Mathematical
Inequalities
& Applications
Volume 13, Number 2 (2010), 305–318

BOUNDEDNESS OF GENERALIZED HARDY OPERATORS ON WEIGHTED AMALGAM SPACES

M. ISABEL AGUILAR CAÑESTRO AND PEDRO ORTEGA SALVADOR

(*Communicated by L.-E. Persson*)

Abstract. Let T_φ^- be the operator defined by

$$T_\varphi^- f(x) = \int_{-\infty}^{x} \varphi(x-y) f(y)\,dy,$$

where φ is a positive function on $(0,\infty)$ verifying $\varphi(a+b) \approx \varphi(a) + \varphi(b)$.

In this paper, we characterize the pairs (u,v) of positive measurable functions such that T_φ^- maps the weighted amalgam $(L^{\overline{p}}(v), \ell^{\overline{q}})$ in $(L^p(u), \ell^q)$ for all values of $p, q, \overline{p}, \overline{q}$ with $1 < p, q, \overline{p}, \overline{q} < \infty$.

As particular cases, we characterize some higher order Hardy inequalities in weighted amalgams.

1. Introduction

If $1 \leqslant p, q < \infty$ and u is a positive measurable function on $\mathbb{R}$, the amalgam space $(L^p(u), \ell^q)$ consists of the measurable functions f on the real line such that the norm

$$||f||_{p,u,q} = \left\{ \sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f|^p u \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$

is finite.

The amalgam spaces were introduced by Wiener ([10]) in 1926. The paper [2] is a survey about the role played by these spaces in Harmonic Analysis.

C. Carton-Lebrun, H. P. Heinig and S. C. Hofmann characterized in [1] the pairs of positive locally integrable functions (u,v) such that the Hardy operator $Pf(x) = \int_{-\infty}^{x} f$ verifies

$$||Pf||_{p,u,q} \leqslant C||f||_{\overline{p},v,\overline{q}} \tag{1.1}$$

in the case $1 < \overline{q} \leqslant q < \infty$. More recently, P. Ortega and C. Ramírez ([7]) have characterized the pairs (u,v) such that (1.1) holds in the case $1 < q < \overline{q} < \infty$.

Mathematics subject classification (2010): 26D10, 26D15 (42B35).

Keywords and phrases: amalgams, generalized Hardy operators, Hardy operators, weighted inequalities, weights.

This research has been supported in part by MEC, grant MTM 2005-08350-C03-02, and Junta de Andalucía, grant FQM354.

Paper MIA-13-24

In this paper, we deal with the operator T_φ^- defined for nonnegative functions f by

$$T_\varphi^- f(x) = \int_{-\infty}^{x} \varphi(x-y) f(y)dy,$$

where φ is a positive function on $(0,\infty)$ such that $\varphi(x+y) \approx \varphi(x) + \varphi(y)$. This means that there exist two positive constants C_1 and C_2 such that

$$C_1(\varphi(x)+\varphi(y)) \leqslant \varphi(x+y) \leqslant C_2(\varphi(x)+\varphi(y))$$

for all $x, y \in (0,\infty)$.

As important particular cases we find the Riemann-Liouville operators

$$T_\alpha^- f(x) = \int_{-\infty}^{x} (x-y)^\alpha f(y)dy \qquad \alpha > 0.$$

Our purpose is to characterize the pairs of positive functions (u,v) such that the inequality

$$||T_\varphi^- f||_{p,u,q} \leqslant C||f||_{\overline{p},v,\overline{q}} \tag{1.2}$$

holds for all nonnegative f with a constant $C > 0$ independent of f, where $1 < p, q, \overline{p}, \overline{q} < \infty$.

The main results will be stated and proved in section 3. They will be extensions to amalgams of well known results due to F. J. Martín-Reyes and E. Sawyer ([5]) and V. D. Stepanov ([9]) on weighted inequalities for T_φ^- in L^p spaces.

In order to characterize (1.2) we proceed essentially by establishing the relationship between inequality (1.2) and the boundedness in suitable weighted spaces of the local operators $T_n f(x) = \int_{n-1}^{x} \varphi(x-y) f(y)dy$ and the discrete operator $T_d(\{a_m\})(n) = \sum_{m=-\infty}^{n-1} \varphi(n-m)a_m$.

As a consequence of our results, we characterize the pairs of weights (u,v) such that the higher order Hardy inequality in amalgams

$$||F||_{p,u,q} \leqslant C||F^{(k)}||_{\overline{p},v,\overline{q}} \tag{1.3}$$

holds for all $F \in AC_L^{(k-1)}(-\infty,\infty)$, where $k \geqslant 2$ and $AC_L^{(k-1)}(-\infty,\infty)$ designs the space consisting of the functions F of one real variable whose $(k-1)$-st derivative is absolutely continuous and verify

$$F(-\infty) = F'(-\infty) = ... = F^{(k-1)}(-\infty) = 0.$$

Some higher order Hardy inequalities in weighted amalgams were studied by H. Heinig and A. Kufner in [3]. Specifically, if k, k_1 and k_2 are integers such that $k = k_1 + k_2$, $k_1, k_2 \geqslant 1$, and $AC_{k_1,k_2}^{(k-1)}(0,\infty)$ is the space of all functions F of one real variable whose $(k-1)$-st derivative is absolutely continuous and verify

$$F(0) = F'(0) = \cdots = F^{(k_1-1)}(0) = 0,$$

$$F^{(k_1)}(\infty) = F^{(k_1+1)}(\infty) = \cdots = F^{(k-1)}(\infty) = 0,$$

Heinig and Kufner characterized the pairs of weights (u,v) such that the higher order Hardy inequality in amalgams

$$\left\{\sum_{n=0}^{\infty}\left(\int_{n}^{n+1}|F|^{p}u\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}} \leqslant C\left\{\sum_{n=0}^{\infty}\left(\int_{n}^{n+1}|F^{(k)}|^{\overline{p}}v\right)^{\frac{\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}}$$

holds for all $F \in AC^{(k-1)}_{k_1,k_2}(0,\infty)$ whenever $1<\overline{q}\leqslant q<\infty$. However, they did not work when $k_1=0$ or $k_2=0$. We deal with these extremal cases in section 4.

