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ON CERTAIN SUBCLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS INVOLVING A LINEAR OPERATOR

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Abstract. The purpose of the present paper is to derive some inclusion relationships and other interesting properties of a certain subclass $\sum_p^+(a,c,A,B)$ of meromorphically p-valent functions with positive coefficients which are defined by means of a linear operator. The familiar concept of neighborhood of analytic functions is extended and applied to meromorphically p-valent functions considered here. We also derive many interesting results on the Hadamard product of functions belonging to the class $\sum_p^+(a,c,A,B)$.

1. Introduction and Definitions

Let $\sum_p$ be the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in \mathbb{N} = \{1,2,3,\cdots\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$.

If f and g are analytic in $\mathbb{U}$, we say that the function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, $z \in \mathbb{U}$, if there exists a Schwarz function ω in $\mathbb{U}$ such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. If g is univalent in $\mathbb{U}$, then the following equivalence relationship holds true:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Given two functions $f, g \in \sum_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \quad (p \in \mathbb{N}; z \in \mathbb{U}^*),$$

the Hadamard product (or convolution) of f and g, denoted by $f * g$, is defined by the power series:

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z) \quad (z \in \mathbb{U}^*).$$

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We note that $f * g \in \sum_p$.

In terms of the Pochhammer symbol (or *shifted factorial*) $(x)_n$ given by

$$(x)_n = \begin{cases} 1 & (n=0) \\ x(x+1)\cdots(x+n-1) & (n \in \mathbb{N}), \end{cases}$$

we define the function ϕ_p by

$$\phi_p(a,c;z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k-p} \quad \left(a \in \mathbb{R},\ c \in \mathbb{R} \setminus \mathbb{Z}_0^-;\ \mathbb{Z}_0^- = \{0,-1,-2,\cdots\};\ z \in \mathbb{U}^*\right). \tag{1.2}$$

Corresponding to the function ϕ_p, we consider the linear operator $\mathscr{L}_p(a,c) : \sum_p \longrightarrow \sum_p$ defined by

$$\mathscr{L}_p(a,c)f(z) = \phi_p(a,c;z) * f(z) \quad \left(f \in \sum\nolimits_p\right). \tag{1.3}$$

The linear operator $\mathscr{L}_p(a,c)$ was introduced and studied by Liu and Srivastava [9]. From (1.2) and (1.3), it follows that

$$z\left(\mathscr{L}_p(a,c)f(z)\right)' = a\,\mathscr{L}_p(a+1,c)f(z) - (a+p)\mathscr{L}_p(a,c)f(z) \quad (z \in \mathbb{U}^*). \tag{1.4}$$

We note that for $f \in \sum_p$,
$\mathscr{L}_p(a,a)f(z) = f(z)$, $\mathscr{L}_p(2,1)f(z) = (p+1)f(z) + zf'(z)$ and for any integer $n > -p$, $\mathscr{L}_p(n+p,1)f(z) = \mathscr{D}^{n+p-1}f(z) = \dfrac{1}{z^p(1-z)^{n+p}} * f(z)$, where $\mathscr{D}^{n+p-1}$ is the differential operator studied by Uralegaddi and Somanatha [12].

Making use of the operator $\mathscr{L}_p(a,c)$, we introduce a subclass of $\sum_p$ as follows:

DEFINITION 1.1. A function $f \in \sum_p$ is said to be in the class $f \in \sum_p(a,c,A,B)$, if it satisfies

$$-z^{p+1}\left(\mathscr{L}_p(a,c)f(z)\right)' \prec \frac{p(1+Az)}{1+Bz} \quad (a \in \mathbb{R}, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, -1 \leqq B < A \leqq 1;\ z \in \mathbb{U}), \tag{1.5}$$

where the symbol "$\prec$" stands for subordination. For the sake of convenience, we write

$$\sum\nolimits_p\left(a,c,1-\frac{2\alpha}{p},-1\right) = \sum\nolimits_p(a,c;\alpha), \tag{1.6}$$

where $\sum_p(a,c;\alpha)$ is the class of functions in $\sum_p$ satisfying the condition

$$-\Re\left\{z^{p+1}\left(\mathscr{L}_p(a,c)f(z)\right)'\right\} > \alpha \quad (0 \leqq \alpha < p;\ z \in \mathbb{U})$$

and for $a = c = 1$ in (1.6), we get the subclass $\sum_p(\alpha)$ consisting of functions in $\sum_p$ and satisfying the inequality:

$$-\Re\left\{z^{p+1}f'(z)\right\} > \alpha \quad (0 \leqq \alpha < p;\ z \in \mathbb{U}),$$

i.e., the class of meromorphically p-valent close-to-convex functions of order α in $\mathbb{U}^*$. Further, we write by $\sum_p^+$, the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} a_k z^k \quad (a_k \geqq 0, p \in \mathbb{N}) \tag{1.7}$$

that are analytic and p-valent in $\mathbb{U}^*$. We say that a function $f \in \sum_p^+$ is in the class $\sum_p^+(a,c,A,B)$, if it satisfies the condition (1.5). We denote

$$\sum\nolimits_p^+(a,c;\alpha) = \sum\nolimits_p^+ \cap \sum\nolimits_p(a,c;\alpha) \quad \text{and} \quad \sum\nolimits_p^+(\alpha) = \sum\nolimits_p^+ \cap \sum\nolimits_p(\alpha) \quad (0 \leqq \alpha < p).$$

In particular, we have the following observations:

(i) $\sum_1(a,a;\alpha) = MC(\alpha)$ $(0 \leqq \alpha < 1)$, the class of meromorphic close-to-convex functions of order α studied by in [7].

(ii) $\sum_p^+(a,a,A,B) = \mathscr{H}(p;A,B)$, the class introduced and studied by Mogra [10].

(iii) $\sum_p(n+p,1,A,B) = \mathscr{C}_{n,p}(A,B)$, the class introduced and studied by Uralegaddi and Somanatha [12].

(iv) $\sum_p\left(n+p,1,1-\dfrac{2\alpha}{p},-1\right) = \sum_{n,p}(\alpha)$ $(0 \leqq \alpha < p)$, the class considered by Cho and Nunokawa [6].

Meromorphically multivalent functions with positive coefficients have also been extensively studied by (for example) Uralegaddi and Ganigi [11] and Aouf [3, 4, 5].

In the present paper, we derive an inclusion relationship of the subclass $\sum_p^+(a,c,A,B)$ which is defined here by means of the linear operator $\mathscr{L}_p(a,c)$. The familiar concept of neighborhoods of analytic functions is extended and applied to the functions belonging to the class $\sum_p^+(a,c,A,B)$. Some interesting results on the Hadamard product for functions in the class $\sum_p^+(a,c,A,B)$ are also obtained.

Unless otherwise mentioned, we assume throughout this paper that $a > 0$, $c > 0$, $p \in \mathbb{N}$ and $-1 \leqq B < A \leqq 1$ $(-1 \leqq B < 0)$.

2. Inclusion relationship of the class $\sum_p^+(a,c,A,B)$

To prove our results, we need the following lemma. A more general form of this lemma can be found in [5, Theorem 4].

LEMMA 2.1. *Let $f \in \sum_p^+$ be given by* (1.7). *Then $f \in \sum_p^+(a,c,A,B)$, if and only if*

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}} a_k \leqq 1.$$

The result is the best possible for the functions f_k, given by

$$f_k(z) = z^{-p} + \frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}} z^k \quad (k \geqq p;\ z \in \mathbb{U}^*).$$

THEOREM 2.1. *We have*

$$\sum_p^+(a+1,c,A,B) \subset \sum_p^+(a,c,A',B) \quad \left(A' = B + \frac{a(A-B)}{2p+a}\right).$$

The result is the best possible.

Proof. Let f, defined by (1.7) be in the class $\sum_p^+(a+1,c,A,B)$. Then by Lemma 2.1,

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a+1)_{p+k}}{p(A-B)(c)_{p+k}} a_k \leqq 1. \tag{2.1}$$

To prove that $f \in \sum_p^+(a,c,A',B)$, we need to find the largest A' such that

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A'-B)(c)_{p+k}} a_k \leqq 1.$$

In view of (2.1), it is enough to show that

$$\frac{k(1-B)(a)_{p+k}}{p(A'-B)(c)_{p+k}} \leqq \frac{k(1-B)(a+1)_{p+k}}{p(A-B)(c)_{p+k}} \quad (k \geqq p),$$

which is equivalent to

$$B + \frac{(A-B)(a)_{p+k}}{(a+1)_{p+k}} \leqq A' \quad (k \geqq p). \tag{2.2}$$

Since $(a)_{p+k}/(a+1)_{p+k}$ is a decreasing function of k, putting $k=p$ in (2.2), we get the required result. $\square$

