lambda),x(h(s),\lambda),\lambda))(y(h(s),\lambda))\, ds \\ & + \int_{-t}^{t} \tfrac{\partial K}{\partial \lambda}(t,s,x(s,\lambda),x(h(s),\lambda),\lambda)\, ds \end{aligned}$$

By this procedure, we generate the triangular operator

$$\begin{gathered} A : X \times X \to X \times X \\ A(x,y) = (B(x), C(x,y)). \end{gathered}$$

From (iv)–(vii) it follows that $C(x,\cdot) : X \to X$, $x \in X$ are contractions. Indeed, we have:

$$\begin{aligned} & |C(x,y)(t,\lambda) - C(x,z)(t,\lambda)| \\ & \quad \leqslant |(D_2 G(t,x(s,\lambda),\beta^*,\lambda))(y(t,\lambda) - z(t,\lambda))| \\ & \qquad + \left| \int_{-t}^{t} |(D_3 K(t,s,x(s,\lambda),x(h(s),\lambda),\lambda))(y(s,\lambda) - z(s,\lambda))|\, ds \right| \\ & \qquad + \left| \int_{-t}^{t} |(D_4 K(t,s,x(s,\lambda),x(h(s),\lambda),\lambda))(y(h(s),\lambda) - z(h(s),\lambda))|\, ds \right| \\ & \quad \leqslant M_1 |y(t,\lambda) - z(t,\lambda)| + M_2 \int_{-|t|}^{|t|} |y(s,\lambda) - z(s,\lambda)|\, ds \end{aligned}$$

$$
\begin{aligned}
&+M_2\int_{-|t|}^{|t|}|y(h(s),\lambda)-z(h(s),\lambda)|\,ds\\
&\leqslant\left(M_1e^{\tau|t|}+2M_2\int_{-|t|}^{|t|}e^{\tau|s|}ds\right)\cdot d_n(y,z)\\
&\leqslant\left(M_1+\tfrac{4M_2}{\tau}\right)\cdot d_n(y,z)\,e^{\tau|t|}\\
&\leqslant\left(l_{1G}+\tfrac{4l_K}{\tau}\right)\cdot d_n(y,z)\,e^{\tau|t|},\ \ t\in[-n;n].
\end{aligned}
$$

Therefore

$$
d_n(C(x,y),C(x,z))\leqslant\left(l_{1G}+\frac{4l_K}{\tau}\right)\cdot d_n(y,z)
$$

$\forall x,y,z\in X$.

For suitable choices of the parameter $\tau>0$, the operator B and the operators $C(x,\cdot)$ are contractions. Using Theorem 2.6 we conclude that A is a PO and the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ defined by:

$$
x_{n+1}(t,\lambda)=G(t,x_n(t,\lambda),\beta^*,\lambda)+\int_{-t}^{t}K(t,s,x_n(s,\lambda),x_n(h(s),\lambda),\lambda)\,ds,\ \ t\in\mathbb{R},\ \lambda\in J,
$$

$$
\begin{aligned}
y_{n+1}(t,\lambda)=&\,(D_2G(t,x_n(s,\lambda),\beta^*,\lambda))(y_n(t,\lambda))+\tfrac{\partial G}{\partial\lambda}(t,x_n(s,\lambda),\beta^*,\lambda)\\
&+\int_{-t}^{t}(D_3K(t,s,x_n(s,\lambda),x_n(h(s),\lambda),\lambda))(y_n(s,\lambda))\,ds\\
&+\int_{-t}^{t}(D_4K(t,s,x_n(s,\lambda),x_n(h(s),\lambda),\lambda))(y_n(h(s),\lambda))\,ds\\
&+\int_{-t}^{t}\tfrac{\partial K}{\partial\lambda}(t,s,x_n(s,\lambda),x_n(h(s),\lambda),\lambda)\,ds
\end{aligned}
$$

converge uniformly on each compact of $\mathbb{R}\times J$ to $(x^*,y^*)\in F_A$, for all $x_0,y_0\in X$. Notice that, for fixed $x_0,y_0\in X$ such that $y_0=\frac{\partial x_0}{\partial\lambda}$ we have that $y_1=\frac{\partial x_1}{\partial\lambda}$ and thus, by induction, we can prove that $y_n=\frac{\partial x_n}{\partial\lambda}$. Hence $\frac{\partial x_n}{\partial\lambda}$ converges uniformly on each compact of $\mathbb{R}\times J$ to y^*. The above relations imply that there exists $\frac{\partial x^*}{\partial\lambda}$ and $\frac{\partial x^*}{\partial\lambda}=y^*$. $\square$

5. The modified equation (1.1)

In this section we will consider the equation:

$$
x(t)=G(t,g(x)(t),x(t),x(0))+\int_{-|t|}^{|t|}K(t,s,x(s),x(h(s)))\,ds. \tag{5.1}
$$

From Theorem 3.1 we get:

THEOREM 5.1. *If* $(C1)-(C7)$ *hold, then*

$$B\Big|_{\bigcup\limits_{\beta\in S_G} X_\beta} : \bigcup_{\beta\in S_G} X_\beta \to \bigcup_{\beta\in S_G} X_\beta$$

is WPO and $Card(F_B) = Card(S_G)$.