Similar results can be obtained for the operator T^{+}_{φ} defined by

$$T^{+}_{\varphi}f(x)=\int_{x}^{\infty}\varphi(x-y)f(y)dy$$

and for the higher order Hardy inequalities in $AC^{(k-1)}_{R}(-\infty,\infty)$, i.e., the space of the functions F whose $(k-1)$-st derivative is absolutely continuous and verify

$$F(\infty)=F'(\infty)=...=F^{(k-1)}(\infty)=0.$$

2. Notations and preliminaries

Throughout the paper, φ will design a positive function defined on $(0,\infty)$ such that $\varphi(x+y)\approx\varphi(x)+\varphi(y)$. As a consequence of this property, we have that, up to a constant, φ increases, i.e., there exists $C>0$ such that $\varphi(x)\leqslant C\varphi(y)$ for all $x\leqslant y$.

In the statements and proofs of the results we will use the following notations, where u and v are positive locally integrable functions on the real line:

(i) If $1<\overline{p}\leqslant p<\infty$,

$$A^0_n=\sup_{\beta\in(n-1,n+1)}\left(\int_{\beta}^{n+1}\varphi^{p}(t-\beta)u(t)dt\right)^{\frac{1}{p}}\left(\int_{n-1}^{\beta}v^{1-\overline{p}'}(t)dt\right)^{\frac{1}{\overline{p}'}};$$

$$A^1_n=\sup_{\beta\in(n-1,n+1)}\left(\int_{\beta}^{n+1}u(t)dt\right)^{\frac{1}{p}}\left(\int_{n-1}^{\beta}\varphi^{\overline{p}'}(\beta-t)v^{1-\overline{p}'}(t)dt\right)^{\frac{1}{\overline{p}'}};$$

$$C_n=\max\{A^0_n,A^1_n\}.$$

(ii) If $1<p<\overline{p}<\infty$ and $\frac{1}{r}=\frac{1}{p}-\frac{1}{\overline{p}}$,

$$B^0_n=\left\{\int_{n-1}^{n+1}\left(\int_{x}^{n+1}\varphi^{p}(t-x)u(t)dt\right)^{\frac{r}{p}}\left(\int_{n-1}^{x}v^{1-\overline{p}'}(t)dt\right)^{\frac{r}{p'}}v^{1-\overline{p}'}(x)dx\right\}^{\frac{1}{r}};$$

$$B^1_n=\left\{\int_{n-1}^{n+1}\left(\int_{x}^{n+1}u(t)dt\right)^{\frac{r}{\overline{p}}}\left(\int_{n-1}^{x}\varphi^{\overline{p}'}(x-t)v^{1-\overline{p}'}(t)dt\right)^{\frac{r}{\overline{p}'}}u(x)dx\right\}^{\frac{1}{r}};$$

$$D_n=\max\{B^0_n,B^1_n\}.$$

(iii) If $1 < \overline{q} \leqslant q < \infty$,

$$A_0 = \sup_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} \varphi^q(k-n)u_k\right)^{\frac{1}{q}} \left(\sum_{k=-\infty}^{n} v_k^{1-\overline{q}'}\right)^{\frac{1}{\overline{q}'}};$$

$$A_1 = \sup_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} u_k\right)^{\frac{1}{q}} \left(\sum_{k=-\infty}^{n} \varphi^{\overline{q}'}(n-k)v_k^{1-\overline{q}'}\right)^{\frac{1}{\overline{q}'}};$$

$$A = \max\{A_0, A_1\}.$$

(iv) If $1 < q < \overline{q} < \infty$ and $\frac{1}{s} = \frac{1}{q} - \frac{1}{\overline{q}}$,

$$B_0 = \left\{\sum_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} \varphi^q(k-n)u_k\right)^{\frac{s}{q}} \left(\sum_{k=-\infty}^{n} v_k^{1-\overline{q}'}\right)^{\frac{s}{q'}} v_n^{1-\overline{q}'}\right\}^{\frac{1}{s}};$$

$$B_1 = \left\{\sum_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} u_k\right)^{\frac{s}{\overline{q}}} \left(\sum_{k=-\infty}^{n} \varphi^{\overline{q}'}(n-k)v_k^{1-\overline{q}'}\right)^{\frac{s}{\overline{q}'}} u_n\right\}^{\frac{1}{s}};$$

$$B = \max\{B_0, B_1\}.$$

(v) By $\tilde{A}$ and $\tilde{B}$ we mean, respectively, the numbers A and B defined above but corresponding to the particular sequences $u_k = \left(\int_k^{k+1} u\right)^{\frac{q}{p}}$ and $v_k = \left(\int_{k-1}^{k} v^{1-\overline{p}'}\right)^{-\frac{\overline{q}}{\overline{p}'}}$.

(vi) If $k \geqslant 2$, by C_n^k, D_n^k, A^k, $\tilde{A}^k$, B^k and $\tilde{B}^k$ we design, respectively, the numbers C_n, D_n, A, $\tilde{A}$, B and $\tilde{B}$ defined above but corresponding to the particular function $\varphi(t) = t^{k-1}$.

We will apply the following results which provide the characterizations of the weighted inequalities for the operators T_n and T_d.

