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A REFINED REVERSE ISOPERIMETRIC INEQUALITY IN THE PLANE

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Abstract. It is proved that if γ is a closed strictly convex curve in the plane with length L and area A, then

$$L^2 \leqslant 4\pi A + 2\pi|\tilde{A}|,$$

with equality holding if and only if γ is a circle, where $\tilde{A}$ denotes the oriented area enclosed by the locus of curvature centers of γ.

1. Introduction

The *classical isoperimetric inequality* in the Euclidean plane $\mathbb{R}^2$ states that for a simple closed curve γ of length L, enclosing a region of area A, one gets

$$L^2 - 4\pi A \geqslant 0, \tag{1.1}$$

and the equality holds if and only if γ is a circle. This fact was known to the ancient Greeks, the first complete mathematical proof was only given in 1882 by Edler [7] (based on the arguments of Steiner [21]). Since then, there have been various proofs, sharpened forms, generalizations and applications of this inequality, see, for instance, papers by Gardner [8], Klain [12], Lawlor [13], survey articles by Blasjö [3], Osserman [14], [15] and Talenti [22], and books by Bandle [2], Bonnesen-Fenchel [4], Burago-Zelgaller [5], Chavel [6], Santaló [19], and Schneider [20] and the literature therein.

In 1995, Howard and Treibergs ([10]) gave a reverse isoperimetric inequality for the plane curves under some assumption on curvature. In the paper [18] by Pan and Zhang there is a reverse isoperimetric inequality for closed strictly convex plane curve γ with length L and area A,

$$L^2 \leqslant 4\pi(A + |\tilde{A}|), \tag{1.2}$$

where $\tilde{A}$ denotes the oriented area of the domain enclosed by the locus of curvature centers of γ, and the equality in (1.2) holds if and only if γ is a circle. In the present note, it shows a new reverse isoperimetric inequality

$$L^2 \leqslant 4\pi A + 2\pi|\tilde{A}|, \tag{1.3}$$

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and the equality holds if and only if γ is a circle. (1.3) is a strengthened form of (1.2) and is a direct consequence of Proposition 2.2 below and the following isoperimetric inequality (Theorem 3.1): for closed strictly convex planar curves,

$$\int_0^{2\pi} \rho^2(\theta)d\theta \geqslant \frac{L^2 - 2\pi A}{\pi} \tag{1.4}$$

where ρ is the radius of curvature of the curve γ, and furthermore, the equality in (1.4) holds if and only if γ is a circle. To our surprise, inequality (1.4) is also essential to the perimeter-preserving curve expanding flow in the plane studied in [17].

REMARKS. (i) It should be pointed out that the above reverse isoperimetric inequalities (1.2) and (1.3) are obtained by the integration of the radius of curvature, our curves must be strictly convex. We wonder if this sort of inequalities can be obtained for any (simple) closed plane curves. And furthermore, it would be interesting to generalize these inequalities to the higher dimensional spaces or manifolds. (ii) Another open problem is that if there is a *best constant* C such that

$$L^2 \leqslant 4\pi A + C|\tilde{A}|, \tag{1.5}$$

where the equality holds if and only if γ is a circle. we conjecture that the best constant in (1.5) would be π; (iii) In higher dimensions, Ball [1] gave a reverse isoperimetric inequality through volume ratios.

This note is arranged as follows. §2 contains some preliminaries. §3 provides a detailed outline of the proof of inequality (1.4), and finally, §4 gives the strengthened reverse isoperimetric inequality (1.3) and its corollary.

2. Preliminaries

2.1. Minkowski's Support Function

From now on, without loss of generality, suppose that γ is a smooth regular positively oriented and closed strictly convex curve in the plane $\mathbb{R}^2$. Take a point O inside γ as the origin of our frame. Let p be the *Minkowski support function* of γ which measures the oriented perpendicular distance from O to the tangent line at a point on γ, and let θ be the oriented angle from the positive x_1-axis to this perpendicular ray. Clearly, p is a single-valued 2π-periodic function of θ and γ can be parameterized in terms of θ and $p(\theta)$ as follows

$$\gamma(\theta) = \big(\gamma_1(\theta),\, \gamma_2(\theta)\big) = \big(p(\theta)\cos\theta - p'(\theta)\sin\theta,\, p(\theta)\sin\theta + p'(\theta)\cos\theta\big), \tag{2.1}$$

(see for instance [11]).