THEOREM 5.2. *We consider the equation* (5.1) *such that all the assumptions of Theorem* 5.1 *hold. In addition, we suppose that :*

(C_8) $\mathbb{B}$ *is an ordered Banach space;*

(C_9) *the operators* $G(t,\cdot,\cdot,\cdot) : \mathbb{B}\times\mathbb{B}\times\mathbb{B} \to \mathbb{B}$, $K(t,s,\cdot,\cdot) : \mathbb{B}\times\mathbb{B} \to \mathbb{B}$, $g : \mathbb{B} \to \mathbb{B}$ *are increasing.*

Let x *and* y *be two solutions of the equation* (5.1)*. If* $x(0) \leqslant y(0)$ *then* $x(t) \leqslant y(t)$ *for all* $t \in \mathbb{R}$.

Proof. We remark that $x \in X_{x(0)}$ and $y \in X_{y(0)}$. If $u \in \mathbb{B}$, then we denote by $\tilde{u}$ the constant function

$$\tilde{u} : B \to B,\ \ \tilde{u}(t) = u,\ \ \text{for all } t \in \mathbb{R}.$$

It is obvious that

$$\widetilde{x(0)} \in X_{x(0)} \text{ and } \widetilde{y(0)} \in X_{y(0)}.$$

thus we have

$$x = B^\infty\left(\widetilde{x(0)}\right),\ \ y = B^\infty\left(\widetilde{y(0)}\right).$$

From the monotonicity of the operator B^∞, we get $x \leqslant y$. □

THEOREM 5.3. *Let* G_i, K_i, g, h *be as in Theorem 4.1,* $i \in \{1,2,3\}$*. In addition, we suppose that:*

$(C_9^{'})$ *the operators* $G_2(t,\cdot,\cdot,\cdot)$, $K_2(t,s,\cdot,\cdot)$ *and* g *are increasing;*

(C_{10}) $G_1 \leqslant G_2 \leqslant G_3$, $K_1 \leqslant K_2 \leqslant K_3$*;*

(C_{11}) $S_{G_1} = S_{G_2} = S_{G_3}$.

If x_i *is a solution of equation* (5.1) *corresponding to* G_i *and* K_i*, for* $i \in \{1,2,3\}$*, then*

$$x_1(0) \leqslant x_2(0) \leqslant x_3(0) \Longrightarrow x_1 \leqslant x_2 \leqslant x_3.$$

Proof. Let B_i be the operator corresponding to G_i and K_i, for $i \in \{1,2,3\}$. We have that

$$x_i = B_i^\infty\left(\widetilde{x_i(0)}\right),\ \ i = 1,2,3.$$

The proof follows now from Theorem 2.3. □

THEOREM 5.4. *We suppose that all the hypothesis of Theorem* 5.2 *hold. Let* x *be a lower solution of the equation (*5.1*), i.e.*

$$x(t) \leqslant G(t,g(x)(t),x(t),x(0)) + \int_{-|t|}^{|t|} K(t,s,x(s),x(h(s)))\,ds.$$

Then $x \leqslant B^{\infty}(x)$.

Proof. The conclusion follows from Theorem 2.2. □

In similar way to the previous section, we can give a result concerning the derivability with respect to the parameter λ. We, also, assume that $G = G(t,x(t,\lambda),x(0,\lambda),\lambda)$, $S_G = \{\beta^*\}$ and the integral equation (5.1) depending on the parameter λ is given by:

$$x(t,\lambda) = G(t,x(t,\lambda),\beta^*,\lambda) + \int_{-|t|}^{|t|} K(t,s,x(s,\lambda),x(h(s),\lambda),\lambda)\,ds. \qquad (5.2)$$

In the same manner as in the proof of Theorem 4.2, we can prove the following result:

THEOREM 5.5. *Suppose that:*

(i) J *is a compact interval of* $\mathbb{R}$ *and* $K \in C\left(\mathbb{R}^2 \times \mathbb{B}^2 \times J, \mathbb{B}\right)$*;*

(ii) h *satisfies* (C_3)*;*

(iii) there exists a unique $\beta^* \in \mathbb{B}$ *such that*

$$\beta^* = G(0,\beta^*,\beta^*,\lambda), \quad \text{for all } \lambda \in J.$$

(iv) $G(t,\cdot,\beta^*,\lambda) \in C^1(\mathbb{B},\mathbb{B})$ *and there exists* $M_1 > 0$ *such that*

$$\|D_2 G(t,\cdot,\beta^*,\lambda)\| \leqslant M_1, \text{ for all } t \in \mathbb{R},\ \lambda \in J;$$