It is easily seen that the result is the best possible for the function

$$f(z) = z^{-p} + \frac{(A-B)(c)_{2p}}{(1-B)(a+1)_{2p}} z^p \quad (z \in \mathbb{U}^*). \tag{2.3}$$

Setting $a=c=1$, $A=1-(2\alpha/p)$ and $B=-1$ in Theorem 2.1, we get

COROLLARY 2.1. *If $f \in \sum_p^+$ satisfies*

$$-\Re\left[z^{p+1}\left\{(p+2)f'(z)+zf''(z)\right\}\right] > \alpha \quad (0 \leqq \alpha < p;\ z \in \mathbb{U}), \tag{2.4}$$

then $f \in \sum_p^+(\beta)$, where

$$\beta = p - \frac{p-\alpha}{2p+1}.$$

The result is the best possible.

3. Neighborhoods results

Following the earlier work (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [8], Ruscheweyh [13], Altintas [1] and also by Altintas et al. [2], we define here the T_δ^+ and N_δ^+ neighborhoods of a function $f \in \sum_p^+$ of the form (1.7) as follows:

$$T_\delta^+(f) = \left\{ g \in \sum\nolimits_p^+ : g(z) = z^{-p} + \sum_{k=p}^{\infty} b_k z^k \text{ and } \sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}} |b_k - a_k| \leqq \delta; \delta > 0 \right\}. \tag{3.1}$$

and

$$N_\delta^+(f) = \left\{ g \in \sum\nolimits_p^+ : g(z) = z^{-p} + \sum_{k=p}^{\infty} b_k z^k \text{ and } \sum_{k=p}^{\infty} k|b_k - a_k| \leqq \delta; \delta > 0 \right\}. \tag{3.2}$$

For a function $f \in \sum_p$, given by (1.1), we define a linear operator $\mathscr{F}_\lambda : \sum_p \longrightarrow \sum_p$ by

$$\mathscr{F}_\lambda(f)(z) = \frac{\lambda}{z^{\lambda+p}} \int_0^z t^{\lambda+p-1} f(t)dt = z^{-p} + \sum_{k=1}^{\infty} \frac{\lambda}{\lambda+k} a_k z^{k-p} \quad (\lambda > 0;\ z \in \mathbb{U}^*). \tag{3.3}$$

If f is given by (1.7), then from (3.3), it follows that

$$\mathscr{F}_\lambda(f)(z) = z^{-p} + \sum_{k=p}^{\infty} \frac{\lambda}{\lambda+p+k} a_k z^k \quad (\lambda > 0;\ z \in \mathbb{U}^*). \tag{3.4}$$

Now, by employing the techniques that proved Theorem 2.1 and using (3.4), it can be shown that

THEOREM 3.1. *If $f \in \sum_p^+(a+1,c,A,B)$ and $\mathscr{F}_\lambda(f)$ is defined by* (3.4)*, then*

$$\mathscr{F}_\lambda(f) \in \sum\nolimits_p^+(a,c,A^*,B) \quad \left(A^* = B + \frac{\lambda\, a(A-B)}{(2p+a)(2p+\lambda)} \right).$$

The result is the best possible for the function f, given by (2.3).

Letting $a = c = 1$, $A = 1 - (2\alpha/p)$ $(0 \leqq \alpha < p)$ and $B = -1$ in Theorem 3.1, we get

COROLLARY 3.1. *If $f \in \sum_p^+$ satisfies the condition* (2.4)*, then the function $\mathscr{F}_\lambda(f) \in \sum_p^+(\gamma)$, where*

$$\gamma = p - \frac{\lambda(p-\alpha)}{(2p+1)(2p+\lambda)}.$$

The result is the best possible.