The geometric quantities of γ can be expressed in terms of its Minkowski support function $p(\theta)$. Its curvature k can be calculated according to

$$k(\theta) = \frac{d\theta}{ds} = \frac{1}{p(\theta) + p''(\theta)} > 0, \tag{2.2}$$

where s is the arc-length parameter of γ. Or equivalently, the radius of curvature ρ is given by

$$\rho(\theta) = \frac{ds}{d\theta} = p(\theta) + p''(\theta). \tag{2.3}$$

Denote L and A the length of γ and the area it bounds, respectively. Then one can get the following Cauchy's and Blaschke's formulae

$$L = \int_\gamma ds = \int_0^{2\pi} \rho(\theta)d\theta = \int_0^{2\pi} p(\theta)d\theta, \tag{2.4}$$

$$\begin{aligned} A &= \frac{1}{2}\int_\gamma p(\theta)ds = \frac{1}{2}\int_0^{2\pi} p(\theta)\Big(p(\theta)+p''(\theta)\Big)d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \Big[p^2(\theta) - p'^2(\theta)\Big]d\theta. \end{aligned} \tag{2.5}$$

2.2. Some Properties of the Locus of Curvature Centers

Let β represent the locus of curvature centers of a closed and strictly convex plane curve γ which is given by (2.1). Then $\beta(\theta) = \big(\beta_1(\theta),\ \beta_2(\theta)\big)$ can be expressed as

$$\beta(\theta) = \gamma(\theta) - \rho(\theta)\mathbf{N}(\theta) = \big(-p'(\theta)\sin\theta - p''(\theta)\cos\theta, p'(\theta)\cos\theta - p''(\theta)\sin\theta\big), \tag{2.6}$$

where $\mathbf{N}(\theta) = (\cos\theta, \sin\theta)$ is the unit outward normal vector field along γ.

The following two propositions are essential to the main result of this note, whose proofs are contained in [18].

PROPOSITION 2.1. *The oriented area of the domain enclosed by β is nonpositive. And moreover, if β is simple, then the orientation of β is the reverse direction against that of the original curve γ and the total curvature of β is equal to -2π.*

PROPOSITION 2.2. *Let γ be a C^2 closed and strictly convex curve in the plane, ρ the radius of curvature of γ, A the area enclosed by γ and $\tilde{A}$ the oriented area enclosed by β. Then we have*

$$\int_0^{2\pi} \rho^2 d\theta = 2(A + |\tilde{A}|). \tag{2.7}$$

2.3. The Unit-speed Outward Normal Flow

Consider a closed convex plane curve $\gamma(\theta) \triangleq \gamma_0(\theta)$ and suppose it moves so that the velocity vector at each point on the curve is equal to its outward unit normal vector at that point. The unit-speed outward normal flow is to find a family of simple closed plane curves $\gamma(\theta,t)$ with initial curve $\gamma(\theta,0)$ being equal to $\gamma_0(\theta)$. Thus the evolution problem in question can be expressed as follows

$$\begin{cases} \dfrac{\partial\gamma(\theta,t)}{\partial t} = \mathbf{N}(\theta), \\ \gamma(\theta,0) = \gamma_0(\theta). \end{cases} \tag{2.8}$$

In this case, the first author of the present note has shown in [16] that the tangent vector field $\mathbf{T}$ and the unit outward normal vector field $\mathbf{N}$ are independent of the time t. And futthermore

LEMMA 2.3. *Under the evolution defined by (2.8), let $\gamma(\theta,t)$ be the curve at time $t \geqslant 0$, we have the following formulas:*

$$\rho(\theta,t) = \rho(\theta,0) + t; \tag{2.9}$$

$$k(\theta,t) = \frac{k(\theta,0)}{1+k(\theta,0)t}; \tag{2.10}$$

$$L(t) = L(0) + 2\pi t; \tag{2.11}$$

$$A(t) = A(0) + L(0)t + \pi t^2, \tag{2.12}$$

where $\rho(\theta,t)$ and $k(\theta,t)$ are the radius of curvature and the curvature, $L(t)$ and $A(t)$ are the length of the evolving curve and the area it encloses at time t, respectively.

