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SOME REMARKS ON CESÀRO–ORLICZ SEQUENCE SPACES

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Abstract. In this paper Cesàro-Orlicz spaces, theory of which started in the papers [7], [28] and [10], are investigated. The problem of the necessity of condition δ_2 for some fundamental topological and geometrical properties is considered again. Criteria for the Kadec-Klee property with respect to the coordinatewise convergence as well as for local uniform convexity of the spaces are given. In the last part, finite dimensional subspaces of Cesàro-Orlicz spaces are investigated.

1. Preliminaries

A map $\varphi : \mathbb{R} \to [0, +\infty]$ is said to be an Orlicz function if φ is even, convex, left continuous on $\mathbb{R}_+$, continuous at zero, $\varphi(0) = 0$ and $\varphi(u) \to \infty$ as $u \to \infty$ (see [3], [20], [24], [26], [27], [29] and [30]). For any Orlicz function φ we denote:

$$a_\varphi = \sup\{u \geqslant 0 : \varphi(u) = 0\} \quad \text{and} \quad b_\varphi = \sup\{u \geqslant 0 : \varphi(u) < \infty\}.$$

Given any Orlicz function φ, we define on l^0 (the space of all real sequences) the following convex modular $\mathrm{I}_\varphi : l^0 \to [0, \infty]$:

$$\mathrm{I}_\varphi(x) = \sum_{i=1}^{\infty} \varphi(x(i)).$$

The space

$$l_\varphi = \{x \in l^0 : \mathrm{I}_\varphi(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

is called the Orlicz sequence space (see [3], [20] [24], [26], [27], [29] and [30]). We equip this space with the Luxemburg norm

$$|||x|||_\varphi = \inf\left\{\lambda > 0 : \mathrm{I}_\varphi\left(\frac{x}{\lambda}\right) \leqslant 1\right\}.$$

The arithmetic mean map σ is defined on l^0 by the formula:

$$\sigma x = (\sigma x(i))_{i=1}^{\infty}, \quad \text{where} \quad \sigma x(i) = \frac{1}{i}\sum_{j=1}^{i} |x(j)|$$

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for any $i \in \mathbb{N}$ and $x = (x(i))_{i=1}^{\infty} \in l^0$. Given any Orlicz function φ, we define on l^0 another convex modular $\rho_\varphi : l^0 \to [0,\infty]$, by

$$\rho_\varphi(x) = I_\varphi(\sigma x).$$

and the Cesàro-Orlicz sequence space

$$ces_\varphi = \{x \in l^0 : \sigma x \in l_\varphi\},$$

(see [7], [28]). We equip this space with the norm $\|x\|_\varphi = \|\|\sigma x\|\|_\varphi$. The Cesàro-Orlicz sequence spaces $ces_\varphi = (ces_\varphi, \|\cdot\|_\varphi)$ have the Fatou property (see [7]). Consequently, ces_φ are Banach spaces (see [26]).

We also define a subspace $(ces_\varphi)_a$ of ces_φ by the following formula

$$(ces_\varphi)_a = \{x \in ces_\varphi : \forall\, k > 0\ \exists\, i_k \in \mathbb{N} \text{ such that } \sum_{i=i_k}^{\infty} \varphi(\sigma(kx)(i)) < \infty\}.$$

The space $(ces_\varphi)_a$ is a closed and separable subspace of ces_φ and it is the subspace of all order continuous elements of ces_φ. For the definition of order continuous elements in a Banach lattice we refer to [19] and [25].

In particular cases, when $\varphi(u) = |u|^p$ for $1 < p < \infty$ or $\varphi(u) = 0$ if $|u| \leqslant 1$ and $\varphi(u) = \infty$ if $|u| > 1$ (which corresponds to $p = \infty$), we get the well known Cesàro sequence spaces ces_p and ces_∞. They appeared in 1968 as the problem of the Dutch Mathematical Society to find their duals (see [1], Problem 2). A regular investigation of Cesàro sequence spaces was done in [31] (see also [2], [16] and [23]). At the end of the previous century several authors studied some geometric properties of these spaces (see [4], [5], [6], [8], [9] and [22]).

In recent years the theory of Cesàro-Orlicz sequence spaces has been studied intensively. Some basic topological properties (nontriviality, order continuity, separability and relationships between the modular and the norm defined itself) as well as some geometric properties (Fatou property, strict monotonicity and rotundity) were considered in [7]. Maligranda, Petrot and Suantai calculated in [28] n-dimensional James constant of Cesàro and Cesàro-Orlicz sequence spaces. They concluded from this result that neither Cesàro sequence spaces ces_p for $1 < p \leqslant \infty$ nor Cesàro-Orlicz sequence spaces ces_φ generated by Orlicz functions φ satisfing condition δ_2 are uniformly nonsquare (they are not even B-convex). In [10] criteria for extreme points and SU-points of ces_φ were given. After sending this paper for publication the paper [21] which also concerns Cesàro-Orlicz spaces was published.

This paper is organized as follows. In the second section we first give an example of Orlicz function φ which does not satisfy condition δ_2 and the space ces_φ generated itself contains an order linearly isometric copy of l^∞. This result is related to the problem of the necessity of condition δ_2 for some fundamental topological and geometrical properties of ces_φ (see [7] and [28]). In the third section the Kadec-Klee property with respect to the coordinatevise convergence is considered. In the fourth section some convexity properties of ces_φ are investigated. The last section is devoted to finite dimensional subspaces of ces_φ.

REMARK 1. Recall that the space ces_φ is nontrivial if and only if there exists $n_1 \in \mathbb{N}$ such that $\sum_{n=n_1}^{\infty} \varphi\left(\frac{1}{n}\right) < \infty$, or equivalently, for any $k > 0$ there exists $n_k \in \mathbb{N}$ such that $\sum_{n=n_k}^{\infty} \varphi\left(\frac{k}{n}\right) < \infty$ (see Theorem 2.1 in [7]). In the whole paper (excluding Lemma 6 and Theorems 5 and 7, see also Remark 3) we will assume that the Orlicz function φ satisfies this condition in order to have $ces_\varphi \neq \{\emptyset\}$.

2. The δ_2 condition

Recall that the Orlicz function φ is said to satisfy condition δ_2 ($\varphi \in \delta_2$ for short) if there exist $u_0 > 0$ and $K > 0$ such that $\varphi(u_0) > 0$ and $\varphi(2u) \leqslant K\varphi(u)$ for any $u \in [0, u_0]$. It is well known that if φ does not satisfy condition δ_2, then the Orlicz space $(l_\varphi, \|\!| \cdot |\!\|_\varphi)$ contains an order linearly isometric copy of l^∞. We do not know, if for arbitrary Orlicz function φ, analogous implication is true for the Cesàro-Orlicz space $(ces_\varphi, \|\cdot\|_\varphi)$. However, we will show an example of Orlicz function φ such that $\varphi \notin \delta_2$ and $(ces_\varphi, \|\cdot\|_\varphi)$ contains an order linearly isometric copy of l^∞.