THEOREM 3.2. *Under the hypothesis of Theorem* 3.1*, we have*

$$T^{+}_{\delta_1}(\mathscr{F}_\lambda(f)) \subset \textstyle\sum_p^+(a,c,A,B),$$

where

$$\delta_1 = \frac{2p(2p+a+\lambda)}{(2p+a)(2p+\lambda)}. \tag{3.5}$$

The result is the best possible in the sense that δ_1 *cannot be increased.*

Proof. Let f, defined by (1.7) be in the class $\sum_p^+(a+1,c,A,B)$. By Theorem 3.1, we obtain

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A-B)(p+k+\lambda)(c)_{p+k}} a_k \leqq \frac{a}{(2p+a)(2p+\lambda)}. \tag{3.6}$$

Suppose that

$$g(z) = z^{-p} + \sum_{k=p}^{\infty} b_k z^k \quad (b_k \geqq 0;\ z \in \mathbb{U}^*) \tag{3.7}$$

and $g \in T^{+}_{\delta_1}(\mathscr{F}_\lambda(f))$ for δ_1 given by (3.5). By (3.1), we get

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}} \left| b_k - \frac{\lambda}{p+k+\lambda} a_k \right| \leqq \delta_1. \tag{3.8}$$

Using (3.6) and (3.8), we deduce that

$$\begin{aligned}
&\sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}} b_k \\
&\quad \leqq \sum_{k=p}^{\infty} \frac{k(1-B)\lambda(a)_{p+k}}{p(A-B)(p+k+\lambda)(c)_{p+k}} a_k + \sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}} \left| b_k - \frac{\lambda}{p+k+\lambda} a_k \right| \\
&\quad \leqq \frac{\lambda a}{(2p+a)(2p+\lambda)} + \delta_1 = 1,
\end{aligned}$$

which in view of Lemma 2.1 implies that $g \in \sum_p^+(a,c,A,B)$.

To show that the result is the best possible, we consider the functions f, g defined in $\mathbb{U}^*$ by

$$f(z) = z^{-p} + \frac{(A-B)(c)_{2p}}{(1-B)(a+1)_{2p}} z^p$$

and

$$g(z) = z^{-p} + \left\{ \frac{\lambda(A-B)(c)_{2p}}{(2p+\lambda)(1-B)(a+1)_{2p}} + \frac{(A-B)(c)_{2p}\delta'}{(1-B)(a)_{2p}} \right\} z^p \quad (\delta' > \delta_1).$$

It is easily seen that $f \in \sum_p^+(a+1,c,A,B)$ and $g \in T^{+}_{\delta'}(\mathscr{F}_\lambda(f))$. But, $g \notin \sum_p^+(a,c,A,B)$. This evidently completes the proof of Theorem 3.2. □

Substituting $a=c=1$, $A = 1-(2\alpha/p)$ and $B=-1$ in Theorem 3.2, we get

COROLLARY 3.2. *If f, given by* (1.7) *satisfies the condition* (2.4) *and g defined by* (3.7) *satisfies the inequality*

$$\sum_{k=p}^{\infty} k\left|b_k - \frac{\lambda}{p+\lambda+k}a_k\right| \leqq \frac{2p(2p+\lambda+1)(p-\alpha)}{(2p+1)(2p+\lambda)} \quad (0 \leqq \alpha < p;\ \lambda > 0),$$

then $g \in \sum_p^+(\alpha)$. The result is the best possible.

For the subsets $\mathscr{A}, \mathscr{B}$ of $\sum_p^+$, we denote

$$\mathscr{A} \otimes \mathscr{B} = \{f * g : f \in \mathscr{A} \text{ and } g \in \mathscr{B}\}.$$

Making use of this notation, we now prove Theorem 3.3 below.

THEOREM 3.3. *If $a \geqq c > 0$, then*

$$\text{(i) } T_{\delta_2}^+(z^{-p}) \otimes T_{\delta_2}^+(z^{-p}) \subset \sum_p^+(a,c,A,B) \quad \left(\delta_2 = \sqrt{\frac{(1-B)(a)_{2p}}{(A-B)(c)_{2p}}}\right)$$

and

$$\text{(ii) } T_{\delta_3}^+(z^{-p}) \otimes N_{\delta_3}^+(z^{-p}) \subset \sum_p^+(a,c,A,B) \quad (\delta_3 = \sqrt{p}).$$

The result in (i) *and* (ii) *are the best possible.*

Proof. Let f be given by (1.7) and g be defined by (3.7). Assuming that $f, g \in T_{\delta_2}^+(z^{-p})$, it follows from (3.1) that

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}} a_k \leqq \delta_2 \quad \text{and} \quad \sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}} b_k \leqq \delta_2.$$

Since $a \geqq c > 0$, $\{k(a)_{p+k}\}/(c)_{p+k}$ is an increasing function of k, so that the first inequality give

$$a_k \leqq \frac{\delta_2(A-B)(c)_{2p}}{(1-B)(a)_{2p}} \quad (k \geqq p).$$

Thus

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}} a_k b_k \leqq \delta_2^2 \frac{(A-B)(c)_{2p}}{(1-B)(a)_{2p}} = 1,$$

which in view of Lemma 2.1 implies that $(f*g) \in \sum_p^+(a,c,A,B)$.