(2.12) is usually called *the Steiner polynomial* for the evolving curve. It is easy to check that the isoperimetric defect $L^2 - 4\pi A$ of the evolving curve is invariant under the unit-speed outward normal follow.

3. A Newly Obtained Isoperimetric-type Inequality

The following isoperimetric inequality has appeared in [17], for completeness of the present note, we outline its proof below.

THEOREM 3.1. *For a C^2 closed and strictly convex curve γ, L and A are the length of γ and the area it encloses, one gets*

$$\int_0^{2\pi} \rho^2(\theta)d\theta \geqslant \frac{L^2 - 2\pi A}{\pi} \tag{3.1}$$

And moreover, the equality in (3.1) holds if and only if γ is a circle.

Proof. It is obvious that the equality in (3.1) holds when γ is a circle. If we can prove that $\int_0^{2\pi} \rho^2(\theta)d\theta > \frac{L^2-2\pi A}{\pi}$ when γ is not a circle, then the result holds. This can be concluded by proving the following theorem. □

THEOREM 3.2. *If γ is a C^2 closed strictly convex and non-circular curve in the plane, then*

$$\int_0^{2\pi} \rho^2(\theta)d\theta > \frac{L^2 - 2\pi A}{\pi} \tag{3.2}$$

holds.

We will make full use of the unit-speed outward normal flow to prove Theorem 3.2. To this end, we need some definitions.

DEFINITION 3.3. Let $t_1 \geqslant t_2$ be the roots of the Steiner polynomial $A(t)$ (see (2.12) above), r_i and r_e be the radii of the largest inscribed and the smallest circumscribed circles of γ (called the *inradius* and the *outradius* of γ), respectively. Let k be the curvature of γ , $\rho = \frac{1}{k}$ the radius of curvature , and $\rho_{\max}$ and $\rho_{\min}$ the maximum and the minimum values of ρ . These quantities are all equal if the curve γ is a circle.

LEMMA 3.4. *If γ is convex and non-circle, then*

$$-\rho_{\max} < t_2 < -r_e < -\frac{L}{2\pi} < -r_i < t_1 < -\rho_{\min}. \tag{3.3}$$

Proof. See Green and Osher [9]. □

DEFINITION 3.5. Consider

$$\sup\left\{\int_I \rho(\theta)d\theta | I \subset S^1, \int_I d\theta = \pi\right\}.$$

Let I_1 denote a subset of S^1 of measure π realizing this bound , and let I_2 be its complement. There exists a real number a such that

$$I_1 \subseteq \{\theta | \rho(\theta) \geqslant a\}, \qquad I_2 \subseteq \{\theta | \rho(\theta) \leqslant a\}.$$

We set

$$\rho_1 = \frac{1}{\pi}\int_{I_1} \rho(\theta)d\theta, \qquad \rho_2 = \frac{1}{\pi}\int_{I_2} \rho(\theta)d\theta,$$

Note that

$$\rho_1 + \rho_2 = \frac{L}{\pi}, \qquad \rho_1 \geqslant \rho_2.$$

PROPOSITION 3.6. *Let γ be a strictly convex curve, if it is not a circle, then*

$$\rho_1 > \rho_2.$$

In other words, there exists a real number $b > 0$ such that

$$\rho_1 = \frac{L}{2\pi} + b, \qquad \rho_2 = \frac{L}{2\pi} - b.$$

Proof. By Definition 3.5 we know that $\rho_1 \geqslant \rho_2$, thus it suffices to prove that $\rho_1 = \rho_2$ implies that γ is a circle. □