EXAMPLE 1. Let $\varphi(u) = \int_0^u p(t)dt$, where

$$p(t) = \begin{cases} \frac{1}{2} & \text{for } t \in \left[\frac{1}{2}, \infty\right), \\ \frac{1}{n!2^{1+2+\ldots+n}} & \text{for } t \in \left[\frac{1}{n+1}, \frac{1}{n}\right) \text{ and } n \geqslant 2. \end{cases}$$

Note that $\varphi \notin \delta_2$. Indeed, for any $n \geqslant 2$ we have

$$\begin{aligned} \varphi\left(\left(1+\frac{1}{n}\right)\frac{1}{n}\right) &= \int_0^{\left(1+\frac{1}{n}\right)\frac{1}{n}} p(t)dt > \int_{\frac{1}{n}}^{\left(1+\frac{1}{n}\right)\frac{1}{n}} p(t)dt \\ &= \frac{1}{n^2}\cdot\frac{1}{(n-1)!2^{1+2+\ldots+n-1}} = \frac{2^n}{n}\cdot\frac{1}{n!2^{1+2+\ldots+n}} \\ &> 2^n \int_0^{\frac{1}{n}} p(t)dt = 2^n\varphi\left(\frac{1}{n}\right). \end{aligned} \tag{1}$$

Denoting

$$k_n = \frac{n}{2}(n+1)!2^{1+2+\ldots+(n-1)} \quad \text{for} \quad n \geqslant 2$$

and

$$\begin{aligned} &a_1 = 0, \quad a_2 = k_2, \quad a_n = 2k_2 + \ldots + 2k_{n-1} + k_n \quad \text{for} \quad n \geqslant 3, \\ &b_1 = 1, \quad b_n = 2k_2 + \ldots + 2k_n + 1 \quad \text{for} \quad n \geqslant 2, \end{aligned}$$

we define $x = (x(m))_{m=1}^{\infty}$ by the formula

$$x(m) = \begin{cases} \frac{b_{n-1}}{n} - \frac{a_{n-1}}{n-1} & \text{for} \quad m = b_{n-1} \\ \frac{1}{n} & \text{for} \quad m = b_{n-1}+1, \ldots, a_n \\ 0 & \text{for} \quad m = a_n + 1, \ldots, b_n - 1 \end{cases}$$

for $n \geqslant 2$.

First we will show that $\sigma x(m) = \frac{1}{n}$ for $m = b_{n-1}, b_{n-1}+1, \ldots, a_n$ and $n \geqslant 2$. We will proceed the proof by induction. Since $x(m) = \frac{1}{2}$ for $n = 2$ and $m = b_1, b_1 + 1, \ldots, a_2$ $(b_1 = 1, a_2 = 12)$, we have $\sigma x(b_1) = \sigma x(b_1+1) = \ldots = \sigma x(a_2) = \frac{1}{2}$. Assume now that $\sigma x(m) = \frac{1}{n}$ for $m = b_{n-1}, \ldots a_n$. We will show that $\sigma x(m) = \frac{1}{n+1}$ for $m = b_n, \ldots, a_{n+1}$.

By the induction assumption, we have $\sum_{n=1}^{a_n} x(m) = \frac{a_n}{n}$. Simultaneously, $x(m) = 0$ for $m = a_n + 1, \ldots, b_{n-1}$. Therefore

$$\sum_{m=1}^{b_n} x(m) = \sum_{m=1}^{a_n} x(m) + x(b_n) = \frac{a_n}{n} + \frac{b_n}{n+1} - \frac{a_n}{n} = \frac{b_n}{n+1},$$

whence $\sigma x(b_n) = \frac{1}{n+1}$. By the implication: if $\sigma x(k) = \alpha$ and $|x(k+1)| = \alpha$, then $\sigma x(k+1) = \alpha$ $(\alpha \geqslant 0)$, we get that $\sigma x(m) = \frac{1}{n+1}$ for $m = b_n + 1, \ldots, a_{n+1}$.

Now, we will show that $\rho_\varphi(x) < 1$. Since $\sigma x(m) \leqslant \frac{1}{n}$ for $m = a_n + 1, \ldots, b_n - 1$, $n \geqslant 2$ and

$$\begin{aligned}
\varphi\left(\frac{1}{n}\right) &= \int_0^{\frac{1}{n+1}} p(t)dt + \int_{\frac{1}{n+1}}^{\frac{1}{n}} p(t)dt \\
&< \frac{1}{n+1} \cdot \frac{1}{(n+1)!2^{1+\ldots+(n+1)}} + \frac{1}{n(n+1)} \cdot \frac{1}{n!2^{1+\ldots+n}} \\
&< \frac{2}{n(n+1)!2^{1+\ldots+n}}
\end{aligned}$$

for $n \geqslant 2$, we get

$$\begin{aligned}
\rho_\varphi(x) &= \sum_{n=1}^{\infty} \varphi(\sigma x(n)) = \sum_{n=2}^{\infty} \left(\sum_{m=b_{n-1}}^{b_n - 1} \varphi(\sigma x(m)) \right) \leqslant \sum_{n=2}^{\infty} (b_n - b_{n-1}) \varphi\left(\frac{1}{n}\right) \\
&= \sum_{n=2}^{\infty} 2k_n \varphi\left(\frac{1}{n}\right) < \sum_{n=2}^{\infty} 2k_n \frac{2}{n(n+1)!2^{1+\ldots+n}} \\
&= \sum_{n=2}^{\infty} 2\frac{n}{2}(n+1)!2^{1+\ldots+(n-1)} \frac{2}{n(n+1)!2^{1+\ldots+n}} \\
&= \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} = \sum_{l=1}^{\infty} \frac{1}{2^l} = 1.
\end{aligned}$$

Finally, we will show that $\rho_\varphi(\lambda x) = \infty$ for any $\lambda > 1$. For any $\lambda > 1$ there exists $k \in \mathbb{N}$ such that $1 + \frac{1}{k} < \lambda$. By (1) and

$$\varphi\left(\frac{1}{n}\right) = \int_0^{\frac{1}{n}} p(t)dt > \int_{\frac{1}{n+1}}^{\frac{1}{n}} p(t)dt = \frac{1}{n(n+1)} \cdot \frac{1}{n!2^{1+\ldots+n}} = \frac{1}{n(n+1)!2^{1+\ldots+n}},$$

we have

$$\begin{aligned}
\rho_{\varphi}(\lambda x) &> \rho_{\varphi}\left(\left(1+\frac{1}{k}\right)x\right) = \sum_{n=1}^{\infty}\varphi\left(\sigma\left(1+\frac{1}{k}\right)x(n)\right) \\
&> \sum_{n=2}^{\infty}\left(\sum_{m=b_{n-1}}^{a_n}\varphi\left(\sigma\left(1+\frac{1}{k}\right)x(m)\right)\right) \\
&\geqslant \sum_{n=k}^{\infty}\left(\sum_{m=b_{n-1}}^{a_n}\varphi\left(\left(1+\frac{1}{k}\right)\sigma x(m)\right)\right) \\
&= \sum_{n=k}^{\infty}k_n\varphi\left(\left(1+\frac{1}{k}\right)\frac{1}{n}\right) > \sum_{n=k}^{\infty}k_n\varphi\left(\left(1+\frac{1}{n}\right)\frac{1}{n}\right) > \sum_{n=k}^{\infty}k_n\cdot 2^n\varphi\left(\frac{1}{n}\right) \\
&> \sum_{n=k}^{\infty}\frac{n}{2}(n+1)!2^{1+\ldots+(n-1)}\cdot 2^n\cdot\frac{1}{n(n+1)!2^{1+\ldots+n}} = \sum_{n=k}^{\infty}\frac{1}{2} = \infty.
\end{aligned}$$

From Theorem 2 in [13] and Theorem 2.1 in [11], we get for the above Orlicz function φ that $(ces_{\varphi}, \|\cdot\|_{\varphi})$ contains an order linearly isometric copy of l^{∞}.

3. The Kadec-Klee property with respect to the coordinatewise convergence

Recall that a Banach space X has the Kadec-Klee property with respect to the coordinatewise convergence if for any $x \in X$ and any sequence (x_m) in X the conditions $\|x_m\| \to \|x\|$ and $x_m(n) \to x(n)$ for all $n \in \mathbb{N}$, yield $\|x - x_m\| \to 0$.