THEOREM A. ([4]) *If $n \in \mathbb{Z}$, $1 < p, \overline{p} < \infty$ and u, v are positive locally integrable functions, then the operator T_n is bounded from $L^{\overline{p}}(v,(n-1,n+1))$ to $L^p(u,(n-1,n+1))$ if and only if*

(i) *in the case $1 < \overline{p} \leqslant p < \infty$, $C_n < \infty$;*

(ii) *in the case $1 < p < \overline{p} < \infty$, $D_n < \infty$.*

THEOREM B. ([4]) *Let $1 < q, \overline{q} < \infty$ and suppose that $\{u_n\}$ and $\{v_n\}$ are sequences of positive numbers. Then the operator T_d is bounded from $\ell^{\overline{q}}(\{v_n\})$ to $\ell^q(\{u_n\})$ if and only if*

(i) *in the case $1 < \overline{q} \leqslant q < \infty$, $A < \infty$;*

(ii) *in the case $1 < q < \overline{q} < \infty$, $B < \infty$.*

We will also need two lemmas. The first one is essentially due to Y. Rakotondratsimba, who studied in [8] weighted inequalities in amalgams for fractional integrals and fractional maximal operators. It reads as follows:

LEMMA 1. *If f is a nonnegative measurable function, $n \in \mathbb{Z}$ and $x \in (n, n+1)$, then*

$$T_{\varphi}^{-}(f\chi_{(-\infty,n-1)})(x) \approx \sum_{m=-\infty}^{n-1} \varphi(n-m)a_m,$$

where $a_m = \int_{m-1}^{m} f$.

Proof. If $k \geqslant 2$, $x \in (n, n+1)$ and $y \in (n-k, n+1-k)$ then $\frac{k}{2} \leqslant x - y \leqslant 2k$ and therefore $\varphi(x-y) \approx \varphi(k)$. On the other hand, since $k-1 < k \leqslant 2(k-1)$, we also have $\varphi(k) \approx \varphi(k-1)$. Then

$$\begin{aligned} T_{\varphi}^{-}(f\chi_{(-\infty,n-1)})(x) &= \int_{-\infty}^{n-1} \varphi(x-y)f(y)dy = \sum_{k=2}^{\infty} \int_{n-k}^{n+1-k} \varphi(x-y)f(y)dy \\ &\approx \sum_{k=2}^{\infty} \int_{n-k}^{n+1-k} \varphi(k)f(y)dy \approx \sum_{k=2}^{\infty} \int_{n-k}^{n+1-k} \varphi(k-1)f(y)dy \\ &= \sum_{k=2}^{\infty} \varphi(k-1)a_{n+1-k} = \sum_{m=-\infty}^{n-1} \varphi(n-m)a_m. \end{aligned}$$

□

The second lemma we will apply characterizes the embedding of the sequence space $\ell^{\overline{q}}(\{v_n^{\overline{q}}\})$ into $\ell^{q}(\{u_n^{q}\})$ for $1 < q < \overline{q} < \infty$.

LEMMA 2. *Let $1 < q < \overline{q} < \infty$ and $\frac{1}{s} = \frac{1}{q} - \frac{1}{\overline{q}}$. Suppose that $\{u_n\}$ and $\{v_n\}$ are sequences of positive real numbers. The following statements are equivalent:*

(i) *There exists $C > 0$ such that the inequality*

$$\left\{\sum_{n\in\mathbf{Z}} (|a_n|u_n)^q\right\}^{\frac{1}{q}} \leqslant C \left\{\sum_{n\in\mathbf{Z}} (|a_n|v_n)^{\overline{q}}\right\}^{\frac{1}{\overline{q}}}$$

holds for all sequences $\{a_n\}$ of real numbers.

(ii) *The sequence $\{u_n v_n^{-1}\}$ belongs to the space ℓ^s.*

3. The main results

Our first result characterizes the pairs of weights (u, v) such that the inequality (1.2) holds in the case $1 < \overline{q} \leqslant q < \infty$.

THEOREM 1. *Let $1 < p, \overline{p} < \infty$ and $1 < \overline{q} \leqslant q < \infty$. Suppose that u, v are locally integrable positive functions on $\mathbb{R}$. Then there exists a constant $C > 0$ such that the inequality (1.2) holds for all nonnegative functions f if and only if*

(i) *in the case $1 < \overline{p} \leqslant p < \infty$, $\sup_{n\in\mathbb{Z}} C_n < \infty$ and $\tilde{A} < \infty$;*

(ii) *in the case $1 < p < \overline{p} < \infty$, $\sup_{n\in\mathbb{Z}} D_n < \infty$ and $\tilde{A} < \infty$.*

Proof. Suppose that the inequality (1.2) holds. Let $n \in \mathbb{Z}$ and let f be a nonnegative function supported in $(n-1,n+1)$. Then

$$||f||_{\overline{p},v,\overline{q}} = \left\{ \left(\int_{n-1}^{n} f^{\overline{p}} v \right)^{\frac{\overline{q}}{\overline{p}}} + \left(\int_{n}^{n+1} f^{\overline{p}} v \right)^{\frac{\overline{q}}{\overline{p}}} \right\}^{\frac{1}{\overline{q}}} \leqslant C_{\overline{p},\overline{q}} \left(\int_{n-1}^{n+1} f^{\overline{p}} v \right)^{\frac{1}{\overline{p}}},$$

$$\begin{aligned} ||T_{\varphi}^{-} f||_{p,u,q} &\geqslant \left\{ \left(\int_{n-1}^{n} (T_{\varphi}^{-} f)^p u \right)^{\frac{q}{p}} + \left(\int_{n}^{n+1} (T_{\varphi}^{-} f)^p u \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\ &\geqslant C_{p,q} \left(\int_{n-1}^{n+1} (T_{\varphi}^{-} f)^p u \right)^{\frac{1}{p}} \\ &= C_{p,q} \left(\int_{n-1}^{n+1} \left(\int_{n-1}^{x} \varphi(x-y) f(y)\, dy \right)^p u(x)\, dx \right)^{\frac{1}{p}} \end{aligned}$$

and (1.2) gives

$$\left(\int_{n-1}^{n+1} \left(\int_{n-1}^{x} \varphi(x-y) f(y)\, dy \right)^p u(x)\, dx \right)^{\frac{1}{p}} \leqslant C \left(\int_{n-1}^{n+1} f^{\overline{p}} v \right)^{\frac{1}{\overline{p}}}$$

for all n, with a constant C independent of n. Therefore the operators T_n are bounded from $L^{\overline{p}}(v,(n-1,n+1))$ to $L^p(u,(n-1,n+1))$ with a constant C independent of n and by Theorem A we have $\sup_{n\in\mathbb{Z}} C_n < \infty$ if $1 < \overline{p} \leqslant p < \infty$ and $\sup_{n\in\mathbb{Z}} D_n < \infty$ if $1 < p < \overline{p} < \infty$.