PROPOSITION 3.7. *For γ a symmetric strictly convex curve and not a circle, then*

$$\rho_1 > -t_2.$$

Proof. The symmetry of the curve dictates that the breadth of the curve $b_{N(\theta)} = 2p(\theta)$ in the direction determined by the normal $\mathbf{N}(\theta)$ is twice the support function $p(\theta)$. Since γ is convex, the breadth satisfies $2r_i \leqslant b_{N(\theta)} \leqslant 2r_e$, and thus, for all θ,

$$r_i \leqslant p(\theta) \leqslant r_e.$$

This together with Lemma 3.4 implies that

$$-t_1 < p(\theta) < -t_2$$

holds for all θ. We recall that

$$-t_1 = \frac{L}{2\pi} - u, \qquad -t_2 = \frac{L}{2\pi} + u.$$

where

$$u = \frac{\sqrt{L^2 - 4\pi A}}{2\pi}.$$

Since γ is not a circle,

$$u > 0.$$

Combining these formulae we have

$$-u < p(\theta) - \frac{L}{2\pi} < u.$$

From Proposition 3.6, we obtain

$$\rho_1 = \frac{L}{2\pi} + b, \qquad \rho_2 = \frac{L}{2\pi} - b.$$

for some $b > 0$. The inequality we are trying to prove, $\rho_1 > -t_2$, is equivalent to

$$b > u.$$

On I_1, $\rho(\theta) \geqslant a$ and on I_2, $\rho(\theta) \leqslant a$, while $\rho(\theta) \equiv a$ holds on at most one interval, unless γ is a circle.

We assume that $\rho(\theta) > a$ on a small interval $I_1^{(1)}$ of I_1, it is true because $\rho(\theta)$ is a continuous function. Thus, on this subinterval $I_1^{(1)}$,

$$-\left(p(\theta) - \frac{L}{2\pi}\right)(\rho(\theta) - a) < u(\rho(\theta) - a).$$

Integrating this on the interval I_1 gives us

$$-\frac{1}{\pi}\int_{I_1}\left(p(\theta) - \frac{L}{2\pi}\right)(\rho(\theta) - a)d\theta < u(\rho_1 - a). \tag{3.4}$$

On I_2, $\rho(\theta) - a \leqslant 0$, we have

$$-\left(p(\theta) - \frac{L}{2\pi}\right)(\rho(\theta) - a) \leqslant -u(\rho(\theta) - a).$$

Integrating this on the interval I_2 gives us

$$-\frac{1}{\pi}\int_{I_2}\left(p(\theta) - \frac{L}{2\pi}\right)(\rho(\theta) - a)d\theta \leqslant -u(\rho_2 - a). \tag{3.5}$$

Adding (3.4) and (3.5) yields

$$-\frac{1}{\pi}\int_{S^1}\left(p(\theta)-\frac{L}{2\pi}\right)(\rho(\theta)-a)d\theta < u(\rho_1-\rho_2).$$

The left-hand side can be simplified to

$$\frac{2}{4\pi^2}(L^2-4\pi A)=2u^2,$$

and the right-hand side is $2ub$. The inequality is then $2u^2<2ub$, and thus $u<b$, as desired. $\square$

PROPOSITION 3.8. *For γ a strictly convex curve and not a circle, then*

$$\rho_1 > -t_2.$$

Proof. We proceed by a symmetrization. Given γ, for any θ, we divide γ into two curves by joining the points on γ corresponding to θ and $\theta+\pi$ by a straight line. Let L_1, L_2 be the lengths of the two pieces of γ and A_1, A_2 the areas of the two halves; thus

$$L=L_1+L_2, \qquad A=A_1+A_2.$$

Choose θ so that

$$(2L_1)^2-8\pi A_1=(2L_2)^2-8\pi A_2, \tag{3.6}$$

Call this quantity β.

Let γ_1 be the symmetric convex curve obtained by joining the first half of γ to a copy of itself rotated by 180 degrees, and γ_2 a symmetric convex curve obtained by doing the same thing to the second half. Thus γ_i has perimeter $2L_i$ and area $2A_i$ for i=1,2.

Now, for symmetric curves, we may take the subset I_1 of the circle to be symmetric. Since $\int_I \rho(\theta)d\theta$ is maximized by I_1 among all subsets of measure π, it follows that

$$\rho_1(\gamma)\geqslant\frac{1}{2\pi}\left[\int_{I_1(\gamma_1)}\rho(\theta)d\theta+\int_{I_1(\gamma_2)}\rho(\theta)d\theta\right]=\frac{1}{2}[\rho_1(\gamma_1)+\rho_1(\gamma_2)].$$

Since γ is not a circle, there exists at least one curve γ_1 or γ_2 not be a circle. We assume that γ_1 is not a circle.