Before presenting the main theorem of this section, concerning the Kadec-Klee property with respect to the coordinatewise convergence of Cesàro-Orlicz spaces ces_{φ}, we wil give some auxiliary lemmas.

LEMMA 1. (cf. Lemmas 2.1 and 2.5 in [7]) *The following assertions are true:*

(i) If $\rho_{\varphi}(x) = 1$, *then* $\|x\|_{\varphi} = 1$ *for any* $x \in ces_{\varphi}$.

(ii) For any $x \in (ces_{\varphi})_a$ *the equality* $\|x\|_{\varphi} = 1$ *implies that* $\rho_{\varphi}(x) = 1$ *if and only if* $\sum_{i=1}^{\infty}\varphi\left(\frac{b_{\varphi}}{i}\right) \geqslant 1$.

(iii) If $\varphi \in \delta_2$, *then for any* $x \in ces_{\varphi}$ *the equality* $\|x\|_{\varphi} = 1$ *implies that* $\rho_{\varphi}(x) = 1$ *if and only if* $\sum_{i=1}^{\infty}\varphi\left(\frac{b_{\varphi}}{i}\right) \geqslant 1$.

Proof. We will show only (ii). Assume that $\sum_{i=1}^{\infty}\varphi\left(\frac{b_{\varphi}}{i}\right) \geqslant 1$ and there exists $x \in (ces_{\varphi})_a$ such that $\|x\|_{\varphi} = 1$ and $\rho_{\varphi}(x) < 1$.

First suppose that there exists $k \in \mathbb{N}$ such that $\sigma x(k) = b_{\varphi}$. We will prove by induction that

$$\sigma x(k+m) \geqslant \frac{b_{\varphi}}{m+1} \quad \text{for any} \quad m \in \mathbb{N}. \tag{2}$$

For $m = 1$ we have

$$\sigma x(k+1) = \frac{|x(1)|+\ldots+|x(k)|+|x(k+1)|}{k+1} \geqslant \frac{|x(1)|+\ldots+|x(k)|}{k+1}$$
$$= \frac{|x(1)|+\ldots+|x(k)|}{k} \cdot \frac{k}{k+1} = b_\varphi \cdot \frac{k}{k+1} \geqslant \frac{b_\varphi}{2}.$$

Now, assume that (2) is satisfied for some $m \in \mathbb{N}$. Then we have

$$\sigma x(k+m+1) \geqslant \frac{|x(1)|+\ldots+|x(k+m)|}{k+m+1} = \frac{|x(1)|+\ldots+|x(k+m)|}{k+m} \cdot \frac{k+m}{k+m+1}$$
$$\geqslant \frac{b_\varphi}{m+1} \cdot \frac{k+m}{k+m+1} \geqslant \frac{b_\varphi}{m+1} \cdot \frac{1+m}{1+m+1} = \frac{b_\varphi}{(m+1)+1}.$$

Hence

$$\rho_\varphi(x) = \sum_{i=1}^{\infty} \varphi(\sigma x(i)) \geqslant \sum_{i=1}^{k-1} \varphi(\sigma x(i)) + \sum_{n=1}^{\infty} \varphi\left(\frac{b_\varphi}{n}\right) \geqslant 1,$$

which contradicts the asumption that $\rho_\varphi(x) < 1$.

Therefore $\sigma x(i) < b_\varphi$ for any $i \in \mathbb{N}$. Let l be a natural number such that

$$\sum_{i=l}^{\infty} \varphi(\sigma(2x)(i)) < \infty.$$

Define the function $f(\lambda) = \rho_\varphi(\lambda x)$ for $\lambda \geqslant 0$. It is clear that f is convex on R_+ and it has finite values on the interval $[0,\lambda_0]$, where $\lambda_0 \in (1,2]$, satisfies the condition: $\sigma(\lambda_0 x)(i) < b_\varphi$ for $i \in \{1,\ldots,l-1\}$. So, f is continuous and consequently, it has the Darboux property on the interval $[0,\lambda_0)$. Since $f(1) = \rho_\varphi(x) < 1$, there exists $\lambda_1 \in (1,\lambda_0)$ such that $f(\lambda_1) = \rho_\varphi(\lambda_1 x) \leqslant 1$. This yields that $\|x\|_\varphi \leqslant 1/\lambda_1 < 1$, a contradiction.

Let now $\sum_{i=1}^{\infty} \varphi\left(\frac{b_\varphi}{i}\right) < 1$. Then for $x = (b_\varphi, 0, 0, \ldots)$ we have

$$\rho_\varphi(x) = \sum_{i=1}^{\infty} \varphi\left(\frac{b_\varphi}{i}\right) < 1$$

and $\rho_\varphi(\lambda x) = \infty$ for any $\lambda > 1$, whence $\|x\|_\varphi = 1$.

Lemma 2 has been proved in [7, Lemma 2.4] under the assumption that $b_\varphi = \infty$. It is easy to show that the Lemma presented below is also true without this assumption. Moreover, Lemma 3 presented below can be proved analogously as Lemma 2.9 in [7].

LEMMA 2. *The following assertions are true:*

(i) If $\|x_m\|_\varphi \to 0$*, then* $\rho_\varphi(x_m) \to 0$ *for any sequence* (x_m) *in* ces_φ.

(ii) If $\varphi \in \delta_2$*, then* $\|x_m\|_\varphi \to 0$ *whenever* $\rho_\varphi(x_m) \to 0$ *for any sequence* (x_m) *in* ces_φ.

LEMMA 3. *The following assertions are true:*

(i) If $\rho_\varphi(x_m) \to 1$*, then* $\|x_m\|_\varphi \to 1$ *for any sequence* (x_m) *in* ces_φ.

(ii) Let $\varphi \in \delta_2$*. Then for any sequence* (x_m) *such that* $\|x_m\|_\varphi \leqslant 1$ *we have* $\rho_\varphi(x_m) \to 1$ *whenever* $\|x_m\|_\varphi \to 1$ *if and only if* $\sum_{i=1}^{\infty} \varphi\left(\frac{b_\varphi}{i}\right) \geqslant 1$.

Lemma 4 can be proved analogously as Lemma 4 in [17]. Lemma 5 can be deduced by Lemma 4 and Theorem 1.39(3) from [3] (cf. Lemma 5 in [17]).

LEMMA 4. *If* $\varphi \in \delta_2$*, then for any* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that for any* $x = (x(n))_{n=1}^{\infty} \in l_\varphi$ *with* $|||x|||_\varphi \leqslant 1$ *and* $|x(n)| \leqslant \frac{1}{3}b_\varphi$ *for any* $n \in \mathbb{N}$ *and any* $y \in l_\varphi$*, we get the implication:*

$$|||x-y|||_\varphi < \delta \Rightarrow |I_\varphi(x) - I_\varphi(y)| < \varepsilon.$$

LEMMA 5. *If* $\varphi \in \delta_2$*, then for every* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that the condition* $I_\varphi(x-y) < \delta$ *implies that* $|I_\varphi(x) - I_\varphi(y)| < \varepsilon$ *for any* $x = (x(n))_{n=1}^{\infty} \in l_\varphi$ *with* $|||x|||_\varphi \leqslant 1$ *and* $|x(n)| \leqslant \frac{1}{3}b_\varphi$ *for any* $n \in \mathbb{N}$*, and any* $y \in l_\varphi$.