On the other hand, if $\{a_m\}$ is a sequence of nonnegative numbers and

$$f = \sum_{m\in\mathbb{Z}} a_m \chi_{(m-1,m)} \left(\int_{m-1}^{m} v^{1-\overline{p}'} \right)^{-1} v^{1-\overline{p}'},$$

then $\int_{m-1}^{m} f = a_m$, $\int_{m-1}^{m} f^{\overline{p}} v = a_m^{\overline{p}} \left(\int_{m-1}^{m} v^{1-\overline{p}'} \right)^{1-\overline{p}}$ and Lemma 1 gives

$$\begin{aligned} &\left\{ \sum_{n\in\mathbb{Z}} \left(\sum_{m=-\infty}^{n-1} \varphi(n-m) a_m \right)^q \left(\int_{n}^{n+1} u \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\ &\quad = \left\{ \sum_{n\in\mathbb{Z}} \left(\sum_{m=-\infty}^{n-1} \varphi(n-m) \int_{m-1}^{m} f \right)^q \left(\int_{n}^{n+1} u \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\ &\quad = \left\{ \sum_{n\in\mathbb{Z}} \left(\int_{n}^{n+1} \left(\sum_{m=-\infty}^{n-1} \varphi(n-m) \int_{m-1}^{m} f \right)^p u(x)\, dx \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \end{aligned}$$

$$\leqslant C\left\{\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1}\left(T_\varphi^-(f\chi_{(-\infty,n-1)})(x)\right)^p u(x)dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

$$\leqslant C\left\{\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1}\left(T_\varphi^- f(x)\right)^p u(x)dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}} \leqslant C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n} f^{\overline{p}}v\right)^{\frac{\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}}$$

$$= C\left\{\sum_{n\in\mathbb{Z}} a_n^{\overline{q}}\left(\int_{n-1}^{n} v^{1-\overline{p}'}\right)^{-\frac{\overline{q}}{\overline{p}'}}\right\}^{\frac{1}{\overline{q}}}.$$

Thus the operator T_d is bounded from $\ell^{\overline{q}}\left(\left\{\left(\int_{n-1}^{n} v^{1-\overline{p}'}\right)^{-\frac{\overline{q}}{\overline{p}'}}\right\}\right)$ to $\ell^q\left(\left\{\left(\int_n^{n+1} u\right)^{\frac{q}{p}}\right\}\right)$ and therefore, by Theorem B, we have $\tilde{A} < \infty$.

Conversely, let us suppose that (i) or (ii) holds depending on the relationship between p and $\overline{p}$. Then, by Lemma 1,

$$\begin{aligned}||T_\varphi^- f||_{p,u,q} &\leqslant C\left\{\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1}(T_\varphi^- f\chi_{(-\infty,n-1)})^p u\right)^{\frac{q}{p}}\right.\\ &\qquad\left.+\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1}(T_\varphi^- f\chi_{(n-1,n+1)})^p u\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\ &\leqslant C\left\{\sum_{n\in\mathbb{Z}} T_d(\{a_m\})^q(n)\left(\int_n^{n+1} u\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}} + C\left\{\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1}(T_n f)^p u\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\ &= C(I_1+I_2),\end{aligned}$$

where $a_m = \int_{m-1}^m f$.

Since (i) or (ii) holds, by Theorem A we know that the operators T_n are uniformly bounded from $L^p(u,(n-1,n+1))$ to $L^{\overline{p}}(v,(n-1,n+1))$ and therefore, taking into account that $1 < \overline{q} \leqslant q < \infty$, we have

$$I_2 \leqslant C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+1} f^{\overline{p}}v\right)^{\frac{q}{\overline{p}}}\right\}^{\frac{1}{q}} \leqslant C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+1} f^{\overline{p}}v\right)^{\frac{\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}} \leqslant C||f||_{\overline{p},v,\overline{q}}.$$

On the other hand, since $\tilde{A} < \infty$, by Theorem B, T_d is bounded from

$\ell^{\overline{q}}\left(\left\{\left(\int_{n-1}^{n} v^{1-\overline{p}'}\right)^{-\frac{\overline{q}}{\overline{p}'}}\right\}\right)$ to $\ell^{q}\left(\left\{\left(\int_{n}^{n+1} u\right)^{\frac{q}{p}}\right\}\right)$ and Hölder inequality gives

$$\begin{aligned} I_1 &\leqslant C\left\{\sum_{n\in\mathbb{Z}} a_n^{\overline{q}}\left(\int_{n-1}^{n} v^{1-\overline{p}'}\right)^{-\frac{\overline{q}}{\overline{p}'}}\right\}^{\frac{1}{\overline{q}}} = C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n} f\right)^{\overline{q}}\left(\int_{n-1}^{n} v^{1-\overline{p}'}\right)^{-\frac{\overline{q}}{\overline{p}'}}\right\}^{\frac{1}{\overline{q}}} \\ &\leqslant C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n} f^{\overline{p}}v\right)^{\frac{\overline{q}}{\overline{p}}}\left(\int_{n-1}^{n} v^{1-\overline{p}'}\right)^{\frac{\overline{q}}{\overline{p}'}}\left(\int_{n-1}^{n} v^{1-\overline{p}'}\right)^{-\frac{\overline{q}}{\overline{p}'}}\right\}^{\frac{1}{\overline{q}}} \\ &= C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n} f^{\overline{p}}v\right)^{\frac{\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}} = C\|f\|_{\overline{p},v,\overline{q}}. \end{aligned}$$

□

The result corresponding to the case $1<q<\overline{q}<\infty$ is the following one:

THEOREM 2. *Let* $1<p,\overline{p}<\infty$, $1<q<\overline{q}<\infty$ *and* $\frac{1}{s}=\frac{1}{q}-\frac{1}{\overline{q}}$. *Suppose that* u, v *are locally integrable positive functions on* $\mathbb{R}$. *Then there exists a constant* $C>0$ *such that the inequality (1.2) holds for all nonnegative functions* f *if and only if*

(i) *in the case* $1<\overline{p}\leqslant p<\infty$, $\{C_n\}\in\ell^s$ *and* $\tilde{B}<\infty$;

(ii) *in the case* $1<p<\overline{p}<\infty$, $\{D_n\}\in\ell^s$ *and* $\tilde{B}<\infty$.