In order to show that the result in (i) is the best possible, we consider the functions f and g defined by

$$f(z) = g(z) = z^{-p} + \sqrt{\frac{(A-B)(c)_{2p}}{(1-B)(a)_{2p}}} z^p \quad (z \in \mathbb{U}^*).$$

Clearly, $f,g \in T^{+}_{\delta_2}(z^{-p})$ and $(f*g) \in \sum^{+}_{p}(a,c,A,B)$. This proves the assertion (i).

Next, we assume that $\delta_3 = \sqrt{p}, f \in T^{+}_{\delta_3}(z^{-p})$ and $g \in N^{+}_{\delta_3}(z^{-p})$. Then, by (3.1) and (3.2), we have

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}} a_k \leqq \delta_3 \text{ and } \sum_{k=p}^{\infty} k b_k \leqq \delta_3.$$

Thus, $b_k \leqq \delta_3/p$ for $k \geqq p$, and

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}} a_k b_k \leqq \frac{\delta_3^2}{p} = 1.$$

The above inequality, again by virtue of Lemma 2.1 implies that $(f*g) \in \sum^{+}_{p}(a,c,A,B)$.

Considering the functions f and g defined in $\mathbb{U}^*$ by

$$f(z) = z^{-p} + \frac{\sqrt{p}(A-B)(c)_{2p}}{(1-B)(a)_{2p}} z^p \quad \text{and} \quad g(z) = z^{-p} + \frac{z^p}{\sqrt{p}},$$

we note that $f \in T^{+}_{\delta_3}(z^{-p}), g \in N^{+}_{\delta_3}(z^{-p})$ and $(f*g) \in \sum^{+}_{p}(a,c,A,B)$. This proves that the assertion (ii) is the best possible and the proof of Theorem 3.3 is completed. □

Putting $a = c = 1$, $A = 1-(2\alpha/p)$ $(0 \leqq \alpha < p)$ and $B = -1$ in Theorem 3.3, we obtain

COROLLARY 3.4. *Let f be given by* (1.7) *and g be defined by* (3.7). *If*
(i) $\sum_{k=p}^{\infty} k a_k \leqq \sqrt{p(p-\alpha)}$ *and* $\sum_{k=p}^{\infty} k b_k \leqq \sqrt{p(p-\alpha)}$, *then* $(f*g) \in \sum^{+}_{p}(\alpha)$.
(ii) $\sum_{k=p}^{\infty} k a_k \leqq \sqrt{p}(p-\alpha)$ *and* $\sum_{k=p}^{\infty} k b_k \leqq \sqrt{p}$, *then* $(f*g) \in \sum^{+}_{p}(\alpha)$.

The result in (i) *and* (ii) *are the best possible.*

4. Hadamard product

In this section, we consider the functions $f_j \in \Sigma^{+}_{p}$ defined by

$$f_j(z) = z^{-p} + \sum_{k=p}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geqq 0, j \in \mathbb{N};\ z \in \mathbb{U}^*).$$

THEOREM 4.1. *If $a \geqq c > 0, -1 \leqq B < A_j \leqq 1 (-1 \leqq B < 0)$ $(j = 1,2,\cdots,n)$ and $f_j \in \sum^{+}_{p}(a,c,A_j,B)$, then the function $(f_1 * f_2 * \cdots * f_n) \in \sum^{+}_{p}(a,c,\rho,B)$, where*

$$\rho = B + \left\{ \frac{(c)_{2p}}{(1-B)(a)_{2p}} \right\}^{n-1} \prod_{j=1}^{n} (A_j - B).$$

The result is the best possible.