THEOREM 1. *If* $\varphi \in \delta_2$*, then* ces_φ *has the Kadec-Klee property with respect to the coordinatewise convergence if and only if* $\sum_{i=1}^{\infty} \varphi\left(\frac{b_\varphi}{i}\right) \geqslant 1$.

Proof. Sufficiency. Assume that $x_m \to x$ coordinatewise and $\|x_m\|_\varphi \to \|x\|_\varphi$. Without loss of generality, we may assume that $\|x\|_\varphi = 1$ and $\|x_m\|_\varphi = 1$ for all $m \in \mathbb{N}$. From Lemma 1, we have that $\rho_\varphi(x) = 1$ and $\rho_\varphi(x_m) = 1$ for all $m \in \mathbb{N}$. We will show that $\rho_\varphi(x - x_m) \to 0$. Hence, from Lemma 2, we immediately will get that $\|x - x_m\|_\varphi \to 0$.

Let $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that

$$\sum_{i=n+1}^{\infty} \varphi(\sigma x(i)) < \frac{\varepsilon}{6K}, \tag{3}$$

where $K > 0$ is the constant from condition δ_2. Since $x_m(i) \to x(i)$ for any $i \in \mathbb{N}$, we have $\sigma x_m(i) \to \sigma x(i)$ for $i \in \mathbb{N}$. Therefore, for any $i = 1, 2, \ldots, n$, we can find $m(i)$ such that

$$|\varphi(\sigma x_m(i)) - \varphi(\sigma x(i))| < \frac{\varepsilon}{6nK} \tag{4}$$

and

$$\varphi(x_m(i) - x(i)) < \frac{\varepsilon}{2n} \tag{5}$$

for $m > m(i)$. Let $m_0 = \max(m(1), m(2), \ldots, m(n))$. Then for $m > m_0$, by (3) and (4), we have

$$\sum_{i=1}^{n} \varphi(\sigma x_m(i)) \geqslant \sum_{i=1}^{n} \left(\varphi(\sigma x(i)) - \frac{\varepsilon}{6nK}\right) > 1 - \frac{\varepsilon}{6K} - \frac{\varepsilon}{6K} = 1 - \frac{\varepsilon}{3K},$$

whence $\sum_{i=n+1}^{\infty}\varphi(\sigma x_m(i)) < \frac{\varepsilon}{3K}$ for $m > m_0$. Since for any $i \in \mathbb{N}$,

$$\begin{aligned}\sigma(x-x_m)(i) &= \frac{|x(1)-x_m(1)|+|x(2)-x_m(2)|+\ldots+|x(i)-x_m(i)|}{i}\\ &\leqslant \frac{|x(1)|+|x_m(1)|+|x(2)|+|x_m(2)|+\ldots+|x(i)|+|x_m(i)|}{i}\\ &= \sigma x(i)+\sigma x_m(i),\end{aligned}$$

we get for $m > m_0$:

$$\begin{aligned}\sum_{i=n+1}^{\infty}\varphi(\sigma(x-x_m)(i)) &\leqslant \sum_{i=n+1}^{\infty}\varphi\left(\frac{2\sigma x(i)+2\sigma x_m(i)}{2}\right)\\ &\leqslant \frac{1}{2}\left(\sum_{i=n+1}^{\infty}\varphi(2\sigma x(i))+\sum_{i=n+1}^{\infty}\varphi(2\sigma x_m(i))\right)\\ &\leqslant \frac{K}{2}\left(\sum_{i=n+1}^{\infty}\varphi(\sigma x(i))+\sum_{i=n+1}^{\infty}\varphi(\sigma x_m(i))\right)\\ &< \frac{K}{2}\left(\frac{\varepsilon}{6K}+\frac{\varepsilon}{3K}\right)=\frac{\varepsilon}{4}.\end{aligned}$$

On the other hand, for $i = 1,2,\ldots,n$, we have

$$\begin{aligned}\sigma(x-x_m)(i) &= \frac{|x(1)-x_m(1)|+\ldots+|x(i)-x_m(i)|}{i}\\ &\leqslant \frac{i\max(|x(1)-x_m(1)|,\ldots,|x(i)-x_m(i)|)}{i}.\end{aligned}$$

Hence for $m > m_0$, by (5), we get

$$\sum_{i=1}^{n}\varphi(\sigma(x-x_m)(i)) \leqslant \sum_{i=1}^{n}\varphi(\max(|x(1)-x_m(1)|,\ldots,|x(i)-x_m(i)|)) < n\cdot\frac{\varepsilon}{2n}=\frac{\varepsilon}{2}.$$

Therefore, we have $\rho_\varphi(x-x_m) = I_\varphi(\sigma(x-x_m)) < \varepsilon$ for $m > m_0$.

Necessity. Assume that $\varphi \in \delta_2$ and $\sum_{i=1}^{\infty}\varphi\left(\frac{b_\varphi}{i}\right) < 1$. We will find $x \in S(ces_\varphi)$ and a sequence (x_m), $\|x_m\|_\varphi = 1$ for $m \in \mathbb{N}$, such that $x_m \to x$ coordinatewise and $\|x_m - x\|_\varphi \geqslant \varepsilon > 0$ for any $m \in \mathbb{N}$.

Let $x = (b_\varphi, 0, 0, \ldots)$. Then $\rho_\varphi(x) = I_\varphi(\sigma x) < 1$. For $\varepsilon = 1 - I_\varphi(\sigma x)$, by Lemma 5, there exists $\delta = \delta(\varepsilon) > 0$ such that $|I_\varphi(z) - I_\varphi(y)| < \varepsilon$ for each $z = (z(n))_{n=1}^{\infty}$ with $|||z|||_\varphi \leqslant 1$ and $|z(n)| \leqslant \frac{1}{3}b_\varphi$ for all $n \in \mathbb{N}$ and each $y \in l_\varphi$ satisfying $I_\varphi(z-y) < \delta$. We may assume without loss of generality that $\delta \leqslant \varphi\left(\frac{1}{2}b_\varphi\right)$. Therefore for each $m \geqslant 3$ we can find $\delta_m > 0$ such that $\sum_{i=m}^{\infty}\varphi\left(\frac{\delta_m}{i}\right) = \frac{\delta}{2}$. Let

$$x_m = (b_\varphi, \underbrace{0,\ldots,0}_{m-2 \text{ times}}, \delta_m, 0, \ldots)$$

for $m \geqslant 3$. It is obvious that $x_m \to x$ coordinatewise. The remaining part of the proof concerns $m \geqslant 3$. Since $\sum_{i=m}^{\infty} \varphi\left(\frac{b_\varphi+\delta_m}{i} - \frac{b_\varphi}{i}\right) = \frac{\delta}{2} < \delta$, by Lemma 5, we get

$$\begin{aligned}\rho_\varphi(x_m) &= \sum_{i=1}^{m-1} \varphi\left(\frac{b_\varphi}{i}\right) + \sum_{i=m}^{\infty} \varphi\left(\frac{b_\varphi+\delta_m}{i}\right) \\ &= \rho_\varphi(x) + \sum_{i=m}^{\infty} \varphi\left(\frac{b_\varphi+\delta_m}{i}\right) - \sum_{i=m}^{\infty} \varphi\left(\frac{b_\varphi}{i}\right) < \rho_\varphi(x) + \varepsilon = 1.\end{aligned}$$

By $\rho_\varphi\left(\frac{x_m}{\lambda}\right) = \infty$ for any $\lambda \in (0,1)$, we have $\|x_m\|_\varphi = 1$. Simulataneously

$$x_m - x = (\underbrace{0,\ldots,0}_{m-1 \text{ times}}, \delta_m, 0, \ldots).$$

Therefore $\rho_\varphi(x_m - x) = \sum_{i=m}^{\infty} \varphi\left(\frac{\delta_m}{i}\right) = \frac{\delta}{2} < \varphi(b_\varphi/2) < 1$, so $\|x_m - x\|_\varphi \geqslant \frac{\delta}{2}$. This finishes the proof.