Proof. Let us suppose that (i) or (ii) holds. As in the proof of Theorem 1, we split the norm of $T_\varphi^- f$ into I_1 and I_2. Following the same steps we prove that I_1 (the global discrete part) is bounded by $C\|f\|_{\overline{p},v,\overline{q}}$. In this case, the relationship between q and $\overline{q}$ is not relevant . But we need to proceed in a different way in order to estimate I_2 (the local continuous part). We will apply the boundedness of T_n from $L^{\overline{p}}(v,(n-1,n+1))$ to $L^p(u,(n-1,n+1))$, Hölder inequality for sums with exponents $\frac{\overline{q}}{q}$ and $\frac{\overline{q}}{\overline{q}-q}$ and $\{J_n\}\in\ell^s$, where $J_n=C_n$ if $1<\overline{p}\leqslant p<\infty$ and $J_n=D_n$ if $1<p<\overline{p}<\infty$. Thus,

$$\begin{aligned} I_2 &\leqslant \left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+1} T_nf(x)^p u(x)dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}} \leqslant C\left\{\sum_{n\in\mathbb{Z}} J_n^q\left(\int_{n-1}^{n+1} f(x)^{\overline{p}}v(x)dx\right)^{\frac{q}{\overline{p}}}\right\}^{\frac{1}{q}} \\ &\leqslant C\left\{\left(\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+1} f(x)^{\overline{p}}v(x)dx\right)^{\frac{\overline{q}}{\overline{p}}}\right)^{\frac{q}{\overline{q}}}\left(\sum_{n\in\mathbb{Z}} J_n^{\frac{q\overline{q}}{\overline{q}-q}}\right)^{\frac{\overline{q}-q}{\overline{q}}}\right\}^{\frac{1}{q}} \\ &= C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+1} f(x)^{\overline{p}}v(x)dx\right)^{\frac{\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}}\left(\sum_{n\in\mathbb{Z}} J_n^s\right)^{\frac{1}{s}} \leqslant C\|f\|_{\overline{p},v,\overline{q}}. \end{aligned}$$

Suppose now that (1.2) holds. Working as in the proof of Theorem 1, we see that the operator T_d is bounded from $\ell^{\overline{q}}\left(\left\{\left(\int_{n-1}^{n} v^{1-\overline{p}'}\right)^{-\frac{\overline{q}}{\overline{p}'}}\right\}\right)$ to $\ell^{q}\left(\left\{\left(\int_{n}^{n+1} u\right)^{\frac{q}{p}}\right\}\right)$ and therefore, by Theorem B, we have $\tilde{B}<\infty$.

Assume that $1<\overline{p}\leqslant p<\infty$. As in Theorem 1, we find that the operators T_n are uniformly bounded from $L^{\overline{p}}(v,(n-1,n+1))$ to $L^{p}(u,(n-1,n+1))$, which gives $\sup_{n\in\mathbb{Z}} A_n^1<\infty$ and $\sup_{n\in\mathbb{Z}} A_n^0<\infty$. By the definition of A_n^1, for every $n\in\mathbb{Z}$ there exists $\beta_n\in(n-1,n+1)$ such that

$$A_n^1-\left(\int_{\beta_n}^{n+1} u(t)dt\right)^{\frac{1}{p}}\left(\int_{n-1}^{\beta_n} \varphi^{\overline{p}'}(\beta_n-t)v^{1-\overline{p}'}(t)dt\right)^{\frac{1}{\overline{p}'}}<\frac{1}{2^{|n|}}.$$

Since we have to prove that $\{A_n^1\}\in\ell^s$, it suffices to show that

$$\left\{\left(\int_{\beta_n}^{n+1} u(t)dt\right)^{\frac{1}{p}}\left(\int_{n-1}^{\beta_n} \varphi^{\overline{p}'}(\beta_n-t)v^{1-\overline{p}'}(t)dt\right)^{\frac{1}{\overline{p}'}}\right\}\in\ell^s.$$

Let $\{a_n\}$ be a sequence of nonnegative numbers and $f(x)=\sum_{k\in\mathbb{Z}} a_k\chi_{(k-1,\beta_k)}(x)$ $\varphi(\beta_k-x)^{\overline{p}'-1}v^{1-\overline{p}'}(x)$. If $n\in\mathbb{Z}$ and $x\in(\beta_n,n+1)$, then

$$\begin{aligned}
T_\varphi^- f(x) &\geqslant \int_{-\infty}^{x} a_n\chi_{(n-1,\beta_n)}\varphi(x-y)\varphi(\beta_n-y)^{\overline{p}'-1}v^{1-\overline{p}'}(y)dy\\
&\geqslant Ca_n\int_{n-1}^{\beta_n}\varphi(\beta_n-y)\varphi(\beta_n-y)^{\overline{p}'-1}v^{1-\overline{p}'}(y)dy\\
&= Ca_n\int_{n-1}^{\beta_n}\varphi(\beta_n-y)^{\overline{p}'}v^{1-\overline{p}'}(y)dy.
\end{aligned}$$