Proof. For $n = 1$, we note that $\rho = A_1$. Let $n = 2$. Then by Lemma 2.1, we get

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}} a_{k,j} \leqq 1 \quad (j = 1,2),$$

which with the help of the Cauchy-Schwarz inequality yields

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p\sqrt{(A_1-B)(A_2-B)}(c)_{p+k}} \sqrt{a_{k,1}a_{k,2}} \leqq 1. \tag{4.1}$$

We need to find the largest ρ such that

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(\rho-B)(c)_{p+k}} a_{k,1}a_{k,2} \leqq 1. \tag{4.2}$$

In view of (4.1) and (4.2), it is sufficient to show that

$$\frac{k(1-B)(a)_{p+k}}{p(\rho-B)(c)_{p+k}} a_{k,1}a_{k,2} \leqq \frac{k(1-B)(a)_{p+k}}{p\sqrt{(A_1-B)(A_2-B)}(c)_{p+k}} \sqrt{a_{k,1}a_{k,2}} \quad (k \geqq p)$$

that is,

$$\sqrt{a_{k,1}a_{k,2}} \leqq \frac{\rho - B}{\sqrt{(A_1-B)(A_2-B)}} \quad (k \geqq p).$$

On the other hand, (4.1) implies that

$$\sqrt{a_{k,1}a_{k,2}} \leqq \frac{p\sqrt{(A_1-B)(A_2-B)}(c)_{p+k}}{k(1-B)(a)_{p+k}} \quad (k \geqq p).$$

Consequently, we need to find the largest ρ such that

$$\frac{p\sqrt{(A_1-B)(A_2-B)}(c)_{p+k}}{k(1-B)(a)_{p+k}} \leqq \frac{\rho - B}{\sqrt{(A_1-B)(A_2-B)}} \quad (k \geqq p),$$

which is equivalent to

$$B + \frac{p(A_1-B)(A_2-B)(c)_{p+k}}{k(1-B)(a)_{p+k}} \leqq \rho \quad (k \geqq p). \tag{4.3}$$

Setting

$$\phi(k) = B + \frac{p(A_1-B)(A_2-B)(c)_{p+k}}{k(1-B)(a)_{p+k}} \quad (k \geqq p),$$

we observe that ϕ is a decreasing function of k, so that by putting $k=p$ in (4.3), we get

$$\rho=\phi(p)=B+\frac{(A_1-B)(A_2-B)(c)_{2p}}{(1-B)(a)_{2p}}.$$

This proves the result for $n=2$. Now, suppose that the result is true for any positive integer m. Then by using the above argument

$$(f_1*f_2*\cdots*f_m*f_{m+1})\in\sum\nolimits_p^+(a,c,\rho',B),$$

where

$$\rho'=B+\frac{(\rho^*-B)(A_{m+1}-B)(c)_{2p}}{(1-B)(a)_{2p}}\quad\text{and}\quad\rho^*=B+\left\{\frac{(c)_{2p}}{(1-B)(a)_{2p}}\right\}^{m-1}\prod_{j=1}^{m}(A_j-B).$$

A simple calculation yields

$$\rho'=B+\left\{\frac{(c)_{2p}}{(1-B)(a)_{2p}}\right\}(\rho-B)(A_{m+1}-B)=B+\left\{\frac{(c)_{2p}}{(1-B)(a)_{2p}}\right\}^{m}\prod_{j=1}^{m+1}(A_j-B).$$

This proves the result for $n=m+1$.

By taking the functions f_j defined in $\mathbb{U}^*$ by

$$f_j(z)=z^{-p}+\frac{(A_j-B)(c)_{2p}}{(1-B)(a)_{2p}}z^p\quad(j=1,2,\cdots,n),\tag{4.4}$$

it is easily seen that

$$(f_1*f_2*\cdots*f_n)(z)=z^{-p}+\left\{\prod_{j=1}^{n}\frac{(A_j-B)(c)_{2p}}{(1-B)(a)_{2p}}\right\}z^p,$$

which shows that

$$\frac{(1-B)(c)_{2p}}{(\rho-B)(a)_{2p}}\prod_{j=1}^{n}\frac{(A_j-B)(c)_{2p}}{(1-B)(a)_{2p}}=1.$$

This completes the proof of Theorem 4.1. □

Putting $A_j=1-(2\alpha_j/p)$ $(j=1,2,\cdots,n)$ and $B=-1$ in Theorem 4.1, we get

COROLLARY 4.1. *If $a\geqq c>0$ and $f_j\in\sum_p^+(a,c;\alpha_j)$ $(0\leqq\alpha_j<p;j=1,2,\cdots,n)$, then the function $(f_1*f_2*\cdots*f_n)\in\sum_p^+(a,c;\nu)$, where*

$$\nu=p-\left\{\frac{(c)_{2p}}{p(a)_{2p}}\right\}^{n-1}\prod_{j=1}^{n}(p-\alpha_j).$$

The result is the best possible.