4. Rotundity properties of Cesàro-Orlicz spaces

In this section convexity properties of Cesàro-Orlicz spaces will be considered. We start with some definitions.

We say that φ is strictly convex on the interval $[a,b]$ if for any u and v ($a \leqslant u < v \leqslant b$) we have $\varphi\left(\frac{v+w}{2}\right) < \frac{1}{2}\{\varphi(v) + \varphi(w)\}$.

For any Orlicz function φ, by φ^* we denote its complementary function in the sense of Young, that is, $\varphi^*(v) = \sup_{u \geqslant 0}\{u|v| - \varphi(u)\}$ for any $v \in \mathbb{R}$.

For any Banach space X we denote by $B(X)$ its closed unit ball and by $S(X)$ - its unit sphere. Recall that X is said to be rotund ($X \in (\mathbf{R})$ for short) if $\|x+y\| < 2$ for every $x,y \in S(X)$ with $x \neq y$. A Banach space X is said to be locally uniformly rotund ($X \in (\mathbf{LUR})$) if for each $x \in B(X)$ and $\varepsilon \in (0,2]$ there is $\delta = \delta(x,\varepsilon) \in (0,1)$ such that for any $y \in B(X)$ the inequality $\|x-y\| \geqslant \varepsilon$ implies that $\|x+y\| \leqslant 2(1-\delta)$. X is said to be uniformly rotund ($X \in (\mathbf{UR})$) if for any $\varepsilon \in (0,2]$ there exists $\delta \in (0,1)$ such that $\left\|\frac{x+y}{2}\right\| \leqslant 1-\delta$ whenever $x,y \in B(X)$ and $\|x-y\| \geqslant \varepsilon$.

We say that X is uniformly nonsquare ($X \in (\mathbf{UNSQ})$ for short) if there exists $\sigma \in (0,1)$ such that $\min(\|\frac{x+y}{2}\|, \|\frac{x-y}{2}\|) \leqslant 1-\sigma$ for every $x,y \in B(X)$.

In [7] it has been shown the following

THEOREM 2. *If $\varphi \in \delta_2$, then ces_φ is rotund if and only if $\sum_{i=1}^{\infty} \varphi\left(\frac{b_\varphi}{i}\right) \geqslant 1$ and φ is strictly convex on the interval $[0,v_2]$, where $2\varphi(v_2) + \sum_{i=3}^{\infty} \varphi\left(\frac{2v_2}{i}\right) = 1$.*

In [28] it has been shown for any Orlicz function φ satisfying condition δ_2 that ces_φ is not uniformly nonsquare, whence it follows immediately that ces_φ is not uniformly rotund. Now we will show the following

THEOREM 3. *If $\varphi \in \delta_2$, then the space ces_φ is locally uniformly rotund if and only if $\sum_{i=1}^{\infty} \varphi\left(\frac{b_\varphi}{i}\right) \geqslant 1$ and there is satisfied at least one of the following conditions:*

(i) φ is strictly convex on the interval $[0, v_1]$, where $\sum_{i=1}^{\infty} \varphi\left(\frac{v_1}{i}\right) = 1$,

(ii) φ is strictly convex on the interval $[0, v_2]$, where $2\varphi(v_2) + \sum_{i=3}^{\infty} \varphi\left(\frac{2v_2}{i}\right) = 1$ and φ^* satisfies the δ_2 condition.

Proof. Sufficiency. Take any $x \in ces_\varphi$ and a sequence (x_m) such that $\|x\|_\varphi = 1$, $\|x_m\|_\varphi = 1$ for all $m \in \mathbb{N}$ and $\|x + x_m\|_\varphi \to 2$ as $m \to \infty$. By Theorem 3 in [14] we can assume that $x \geqslant 0$ and $x_m \geqslant 0$ for all $m \in \mathbb{N}$. From Lemma 1 we have $\rho_\varphi(x) = \rho_\varphi(x_m) = 1$ for any $m \in \mathbb{N}$. If $x_m \to x$ coordinatewise, then by Theorem 1 we immediately get $\|x_m - x\|_\varphi \to 0$, which finishes the proof.

Now, we will show that the assumption that $x_m \not\to x$ coordinatewise leads to a contradiction with the condition $\|x_m + x\|_\varphi \to 2$.

Let i_1 be the smallest natural number such that $x_m(i_1) \not\to x(i_1)$. We can assume without loss of generality that

$$|x_m(i_1) - x(i_1)| > \varepsilon \tag{6}$$

and

$$|\sigma x_m(i_1) - \sigma x(i_1)| > \frac{\varepsilon}{2i_1} \tag{7}$$

for some $\varepsilon > 0$ and any $m \in \mathbb{N}$. If the function φ satisfies condition (i), then by Lemma 0.5 in [18], we have

$$\varphi\left(\sigma \frac{x_m + x}{2}(i_1)\right) = \varphi\left(\frac{\sigma x_m(i_1) + \sigma x(i_1)}{2}\right) \leqslant \frac{1 - p_1(\varepsilon)}{2}\left(\varphi(\sigma x_m(i_1)) + \varphi(\sigma x(i_1))\right) \tag{8}$$

for all $m \in \mathbb{N}$ with some $p_1(\varepsilon) \in (0,1)$. Hence

$$\begin{aligned} \rho_\varphi\left(\frac{x_m + x}{2}\right) &= I_\varphi\left(\sigma \frac{x_m + x}{2}\right) = I_\varphi\left(\frac{\sigma x_m + \sigma x}{2}\right) \\ &\leqslant \frac{1}{2}(I_\varphi(\sigma x_m) + I_\varphi(\sigma x)) - \frac{p_1(\varepsilon)}{2} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right) = 1 - \frac{p_1(\varepsilon)}{2} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right) \end{aligned}$$

for all $m \in \mathbb{N}$. Therefore, by Lemma 3, we get $\|\frac{x + x_m}{2}\|_\varphi \leqslant 1 - \sigma(\varepsilon)$ for all $m \in \mathbb{N}$ with some $\sigma(\varepsilon) \in (0,1)$, which is a contradiciton.

Now, we assume that φ does not satisfy condition (i) and $\min(\sigma x_m(i_1), \sigma x(i_1)) \leqslant s < v_2$ for all $m \in \mathbb{N}$ (we can pass to a subsequence if necessary to have such a situation). From Lemma 0.5 or Lemma 0.6 in [18] it follows that we can find $p_2(\varepsilon) \in (0,1)$ such that inequality (8) is satisfied. As above, we have $\rho_\varphi\left(\frac{x_m + x}{2}\right) \leqslant 1 - \frac{p_2(\varepsilon)}{2} \cdot \varphi(\varepsilon/(2i_1))$, which is again a contradiction.

Let now φ does not satisfy condition (i) and

$$\liminf_{m \to \infty}(\min(\sigma x_m(i_1), \sigma x(i_1))) \geqslant v_2. \tag{9}$$

If there exists a subsequence (m_n) such that $\sigma x_{m_n}(i_1) < \sigma x(i_1)$, then by (9) we get

$\sigma x(i_1) \geqslant v_2 + \frac{\varepsilon}{2i_1}$. Let now $\sigma x_m(i_1) > \sigma x(i_1)$ for $m \geqslant m_0$. By (9) we have $\sigma x(i_1) \geqslant v_2$, whence $\sigma x_m(i_1) \geqslant v_2 + \frac{\varepsilon}{2i_1}$ for $m \geqslant m_0$. Therefore, we can assume that

$$\max(\sigma x_m(i_1), \sigma x(i_1)) \geqslant v_2 + \frac{\varepsilon}{2i_1}$$

for $m \geqslant m_0$ with some $m_0 \in \mathbb{N}$. We will consider two cases separately.