This inequality implies

$$\begin{aligned}
||T_\varphi^- f||_{p,u,q} &= \left\{\sum_{n\in\mathbb{Z}}\left(\int_{n}^{n+1} T_\varphi^- f(x)^p u(x)dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\
&\geqslant C_{p,q}\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+1} T_\varphi^- f(x)^p u(x)dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\
&\geqslant C_{p,q}\left\{\sum_{n\in\mathbb{Z}} a_n^q\left(\int_{\beta_n}^{n+1}\left(\int_{n-1}^{\beta_n}\varphi(\beta_n-y)^{\overline{p}'}v^{1-\overline{p}'}(y)dy\right)^{p} u(x)dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\
&= C_{p,q}\left\{\sum_{n\in\mathbb{Z}} a_n^q\left(\int_{n-1}^{\beta_n}\varphi(\beta_n-y)^{\overline{p}'}v^{1-\overline{p}'}(y)dy\right)^{q}\left(\int_{\beta_n}^{n+1} u(x)dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}.
\end{aligned}$$

On the other hand,

$$||f||_{\overline{p},v,\overline{q}} \leqslant C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+1} f^{\overline{p}}v\right)^{\frac{\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}} \leqslant C\left\{\sum_{n\in\mathbb{Z}} a_n^{\overline{q}}\left(\int_{n-1}^{\beta_n} \varphi(\beta_n-y)^{\overline{p}'}v^{1-\overline{p}'}(y)dy\right)^{\frac{\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}}.$$

Therefore, by (1.2), we have

$$\left\{\sum_{n\in\mathbb{Z}} a_n^q\left(\int_{n-1}^{\beta_n}\varphi(\beta_n-y)^{\overline{p}'}v^{1-\overline{p}'}(y)dy\right)^q\left(\int_{\beta_n}^{n+1}u(x)dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

$$\leqslant C\left\{\sum_{n\in\mathbb{Z}} a_n^{\overline{q}}\left(\int_{n-1}^{\beta_n}\varphi(\beta_n-y)^{\overline{p}'}v^{1-\overline{p}'}(y)dy\right)^{\frac{\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}}$$

for all sequences $\{a_n\}$, i.e., the identity is bounded from $\ell^{\overline{q}}\left(\left\{\left(\int_{n-1}^{\beta_n}\varphi(\beta_n-y)^{\overline{p}'}v^{1-\overline{p}'}(y)dy\right)^{\frac{\overline{q}}{\overline{p}}}\right\}\right)$ to $\ell^q\left(\left\{\left(\int_{n-1}^{\beta_n}\varphi(\beta_n-y)^{\overline{p}'}v^{1-\overline{p}'}(y)dy\right)^q\left(\int_{\beta_n}^{n+1}u(x)dx\right)^{\frac{q}{p}}\right\}\right)$.
Applying Lemma 2 we obtain

$$\left\{\left(\int_{n-1}^{\beta_n}\varphi(\beta_n-y)^{\overline{p}'}v^{1-\overline{p}'}(y)dy\right)^{\frac{1}{\overline{p}'}}\left(\int_{\beta_n}^{n+1}u(x)dx\right)^{\frac{1}{p}}\right\}\in\ell^s.$$

Let us prove now that $\{A_n^0\}\in\ell^s$. In order to do this, we observe that (1.2) is equivalent to the dual inequality

$$\left\{\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1}((T_\varphi^-)^*g)^{\overline{p}'}v^{1-\overline{p}'}\right)^{\frac{\overline{q}'}{\overline{p}'}}\right\}^{\frac{1}{\overline{q}'}} \leqslant C\left\{\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1}g^{p'}u^{1-p'}\right)^{\frac{q'}{p'}}\right\}^{\frac{1}{q'}},$$

where $(T_\varphi^-)^*g(x)=\int_x^\infty\varphi(y-x)g(y)dy$.

Working as above, in order to prove that $\{A_n^0\}\in\ell^s$ it suffices to show that

$$\left\{\left(\int_{\beta_n}^{n+1}\varphi^p(t-\beta_n)u(t)dt\right)^{\frac{1}{p}}\left(\int_{n-1}^{\beta_n}v^{1-\overline{p}'}(t)dt\right)^{\frac{1}{\overline{p}'}}\right\}\in\ell^s$$

where $\beta_n\in(n-1,n+1)$ verifies

$$A_n^0-\left(\int_{\beta_n}^{n+1}\varphi^p(t-\beta_n)u(t)dt\right)^{\frac{1}{p}}\left(\int_{n-1}^{\beta_n}v^{1-\overline{p}'}(t)dt\right)^{\frac{1}{\overline{p}'}}<\frac{1}{2^{|n|}}.$$

Let $\{a_n\}$ be a sequence of nonnegative numbers and $f(x) = \sum_{k\in\mathbb{Z}} a_k\chi_{(\beta_k,k+1)}(x)$ $\varphi(x-\beta_k)^{p-1}u(x)$. If $n\in\mathbb{Z}$ and $x\in(n-1,\beta_n)$, then

$$\begin{aligned}(T_\varphi^-)^*f(x) &\geqslant a_n\int_x^\infty \varphi(y-x)\chi_{(\beta_n,n+1)}(y)\varphi(y-\beta_n)^{p-1}u(y)dy\\ &\geqslant a_n\int_{\beta_n}^{n+1}\varphi(y-\beta_n)^p u(y)dy\end{aligned}$$

and we deduce

$$\begin{aligned}||(T_\varphi^-)^*f||_{\overline{p}',v^{1-\overline{p}'},\overline{q}'} &\geqslant C_{\overline{p},\overline{q}}\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+1}((T_\varphi^-)^*f)^{\overline{p}'}v^{1-\overline{p}'}\right)^{\frac{\overline{q}'}{\overline{p}'}}\right\}^{\frac{1}{\overline{q}'}}\\ &\geqslant C_{\overline{p},\overline{q}}\left\{\sum_{n\in\mathbb{Z}}a_n^{\overline{q}'}\left(\int_{n-1}^{\beta_n}v^{1-\overline{p}'}\right)^{\frac{\overline{q}'}{\overline{p}'}}\left(\int_{\beta_n}^{n+1}\varphi(y-\beta_n)^p u(y)dy\right)^{\overline{q}'}\right\}^{\frac{1}{\overline{q}'}}.\end{aligned}$$