Now, by applying Proposition 3.7 to the symmetric curves γ_1, γ_2,

$$\rho_1(\gamma_1)>\frac{2L_1}{2\pi}+\frac{\sqrt{(2L_1)^2-8\pi A_1}}{2\pi}, \quad \rho_1(\gamma_2)\geqslant\frac{2L_2}{2\pi}+\frac{\sqrt{(2L_2)^2-8\pi A_2}}{2\pi}.$$

Thus,

$$\rho_1(\gamma)>\frac{L_1+L_2}{2\pi}+\frac{1}{2}\left\{\frac{\sqrt{(2L_1)^2-8\pi A_1}}{2\pi}+\frac{\sqrt{(2L_2)^2-8\pi A_2}}{2\pi}\right\}.$$

The inequality we want is

$$\rho_1(\gamma) > \frac{L}{2\pi} + \frac{\sqrt{(L_1+L_2)^2 - 4\pi(A_1+A_2)}}{2\pi}.$$

It will be done if we can show that

$$\beta \geqslant (L_1+L_2)^2 - 4\pi(A_1+A_2),$$

Now,

$$\begin{aligned}\beta &= 2L_1^2 - 4\pi A_1 + 2L_2^2 - 4\pi A_2 \\ &= (L_1+L_2)^2 + (L_1-L_2)^2 - 4\pi(A_1+A_2) \\ &\geqslant (L_1+L_2)^2 - 4\pi(A_1+A_2).\end{aligned}$$

This proves the proposition. □

The following two elementary lemmata have appeared in Green and Osher [9], we omitted their proofs here.

LEMMA 3.9. *Let $F(x)$ be a convex function on $(0,+\infty)$, then*

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta \geqslant \frac{1}{2}[F(\rho_1)+F(\rho_2].$$

LEMMA 3.10. *If $F(x)$ is strictly convex on $(0,+\infty)$, then for $b > a > 0$ and c arbitrary, one gets*

$$F(c-a)+F(c+a) < F(c-b)+F(c+b). \qquad \square$$

Proof of Theorem 3.2. We already have that

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))d\theta \geqslant \frac{1}{2}[F(\rho_1)+F(\rho_2)],$$

Now

$$\rho_1 = \frac{L}{2\pi}+b, \quad \rho_2 = \frac{L}{2\pi}-b, \quad -t_1 = \frac{L}{2\pi}-u, \quad -t_2 = \frac{L}{2\pi}+u.$$

By Proposition 3.8, $b > u > 0$, and so by Lemma 3.10,

$$F(\rho_1)+F(\rho_2) > F(-t_1)+F(-t_2). \tag{3.7}$$

Takeing $F(x) = x^2$ and using Lemma 3.9 we get

$$\frac{1}{2\pi}\int_0^{2\pi} \rho^2(\theta)d\theta \geqslant \frac{1}{2}(\rho_1^2+\rho_2^2).$$

The above inequality (3.7) is

$$\rho_1^2+\rho_2^2 > t_1^2+t_2^2,$$

and t_1, t_2 are the roots of $A(t) = \pi t^2 + Lt + A = 0$.Thus

$$t_1^2 + t_2^2 = \frac{L^2 - 2\pi A}{\pi^2}.$$

All the results indicate that

$$\int_0^{2\pi} \rho^2(\theta) d\theta > \frac{L^2 - 2\pi A}{\pi},$$

which proves the theorem. □

4. A Reverse Isoperimetric Inequality

Now, from Theorem 3.1 and Proposition 2.2 above, one can easily get our main result.

THEOREM 4.1. *If γ is a closed strictly convex plane curve with length L and enclosing an area A, let $\tilde{A}$ denote the oriented area bounded by its locus of centers of curvature, then we get*

$$L^2 \leqslant 4\pi A + 2\pi |\tilde{A}|, \tag{4.1}$$

and the equality holds if and only if γ is a circle.

The following corollary is a direct consequence of the classical isoperimetric inequality (1.1) and our reverse isoperimetric inequality (4.1).

COROLLARY 4.2. *Let β be the locus of curvature centers of a closed strictly convex plane curve γ. Then the oriented area $\tilde{A}$ of β is zero if and only if γ is a circle and thus β can only be the center of γ.*

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