First assume that there exists $i_2 > i_1$ such that $|\sigma x_m(i_2) - \sigma x(i_2)| > \eta > 0$ for all $m \in \mathbb{N}$. Since $\max(\sigma x_m(i_1), \sigma x(i_1)) \geqslant v_2 + \frac{\varepsilon}{2i_1}$ for all $m \geqslant m_0$ and $\rho_\varphi(x) = \rho_\varphi(x_m) = 1$ for all $m \in \mathbb{N}$, by the definition of v_2, there exists $t < v_2$ such that $\min(\sigma x_m(i_2), \sigma x(i_2)) \leqslant t$. Proceeding as above, we can get a contradiction.

Finally, we assume that $\sigma x_m(i) \to \sigma x(i)$ for each $i \neq i_1$. Since $\varphi^* \in \delta_2$, there exists $q \in (0,1)$ such that $\varphi\left(\frac{u}{2}\right) \leqslant \frac{1-q}{2} \cdot \varphi(u)$ for all $u \in [0, v_2]$ (see [12]). Define $\varepsilon_1 = \frac{q}{8} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right)$ and let $\delta_1 = \delta_1(\varepsilon_1)$ be the constant from Lemma 5. We may assume without loss of generality that $\delta_1 < \frac{1}{4}\varphi\left(\frac{\varepsilon}{2i_1}\right)$. Note that for any $a \in (0, b_\varphi)$ we have that $\sum_{i=n}^{\infty} \varphi\left(\frac{na}{i}\right) \leqslant \sum_{i=n+1}^{\infty} \varphi\left(\frac{(n+1)a}{i}\right)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \sum_{i=n}^{\infty} \varphi\left(\frac{na}{i}\right) = \infty$. So, we can find $i_3 > i_1$ and $m_1 \in \mathbb{N}$ such that $\sum_{i=i_3}^{\infty} \varphi(\sigma x(i)) < \delta_1$,

$$\sum_{i=i_3}^{\infty} \varphi\left(\frac{i_3 \cdot \min((b_\varphi/3), v_2)}{i}\right) \geqslant 1$$

and

$$\sum_{i=\{1,\ldots,i_3-1\}\setminus\{i_1\}} |\varphi(\sigma x_m(i)) - \varphi(\sigma x(i))| < \frac{1}{2} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right) \tag{10}$$

for $m \geqslant m_1$. Now we will show that $x(i_1) > x_m(i_1)$ for $m \geqslant m_1$ in inequality (6). Assume for the contrary that we have $x(i_1) < x_m(i_1)$ for some $m \geqslant m_1$ in that place. Then for the same m, by inequality (7), we have $\sigma x_m(i_1) - \sigma x(i_1) > \frac{\varepsilon}{2i_1}$, whence, by the superadditivity of the Orlicz function φ on R_+, we get

$$\varphi(\sigma x(i_1)) \leqslant \varphi(\sigma x_m(i_1)) - \varphi(\sigma x_m(i_1) - \sigma x(i_1)) < \varphi(\sigma x_m(i_1)) - \varphi\left(\frac{\varepsilon}{2i_1}\right). \tag{11}$$

Since $\sum_{i=i_3}^{\infty} \varphi(\sigma x(i)) < \delta_1 < \frac{1}{4}\varphi\left(\frac{\varepsilon}{2i_1}\right)$, by inequalities (10) and (11), we have

$$\begin{aligned} 1 &= \varphi(\sigma x(i_1)) + \sum_{i=\{1,\ldots,i_3-1\}\setminus\{i_1\}} \varphi(\sigma x(i)) + \sum_{i=i_3}^{\infty} \varphi(\sigma x(i) \\ &< \varphi(\sigma x_m(i_1)) - \varphi\left(\frac{\varepsilon}{2i_1}\right) + \sum_{i=\{1,\ldots,i_3-1\}\setminus\{i_1\}} \varphi(\sigma x_m(i)) \\ &\quad + \frac{1}{2} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right) + \frac{1}{4} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right) \\ &\leqslant 1 - \frac{1}{4} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right) < 1, \end{aligned}$$

which is a contradiction. Therefore $x(i_1) > x_m(i_1)$ for $m \geqslant m_1$ in inequality (6).

Analogously as inequality (11), we can show that

$$\varphi(\sigma x_m(i_1)) < \varphi(\sigma x(i_1)) - \varphi\left(\frac{\varepsilon}{2i_1}\right)$$

for $m \geqslant m_1$. Hence and from inequality (10), by the equalities $\rho_\varphi(x_m) = \rho_\varphi(x) = 1$ for $m \in N$, we get that the set $N_m = \{i \geqslant i_3 : \sigma x_m(i) > \sigma x(i)\}$ is nonempty for $m \geqslant m_1$. Moreover,

$$\sum_{i \in N_m} \varphi(\sigma x_m(i)) - \sum_{i \in N_m} \varphi(\sigma x(i)) > \frac{1}{2} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right)$$

for $m \geqslant m_1$. Since

$$\sum_{i \in N_m} \varphi\left(\frac{\sigma x_m(i) + \sigma x(i)}{2} - \frac{\sigma x_m(i)}{2}\right) \leqslant \sum_{i=i_3}^{\infty} \varphi(\sigma x(i)) < \delta_1,$$

by Lemma 5, we have

$$\left| \sum_{i \in N_m} \varphi\left(\frac{\sigma x_m(i) + \sigma x(i)}{2}\right) - \sum_{i \in N_m} \varphi\left(\frac{\sigma x_m(i)}{2}\right) \right| < \frac{q}{8} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right)$$

for $m \geqslant m_1$. Hence, for the same m, we have

$$\begin{aligned}
&\rho_\varphi\left(\frac{x + x_m}{2}\right) = I_\varphi\left(\sigma \frac{x + x_m}{2}\right) = I_\varphi\left(\frac{\sigma x + \sigma x_m}{2}\right) \\
&\leqslant \frac{1}{2} \sum_{i \in \mathbb{N} \setminus N_m} \varphi(\sigma x(i)) + \frac{1}{2} \sum_{i \in \mathbb{N} \setminus N_m} \varphi(\sigma x_m(i)) + \sum_{i \in N_m} \varphi\left(\frac{\sigma x(i) + \sigma x_m(i)}{2}\right) \\
&\leqslant \frac{1}{2} \sum_{i \in \mathbb{N} \setminus N_m} \varphi(\sigma x(i)) + \frac{1}{2} \sum_{i \in \mathbb{N} \setminus N_m} \varphi(\sigma x_m(i)) + \sum_{i \in N_m} \varphi\left(\frac{\sigma x_m(i)}{2}\right) + \frac{q}{8} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right) \\
&\leqslant \frac{1}{2} \sum_{i \in \mathbb{N} \setminus N_m} \varphi(\sigma x(i)) + \frac{1}{2} \sum_{i \in \mathbb{N} \setminus N_m} \varphi(\sigma x_m(i)) + \frac{1-q}{2} \sum_{i \in N_m} \varphi(\sigma x_m(i)) + \frac{q}{8} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right) \\
&\leqslant 1 - \frac{q}{4} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right) + \frac{q}{8} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right) = 1 - \frac{q}{8} \cdot \varphi\left(\frac{\varepsilon}{2i_1}\right),
\end{aligned}$$

which is a contradiction.