The function f also verifies

$$\begin{aligned}||f||_{p',u^{1-p'},q'} &\leqslant C\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+1}f^{p'}u^{1-p'}\right)^{\frac{q'}{p'}}\right\}^{\frac{1}{q'}}\\ &\leqslant C\left\{\sum_{n\in\mathbb{Z}}a_n^{q'}\left(\int_{\beta_n}^{n+1}\varphi(y-\beta_n)^p u(y)dy\right)^{\frac{q'}{p'}}\right\}^{\frac{1}{q'}}.\end{aligned}$$

Thus from (1.2) we obtain

$$\left\{\sum_{n\in\mathbb{Z}}a_n^{\overline{q}'}\left(\int_{n-1}^{\beta_n}v^{1-\overline{p}'}\right)^{\frac{\overline{q}'}{\overline{p}'}}\left(\int_{\beta_n}^{n+1}\varphi(y-\beta_n)^p u(y)dy\right)^{\overline{q}'}\right\}^{\frac{1}{\overline{q}'}}$$

$$\leqslant C\left\{\sum_{n\in\mathbb{Z}}a_n^{q'}\left(\int_{\beta_n}^{n+1}\varphi(y-\beta_n)^p u(y)dy\right)^{\frac{q'}{p'}}\right\}^{\frac{1}{q'}}.$$

This means that the identity is bounded from $\ell^{q'}\left(\left\{\left(\int_{\beta_n}^{n+1}\varphi(y-\beta_n)^p u(y)dy\right)^{\frac{q'}{p'}}\right\}\right)$ to $\ell^{\overline{q}'}\left(\left\{\left(\int_{n-1}^{\beta_n}v^{1-\overline{p}'}\right)^{\frac{\overline{q}'}{\overline{p}'}}\left(\int_{\beta_n}^{n+1}\varphi(y-\beta_n)^p u(y)dy\right)^{\overline{q}'}\right\}\right)$ and Lemma 2 gives

$$\left\{ \left(\int_{n-1}^{\beta_n} v^{1-\overline{p}'} \right)^{\frac{1}{\overline{p}'}} \left(\int_{\beta_n}^{n+1} \varphi(y-\beta_n)^p u(y)dy \right)^{\frac{1}{p}} \right\} \in \ell^s.$$

Suppose now that $1 < p < \overline{p} < \infty$. Let us see that $\{B_n^0\} \in \ell^s$. Let $\{a_n\}$ be a sequence of nonnegative numbers and

$$\begin{aligned} f(x) &= \sum_{k\in\mathbb{Z}} a_k \chi_{(k-1,k+1)}(x) \left(\int_x^{k+1} \varphi(y-x)^p u(y)dy \right)^{\frac{r}{p\overline{p}}} \left(\int_{k-1}^x v^{1-\overline{p}'} \right)^{\frac{r}{\overline{p}p'}} v^{1-\overline{p}'}(x) \\ &= \sum_{k\in\mathbb{Z}} f_k(x). \end{aligned}$$

If $n \in \mathbb{Z}$, we have

$$\begin{aligned} \int_{n-1}^{n+1} (T_\varphi^- f(x))^p u(x)dx &= \int_{n-1}^{n+1} T_\varphi^- f(x)(T_\varphi^- f(x))^{p-1} u(x)dx \\ &\geqslant \int_{n-1}^{n+1} \left(\int_{n-1}^x \varphi(x-y) f_n(y)dy \right) \left(\int_{n-1}^x \varphi(x-s) f_n(s)ds \right)^{p-1} u(x)dx \\ &= \int_{n-1}^{n+1} f_n(y) \left(\int_y^{n+1} \varphi(x-y)u(x) \left(\int_{n-1}^x \varphi(x-s) f_n(s)ds \right)^{p-1} dx \right) dy \\ &\geqslant C \int_{n-1}^{n+1} f_n(y) \left(\int_y^{n+1} \varphi(x-y)u(x) \left(\int_{n-1}^y \varphi(x-y) f_n(s)ds \right)^{p-1} dx \right) dy \\ &= C \int_{n-1}^{n+1} f_n(y) \left(\int_y^{n+1} \varphi(x-y)^p u(x)dx \right) \left(\int_{n-1}^y f_n(s)ds \right)^{p-1} dy \\ &= C \int_{n-1}^{n+1} f_n(y) \left(\int_y^{n+1} \varphi(x-y)^p u(x)dx \right) \\ &\quad \times \left(\int_{n-1}^y a_n \left(\int_s^{n+1} \varphi(t-s)^p u(t)dt \right)^{\frac{r}{p\overline{p}}} \left(\int_{n-1}^s v^{1-\overline{p}'} \right)^{\frac{r}{\overline{p}p'}} v^{1-\overline{p}'}(s)ds \right)^{p-1} dy \\ &\geqslant C a_n^{p-1} \int_{n-1}^{n+1} f_n(y) \left(\int_y^{n+1} \varphi(x-y)^p u(x)dx \right)^{1+\frac{r(p-1)}{p\overline{p}}} \\ &\quad \times \left(\int_{n-1}^y \left(\int_{n-1}^s v^{1-\overline{p}'} \right)^{\frac{r}{\overline{p}p'}} v^{1-\overline{p}'}(s)ds \right)^{p-1} dy \end{aligned}$$

$$
\begin{aligned}
&= Ca_n^{p-1}\int_{n-1}^{n+1} f_n(y)\left(\int_y^{n+1}\varphi(x-y)^p u(x)dx\right)^{1+\frac{r}{p'\overline{p}}}\left(\int_{n-1}^{y} v^{1-\overline{p}'}\right)^{\frac{r}{\overline{p}'p'}} dy\\
&= Ca_n^{p}\int_{n-1}^{n+1}\left(\int_y^{n+1}\varphi(t-y)^p u(t)dt\right)^{\frac{r}{p}}\left(\int_{n-1}^{y} v^{1-\overline{p}'}\right)^{\frac{r}{p'}} v^{1-\overline{p}'}(y)dy\\
&= Ca_n^p(B_n^0)^r,
\end{aligned}
$$

which implies

$$
||T_\varphi^- f||_{p,u,q}\geqslant C\left\{\sum_{n\in\mathbb{Z}} a_n^q (B_n^0)^{\frac{rq}{p}}\right\}^{\frac{1}{q}}.
$$