In the special case when $a=c=1$, Corollary 4.1 yields

COROLLARY 4.2. *If* $f_j \in \sum_p^+(\alpha_j)$ $(0 \leqq \alpha_j < p; j = 1,2,\cdots,n)$*, then the function* $(f_1 * f_2 * \cdots * f_n) \in \sum_p^+(\mu)$*, where*

$$\mu = p - \frac{\prod_{j=1}^{n}(p-\alpha_j)}{p^{n-1}}.$$

The result is the best possible.

THEOREM 4.2. *Let* $a \geqq c > 0$ *and* $-1 \leqq B < A_j \leqq 1$ $(-1 \leqq B < 0)$ $(j = 1,2)$*. If* $f_1 \in \sum_p^+(a+1,c,A_1,B)$ *and* $f_2 \in \sum_p^+(a,c,A_2,B)$*, then the function* $(f_1 * f_2) \in \sum_p^+(a,c,\rho,B)$*, where*

$$\rho = B + \frac{(A_1-B)(A_2-B)(c)_{2p}}{(1-B)(a+1)_{2p}}.$$

The result is the best possible.

Proof. We need to find the largest ρ such that

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(\rho-B)(c)_{p+k}} a_{k,1}a_{k,2} \leqq 1.$$

From Lemma 2.1, we get

$$\sum_{k=p}^{\infty} \frac{k(1-B)(a+1)_{p+k}}{p(A_1-B)(c)_{p+k}} a_{k,1} \leqq 1 \quad \text{and} \quad \sum_{k=p}^{\infty} \frac{k(1-B)(a)_{p+k}}{p(A_2-B)(c)_{p+k}} a_{k,2} \leqq 1,$$

which in view of the Cauchy-Schwarz inequality yields

$$\sum_{k=p}^{\infty} \frac{k(1-B)\sqrt{(a)_{p+k}(a+1)_{p+k}}}{p\sqrt{(A_1-B)(A_2-B)}(c)_{p+k}} \sqrt{a_{k,1}a_{k,2}} \leqq 1.$$

Now, by following the techniques used in the proof of Theorem 4.1 (for the case $n = 2$) and using the fact that $(c)_{p+k}/\{k(a+1)_{p+k}\}$ is a decreasing function of k, we get the required result.

By taking the functions

$$f_1(z) = z^{-p} + \frac{(A_1-B)(c)_{2p}}{(1-B)(a+1)_{2p}} z^p \quad \text{and} \quad f_2(z) = z^{-p} + \frac{(A_2-B)(c)_{2p}}{(1-B)(a)_{2p}} z^p$$

defined in $\mathbb{U}^*$, it is easily seen that the result is the best possible. □

Letting $A_j = 1 - (2\alpha_j/p)$ $(0 \leqq \alpha_j < p; j = 1,2)$ and $B = -1$ in Theorem 4.2, we obtain

COROLLARY 4.3. *Let* $a \geqq c > 0$. *If* $f_1 \in \sum_p^+(a+1,c;\alpha_1)$ *and* $f_2 \in \sum_p^+(a,c;\alpha_2)$, *then the function* $(f_1 * f_2) \in \sum_p^+(a,c;\sigma)$, *where*

$$\sigma = p - \frac{(p-\alpha_1)(p-\alpha_2)(c)_{2p}}{p(a+1)_{2p}}.$$

The result is the best possible.

The proof of the following theorem is much akin to that of Theorem 4.2 and we choose to omit the details.

THEOREM 4.3. *If* $a \geqq c > 0, -1 \leqq B < A_j \leqq 1$ $(-1 \leqq B < 0; j = 1,2)$ *and* $f_j \in \sum_p^+(a+1,c,A_j,B)$, *then the function* $(f_1 * f_2) \in \sum_p^+(a,c,\tau,B)$, *where*

$$\tau = B + \frac{(A_1-B)(A_2-B)(a)_{2p}(c)_{2p}}{(1-B)(a+1)_{2p}^2}.$$

The result is the best possible for the functions f_j *defined in* $\mathbb{U}^*$ *by*

$$f_j(z) = z^{-p} + \frac{(A_j-B)(c)_{2p}}{(1-B)(a+1)_{2p}} z^p \quad (j = 1,2).$$

Using Theorem 4.2 and Theorem 4.3, we deduce the following result.