Necessity. Let $\varphi \in \delta_2$. By Theorem 2 we can assume that $\sum_{i=1}^{\infty} \varphi\left(\frac{b_\varphi}{i}\right) \geqslant 1$ and φ is strictly convex on the interval $[0, v_2]$. Assume now that φ is not strictly convex on the interval $[v_2, v_1]$ with $v_2 < v_1$ and $\varphi^* \notin \delta_2$. We will find $x \in S(ces_\varphi)$ and (x_n) in $B(ces_\varphi)$ such that $\|x + x_n\|_\varphi \to 2$ and $\|x - x_n\|_\varphi \geqslant \eta > 0$.

By the assumption, there exist a and b ($v_2 \leqslant a < b \leqslant v_1$) such that φ is affine on the interval $[a, b]$. Let $x = (b, c, 0, \ldots)$, where $c \geqslant 0$ is chosen in such a way that

$$\rho_\varphi(x) = \varphi(b) + \sum_{i=2}^{\infty} \varphi\left(\frac{b+c}{i}\right) = 1.$$

For building the disired sequence (x_n), we will define by induction special fours (i_n, u_n, j_n, k_n) for $n \geqslant 2$. Let $i_2 > 2$ be such that $\sum_{i=i_2}^{\infty} \varphi\left(\frac{b+c}{i}\right) \leqslant \frac{\varepsilon}{2^2 K^2}$, where $\varepsilon = \varphi(b) - \varphi(a)$ and K be the constant from condition δ_2. Since $\varphi^* \notin \delta_2$, we can find $u_2 \leqslant \frac{b+c}{i_2}$ such that

$$\varphi\left(\frac{u_2}{2}\right) \geqslant \frac{1 - \frac{1}{(2+1)!}}{2} \varphi(u_2).$$

Let $j_2 \geqslant i_2$ satisfy the inequalities

$$\frac{b+c}{j_2+1} < u_2 \leqslant \frac{b+c}{j_2}$$

and k_2 be the biggest natural number such that

$$\sum_{i=j_2+1}^{j_2+k_2} \varphi(u_2) + \sum_{i=j_2+k_2+1}^{\infty} \varphi\left(\frac{(j_2+k_2)u_2}{i}\right) \leqslant \varepsilon.$$

Having defined a four $(i_{n-1}, u_{n-1}, j_{n-1}, k_{n-1})$ for $n > 2$ we will define next four (i_n, u_n, j_n, k_n). Let $i_n > j_{n-1} + k_{n-1}$ be such that $\sum_{i=i_n}^{\infty} \varphi\left(\frac{b+c}{i}\right) \leqslant \frac{\varepsilon}{2^n K^n}$. Since $\varphi^* \notin \delta_2$, we can find $u_n \leqslant \frac{b+c}{i_n}$ such that

$$\varphi\left(\frac{u_n}{2}\right) \geqslant \frac{1 - \frac{1}{(n+1)!}}{2} \varphi(u_n). \tag{12}$$

Analogously as for $n = 2$ we chose $j_n \geqslant i_n$ satisfying the condition

$$\frac{b+c}{j_n+1} < u_n \leqslant \frac{b+c}{j_n}$$

and k_n being the biggest natural number satisfying the inequality

$$\sum_{i=j_n+1}^{j_n+k_n} \varphi(u_n) + \sum_{i=j_n+k_n+1}^{\infty} \varphi\left(\frac{(j_n+k_n)u_n}{i}\right) \leqslant \varepsilon.$$

Now let us define $(x_n)_{n=1}^{\infty}$, by

$$x_n = (a, c+(b-a), \underbrace{0, \ldots, 0}_{j_n-2 \text{ times}}, (j_n+1)u_n - (b+c), \underbrace{u_n, \ldots, u_n}_{k_n-1 \text{ times}}, 0, \ldots)$$

for all $n \in \mathbb{N}$. Since

$$\sum_{i=j_n+1}^{j_n+k_n} \varphi(u_n) + \sum_{i=j_n+k_n+1}^{\infty} \varphi\left(\frac{(j_n+k_n)u_n}{i}\right) \to \varepsilon,$$

we get $\rho_\varphi(x_n) \to 1$.

Now for any $n \geqslant 2$ we will show that

$$\varphi\left(\frac{u}{2}\right) \geqslant \frac{1 - \frac{1}{n}}{2} \varphi(u) \tag{13}$$

whenever $\frac{1}{n!}u_n \geqslant u \geqslant u_n$. Note that the functions ψ_n graphs of which are the straight lines

$$\psi_n(u) = \left(1 + \frac{1}{(n+1)!}\right)\frac{\varphi(u_n)}{u_n} \cdot u - \frac{1}{(n+1)!}\varphi(u_n)$$

crosing trough the points $\left(\frac{1}{2}u_n, \frac{1-\frac{1}{(n+1)!}}{2}\varphi(u_n)\right)$ and $(u_n, \varphi(u_n))$ satisfy the inequality

$$\psi_n\left(\frac{u}{2}\right) \geqslant \frac{1-\frac{1}{n}}{2}\psi_n(u),$$

for $u \geqslant \frac{1}{n!}u_n$. In order to see that inequality (13) holds, assume for the contrary that

$$\varphi\left(\frac{v_n}{2}\right) < \frac{1-\frac{1}{n}}{2}\varphi(v_n)$$

for some $v_n \in [\frac{1}{n!}u_n, u_n)$ and $n \geqslant 2$. Since $\varphi(u_n) = \psi_n(u_n)$ and $\varphi(u_n/2) \geqslant \psi_n(u_n/2)$ (see inequality (12)), by convexity of φ, we get $\varphi(v_n/2) > \psi_n(v_n/2)$. Let χ_n be the function graph of which pass trough the point $(\frac{v_n}{2}, \varphi(\frac{v_n}{2}))$ and is parallel to the straight line being the graph of ψ_n. Since

$$\frac{2\chi_n(\frac{v_n}{2})}{\chi_n(v_n)} \geqslant \frac{2\psi_n(\frac{v_n}{2})}{\psi_n(v_n)} \geqslant 1 - \frac{1}{n},$$

we have $\chi_n(v_n) < \varphi(v_n)$. Simultaneously, $\varphi(u_n) = \psi_n(u_n) < \chi_n(u_n)$ for $u_n > v_n$ which contradicts the convexity of φ.

By (13), we get

$$\begin{aligned}\rho_\varphi\left(\frac{x+x_n}{2}\right) &\geqslant \varphi\left(\frac{b+a}{2}\right) + \sum_{i=2}^{j_n}\varphi\left(\frac{b+c}{i}\right) + \sum_{i=j_n+1}^{j_n+k_n}\varphi\left(\frac{u_n}{2}\right)\\ &+ \sum_{i=j_n+k_n+1}^{\infty}\varphi\left(\frac{(j_n+k_n)u_n}{2i}\right) \geqslant \frac{1}{2}\{\varphi(b)+\varphi(a)\} + \sum_{i=2}^{j_n}\varphi\left(\frac{b+c}{i}\right)\\ &+ \frac{1-\frac{1}{n}}{2}\left\{\sum_{i=j_n+1}^{j_n+k_n}\varphi(u_n) + \sum_{i=j_n+k_n+1}^{n!(j_n+k_n)}\varphi\left(\frac{(j_n+k_n)u_n}{i}\right)\right\}\\ &\to \frac{1}{2}\rho_\varphi(x) + \frac{1}{2}\rho_\varphi(x_n) \to 1,\end{aligned}$$

whence, by Lemma 3, we have $\|x+x_n\|_\varphi \to 2$. Simultaneously, $\rho_\varphi(x-x_n) \geqslant \varphi(b-a) = \eta > 0$. Since $\eta < 1$, we have $\|x-x_n\|_\varphi \geqslant \eta$.