Since

$$
||f||_{\overline{p},v,\overline{q}}\leqslant\left\{\sum_{n\in\mathbb{Z}}\left(\int_{n-1}^{n+1} f^{\overline{p}}v\right)^{\frac{\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}}\leqslant C\left\{\sum_{n\in\mathbb{Z}} a_n^{\overline{q}}(B_n^0)^{\frac{r\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}},
$$

(1.2) yields

$$
\left\{\sum_{n\in\mathbb{Z}} a_n^q (B_n^0)^{\frac{rq}{p}}\right\}^{\frac{1}{q}}\leqslant C\left\{\sum_{n\in\mathbb{Z}} a_n^{\overline{q}}(B_n^0)^{\frac{r\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}}
$$

for all sequences $\{a_n\}$ of nonnegative numbers and by Lemma 2, $\{B_n^0\}\in\ell^s$.

The proof of $\{B_n^1\}\in\ell^s$ follows the same pattern, but applying the dual inequality of (1.2) to the function

$$
f(x)=\sum_{k\in\mathbb{Z}} a_k\chi_{(k-1,k+1)}(x)\left(\int_x^{k+1}u\right)^{\frac{r}{p'\overline{p}}}\left(\int_{k-1}^{x} v^{1-\overline{p}'}(y)\varphi(x-y)^{\overline{p}'}dy\right)^{\frac{r}{\overline{p}'p'}} u(x).
$$

□

4. Higher order Hardy inequalities

As we mentioned in the introduction, in this section we characterize the pairs of weights (u,v) such that the higher order Hardy inequality (1.3) holds for all $F\in AC_L^{(k-1)}(-\infty,\infty)$. It is well known ([6]) that (1.3) holds if and only if the operator

$$
Tf(x)=\int_{-\infty}^{x}(x-t)^{k-1}f(t)dt
$$

verifies

$$
||Tf||_{p,u,q}\leqslant C||f||_{\overline{p},v,\overline{q}}.
$$

Since T is one of the operators considered in section 3 (corresponding to $\varphi(t) = t^{k-1}$), by applying Theorems 1 and 2 to this particular φ we obtain the desired characterizations:

THEOREM 3. *Let u, v be positive measurable functions of one real variable. Then there exists a constant $C > 0$ such that the inequality (1.3) holds for all $F \in AC_L^{(k-1)}(-\infty,\infty)$ if and only if*

(i) *in the case $1 < \overline{p} \leqslant p < \infty$ and $1 < \overline{q} \leqslant q < \infty$, $\sup_{n\in\mathbb{Z}} C_n^k < \infty$ and $\tilde{A}^k < \infty$;*

(ii) *in the case $1 < p < \overline{p} < \infty$ and $1 < \overline{q} \leqslant q < \infty$, $\sup_{n\in\mathbb{Z}} D_n^k < \infty$ and $\tilde{A}^k < \infty$;*

(iii) *in the case $1 < \overline{p} \leqslant p < \infty$ and $1 < q < \overline{q} < \infty$, $\{C_n^k\}_n \in \ell^s$ and $\tilde{B}^k < \infty$;*

(iv) *in the case $1 < p < \overline{p} < \infty$ and $1 < q < \overline{q} < \infty$, $\{D_n^k\}_n \in \ell^s$ and $\tilde{B}^k < \infty$.*

REFERENCES

[1] C. CARTON-LEBRUN, H. P. HEINIG AND S. C. HOFMANN, *Integral operators on weighted amalgams*, Studia Math. 109 (2) (1994), 133–157.
[2] J. J. F. FOURNIER AND J. STEWART, *Amalgams of L^p and ℓ^q*, Bull. Amer. Math. Soc. 13 (1) (1985), 1–21.
[3] H. P. HEINIG AND A. KUFNER, *Weighted Friedrichs inequalities in amalgams*, Czechoslovak. Math. J. 43 (118) (1993), no. 2, 285–308.
[4] A. KUFNER AND L. E. PERSSON, *Weighted inequalities of Hardy type*, World Scientific, 2003.
[5] F. J. MARTÍN-REYES AND E. SAWYER, *Weighted inequalities for Riemann-Liouville fractional integrals of order one and greater*, Proc. Amer. Math. Soc. 106 (1989), no. 3, 727–733.
[6] B. OPIC AND A. KUFNER, *Hardy-type inequalities*, Longman, 1990.
[7] P. ORTEGA SALVADOR AND C. RAMÍREZ TORREBLANCA, *Hardy operators on weighted amalgams*, preprint.
[8] Y. RAKOTONDRATSIMBA, *Fractional maximal and integral operators on weighted amalgam spaces*, J. Korean Math. Soc. 36 (1999), no. 5, 855–890.
[9] V. D. STEPANOV, *Two-weighted estimates for Riemann-Liouville integrals*, Math. SSSR Izv. 36 (1991), no. 3, 669–681.
[10] N. WIENER, *On the representation of functions by trigonometric integrals*, Math. Z. 24 (1926), 575–616.

(Received May 28, 2008)

M. Isabel Aguilar Cañestro
Análisis Matemático
Facultad de Ciencias
Universidad de Málaga
29071 Málaga, Spain
e-mail: mi_aguilar@hotmail.com

Pedro Ortega Salvador
Análisis Matemático
Facultad de Ciencias
Universidad de Málaga
29071 Málaga, Spain
e-mail: portega@uma.es