THEOREM 4.4. *If* $a \geqq c > 0$ *and* $f_j \in \sum_p^+(a+1,c,A_j,B)$ $(-1 \leqq B < A_j \leqq 1, -1 \leqq B < 0; j = 1,2,\cdots,n)$, *then the function* $(f_1 * f_2 * \cdots * f_n) \in \sum_p^+(a,c,\kappa,B)$, *where*

$$\kappa = B + \left\{ \frac{(a)_{2p}(c)_{2p}^{n-1}}{(1-B)^{n-1}(a+1)_{2p}^n} \right\} \prod_{j=1}^n (A_j - B).$$

The result is the best possible for the functions f_j *defined in* $\mathbb{U}^*$ *by*

$$f_j(z) = z^{-p} + \frac{(A_j-B)(c)_{2p}}{(1-B)(a+1)_{2p}} z^p \quad (j = 1,2,\cdots,n).$$

Setting $a = c = 1$, $A_j = 1-(2\alpha_j/p)$ $(j = 1,2,\cdots,n)$ and $B = -1$ in Theorem 4.4, we obtain

COROLLARY 4.4. *If* $f_j \in \sum_p^+$ *satisfies*

$$-\Re\left[z^{p+1}\left\{(p+2)f_j'(z) + zf_j''(z)\right\}\right] > \alpha_j \quad (0 \leqq \alpha_j < p, j = 1,2,\cdots,n;\ z \in \mathbb{U}),$$

then the function $(f_1 * f_2 * \cdots * f_n) \in \sum_p^+(\xi)$, *where*

$$\xi = p - \frac{\prod_{j=1}^n (p-\alpha_j)}{p^{n-1}(2p+1)^n}.$$

The result is the best possible.

THEOREM 4.5. *If* $a \geqq c > 0$, $f_j \in \sum_p^+(a,c,A,B)$ $(j = 1,2,\cdots,n)$ *and*

$$h(z) = z^{-p} + \sum_{k=p}^{\infty}\left(\sum_{j=1}^{n} a_{k,j}^2\right) z^k \quad (z \in \mathbb{U}^*), \tag{4.5}$$

then the function $h \in \sum_p^+(a,c,\varkappa,B)$, *where*

$$\varkappa = B + \frac{n(A-B)^2(c)_{2p}}{(1-B)(a)_{2p}}.$$

The result is the best possible.

Proof. Since by Lemma 2.1,

$$\sum_{k=p}^{\infty}\left\{\frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}}\right\}^2 a_{k,j}^2 \leqq \left\{\sum_{k=p}^{\infty}\frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}} a_{k,j}\right\}^2 \leqq 1$$

for $j = 1,2,\cdots,n$, we have

$$\sum_{k=p}^{\infty}\frac{1}{n}\left\{\frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}}\right\}^2\left(\sum_{j=1}^{n} a_{k,j}^2\right) \leqq 1. \tag{4.6}$$

We have to find the largest σ such that

$$\sum_{k=p}^{\infty}\left\{\frac{k(1-B)(a)_{p+k}}{p(\sigma-B)(c)_{p+k}}\right\}\left(\sum_{j=1}^{n} a_{k,j}^2\right) \leqq 1.$$

In view of (4.6), we need to find the largest σ such that

$$\frac{k(1-B)(a)_{p+k}}{p(\sigma-B)(c)_{p+k}} \leqq \frac{1}{n}\left\{\frac{k(1-B)(a)_{p+k}}{p(A-B)(c)_{p+k}}\right\}^2$$

which is equivalent to

$$B + \frac{np(A-B)^2(c)_{p+k}}{k(1-B)(a)_{p+k}} \leqq \sigma \quad (k \geqq p). \tag{4.7}$$

Since $(c)_{p+k}/\{k(a)_{p+k}\}$ is a decreasing function of k, putting $k = p$ in (4.7), we get the required result.

It can be easily verified that the result is the best possible for the functions f_j $(j = 1,2,\cdots,n)$, given by (4.4). This proves Theorem 4.5. □

Putting $a = c = 1$, $A = 1-(2\alpha/p)$ and $B = -1$ in Theorem 4.5, we obtain

COROLLARY 4.5. *If* $f_j \in \sum_p^+(\alpha)$ $(0 \leqq \alpha < p; j = 1,2,\cdots,n)$ *and* h *is given by* (4.5), *then the function* $h \in \sum_p^+(\eta)$, *where*

$$\eta = p - \frac{n(p-\alpha)^2(c)_{2p}}{p(a)_{2p}}.$$

The result is the best possible.

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