(v) $l_{1G} < 1$*;*

(vi) $K(t,s,\cdot,v,\lambda)$, $K(t,s,u,\cdot,\lambda) \in C^1(\mathbb{B},\mathbb{B})$ *and there exists* $M_2 > 0$ *such that*

$$\|D_3 K(t,s,\cdot,v,\lambda)\| \leqslant M_2, \quad \|D_4 K(t,s,u,\cdot,\lambda)\| \leqslant M_2,$$

for all $t,s \in \mathbb{R}$, $u,v \in \mathbb{B}$ *and* $\lambda \in J$;

(vii) $K(t,s,u,v,\cdot) \in C^1(J,\mathbb{B})$, *for all* $t,s \in \mathbb{R}$, $u,v \in \mathbb{B}$.

In these conditions we have:

*(a) the equation (*5.2*) has a unique solution* x^* *in* $C(\mathbb{R} \times J, \mathbb{B})$*;*

(b) for all $x_0 \in C(\mathbb{R} \times J, \mathbb{B})$ *the sequence* $(x_n)_{n\in\mathbb{N}}$ *given by:*

$$x_{n+1}(t,\lambda) = G(t, x_n(t,\lambda), \beta^*, \lambda) + \int_{-|t|}^{|t|} K(t,s,x_n(s,\lambda), x_n(h(s),\lambda),\lambda)\,ds., \; t \in \mathbb{R},\ \lambda \in J,$$

converges uniformly, on each compact of $\mathbb{R} \times J$*, to* x^**;*

(c) $x^*(t,\cdot) \in C^1(J,\mathbb{B})$*, for all* $t \in \mathbb{R}$*.*

6. Examples

EXAMPLE 6.1.

$$x(t) = G(t) + \int_{-t}^{t} K(t,s,x(s),x(h(s))), \;\; t \in \mathbb{R}.$$

In this case, the conditions (C_1)–(C_7) become:

(C_1) $G \in C(\mathbb{R},\mathbb{B})$ and $K \in C\left(\mathbb{R}^2 \times \mathbb{B}^2, \mathbb{B}\right)$;

(C_3) $h \in C(\mathbb{R},\mathbb{R})$ and $|h(t)| \leqslant |t|$, for all $t \in \mathbb{R}$;

(C_6) there exists $l_K > 0$ such that:

$$|K(t,s,u_1,v_1) - K(t,s,u_2,v_2)| \leqslant l_K \max\{|u_1-u_2|, |v_1-v_2|\},$$

for all $t \in \mathbb{R}$, $u_i, v_i \in \mathbb{B}$, $i \in \{1,2\}$;

Notice that $S_G = \{G(0)\}$ and, therefore, the integral equation has a unique solution.

EXAMPLE 6.2.

$$x(t) = x(0) + \int_{-t}^{t} K(t,s,x(s),x(h(s))), \;\; t \in \mathbb{R}.$$

In this case, the conditions (C_1)–(C_7) become:

(C_1) $K \in C\left(\mathbb{R}^2 \times \mathbb{B}^2, \mathbb{B}\right)$;

(C_3) $h \in C(\mathbb{R},\mathbb{R})$ and $|h(t)| \leqslant |t|$, for all $t \in \mathbb{R}$;

(C_6) there exists $l_K > 0$ such that:

$$|K(t,s,u_1,v_1) - K(t,s,u_2,v_2)| \leqslant l_K \max\{|u_1-u_2|, |v_1-v_2|\},$$

for all $t \in \mathbb{R}$, $u_i, v_i \in \mathbb{B}$, $i \in \{1,2\}$.

In this case $S_G = \mathbb{B}$ and the integral equation has an infinite number of solutions.

EXAMPLE 6.3. Let $\mathbb{B} = \mathbb{R}$ and

$$x(t) = x(0)^3 + \int_{-t}^{t} K(t,s,x(s),x(h(s))), \quad t \in \mathbb{R}.$$

The conditions (C_1)–(C_7) become:

(C_1) $K \in C\left(\mathbb{R}^4, \mathbb{R}\right)$;

(C_3) $h \in C(\mathbb{R},\mathbb{R})$ and $|h(t)| \leqslant |t|$, for all $t \in \mathbb{R}$;

(C_6) there exists $l_K > 0$ such that:

$$|K(t,s,u_1,v_1) - K(t,s,u_2,v_2)| \leqslant l_K \max\{|u_1 - u_2|, |v_1 - v_2|\},$$

for all $t, u_i, v_i \in \mathbb{R}$, $i \in \{1,2\}$.

In this case $S_G = \{-1,0,1\}$ and since $Card(S_G) = Card(F_B)$, we get that the integral equation has exactly three solutions.

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