REMARK 2. For any Orlicz function φ we get the implication: if an Orlicz space l_φ is rotund {see Theorem 0.7 in [18]} (locally uniformly rotund {see Theorem 2 in [18]}), then ces_φ is rotund (locally uniformly rotund) either. Examples 2 and 3 show Orlicz functions ψ and φ such that ces_ψ and ces_φ are locally uniformly rotund but l_ψ and l_φ are not even rotund (recall that an Orlicz space l_φ is rotund if and only if $\varphi \in \delta_2$, $\varphi(b_\varphi) \geqslant 1$ and φ is strictly convex on the interval $[0, u_2]$, where $\varphi(u_2) = \frac{1}{2}$; see [7]).

EXAMPLE 2. For the Orlicz function ψ defined by the formula

$$\psi(u)=\begin{cases}u^2 & \text{for} \quad |u|\leqslant \frac{\sqrt{3}}{2},\\ \infty & \text{for} \quad |u|>\frac{\sqrt{3}}{2},\end{cases}$$

we have that $\psi\in\delta_2$, ψ is strictly convex on the interval $[0,b_\psi]$, $\sum_{i=1}^\infty \varphi\left(\frac{b_\psi}{i}\right)=\frac{\pi^2}{8}>1$ and $\varphi(b_\psi)=\frac{3}{4}$, so the space ces_ψ is locally uniformly rotund, but the space l_ψ is not rotund.

EXAMPLE 3. Let us define the function

$$\varphi(u)=\begin{cases}u^2 & \text{for } |u|\leqslant\sqrt{\frac{3}{2\pi^2-9}}\\ 2u\sqrt{\frac{3}{2\pi^2-9}}-\frac{3}{2\pi^2-9} & \text{for } |u|>\sqrt{\frac{3}{2\pi^2-9}}.\end{cases}$$

We have $\varphi\in\delta_2$, $b_\varphi=\infty$ and φ is strictly convex on the interval $\left[0,\sqrt{\frac{3}{2\pi^2-9}}\right]$. Since

$$\begin{aligned}&2\varphi\left(\sqrt{\frac{3}{2\pi^2-9}}\right)+\sum_{i=3}^\infty\varphi\left(\frac{2}{i}\sqrt{\frac{3}{2\pi^2-9}}\right)\\ &=\frac{6}{2\pi^2-9}+\frac{12}{2\pi^2-9}\sum_{i=3}^\infty\frac{1}{i^2}=\frac{6}{2\pi^2-9}+\frac{12}{2\pi^2-9}\left(\frac{\pi^2}{6}-\frac{5}{4}\right)=1,\end{aligned}$$

and $\varphi^*\in\delta_2$, we know that ces_φ is locally uniformly rotund. Simultaneously,

$$\varphi\left(\sqrt{\frac{3}{2\pi^2-9}}\right)=\frac{3}{2\pi^2-9}<\frac{1}{2},$$

so the Orlicz space l_φ is not rotund.

5. Finite dimensional subspaces of Cesàro-Orlicz spaces

For any $n\geqslant 2$ we can define the subspace ces_φ^n of the space ces_φ by the formula

$$ces_\varphi^n=\{x=(x(i))_{i=1}^\infty\in ces_\varphi : x(i)=0 \text{ for all } i>n\}.$$

Obviously, ces_φ^n are subspaces of $(ces_\varphi)_a$ for all $n\geqslant 2$. The spaces ces_φ^n $(n\geqslant 2)$ will be investigated with the original norm $\|\cdot\|_\varphi$ as well as with the norm

$$\|x\|_n=\inf\left\{\lambda>0:\rho_n\left(\tfrac{x}{\lambda}\right)\leqslant 1\right\},$$

where

$$\rho_n(x)=\sum_{i=1}^n\varphi(\sigma x(i)),$$

which has been introduced in [28]. Note that if there exists $n_1 \in \mathbb{N}$ such that $\sum_{n=n_1}^{\infty} \varphi\left(\frac{1}{n}\right) < \infty$ (see Remark 1), then for any $n \geqslant 2$ we have $ces_{\varphi}^{n} = l_n^0$ and there exists $k(n) > 0$ such that

$$\|x\|_n \leqslant \|x\|_{\varphi} \leqslant k(n)\,\|x\|_n$$

for any $x \in ces_{\varphi}^{n}$. In the opposite case ces_{φ}^{n} is trivial.

REMARK 3. Obviously if $\sum_{n=i}^{\infty} \varphi\left(\frac{1}{n}\right) = \infty$ for all $i \in \mathbb{N}$, the space l_n^0 can be investigated with the norm $\|\cdot\|_n$ and then Lemma 6 and Theorems 5 and 7 remain true.

REMARK 4. For some Orlicz function φ one can easily find the smallest constant $k_{\varphi}(n) > 0$ such that

$$\|x\|_{\varphi} \leqslant k_{\varphi}(n)\,\|x\|_n \tag{14}$$

for any $x \in ces_{\varphi}^{n}$, $n \in N$. In the example presented below such the smallest number is found for the Orlicz function $\varphi(u) = u^2$.

EXAMPLE 4. Let $\varphi(u) = u^2$ for $u \in \mathbb{R}$. Then for any $n \in \mathbb{N}$ and any $x = (x_1, \ldots, x_n, 0, 0, \ldots) \in ces_{\varphi}^{n}$, we have

$$\|x\|_n = \sqrt{|x_1|^2 + \ldots + \left(\frac{|x_1| + \ldots + |x_{n-1}|}{n-1}\right)^2 + \left(\frac{|x_1| + \ldots + |x_{n-1}| + |x_n|}{n}\right)^2}$$

and

$$\|x\|_{\varphi} = \sqrt{|x_1|^2 + \ldots + \left(\frac{|x_1| + \ldots + |x_{n-1}|}{n-1}\right)^2 + (|x_1| + \ldots + |x_{n-1}| + |x_n|)^2 \sum_{i=n}^{\infty} \frac{1}{i^2}}.$$

We will show that the smallest number $k_{\varphi}(n)$ for which (14) holds equals $n\sqrt{\sum_{i=n}^{\infty} \frac{1}{i^2}}$, because

$$\|x\|_{\varphi} = n\sqrt{\sum_{i=n}^{\infty} \frac{1}{i^2}} \cdot \left(\frac{|x_1|^2}{n^2 \sum_{i=n}^{\infty} \frac{1}{i^2}} + \ldots + \frac{1}{n^2 \sum_{i=n}^{\infty} \frac{1}{i^2}} \left(\frac{|x_1| + \ldots + |x_{n-1}|}{n-1}\right)^2 \right.$$
$$\left. + \left(\frac{|x_1| + \ldots + |x_{n-1}| + |x_n|}{n}\right)^2\right)^{\frac{1}{2}} \leqslant k_{\varphi}(n)\|x\|_n,$$

for any $x \in ces_{\varphi}^{n}$. Simultaneously for $y = y_n e_n$, we get

$$\|y\|_{\varphi} = \sqrt{|y_n|^2 \sum_{i=n}^{\infty} \frac{1}{i^2}} = n\sqrt{\sum_{i=n}^{\infty} \frac{1}{i^2}} \sqrt{\left(\frac{|y_n|}{n}\right)^2} = k_{\varphi}(n)\,\|y\|_n.$$

Figure 1 shows spheres of 2-dimensional Cesàro-Orlicz sequence spaces generated by the function $\varphi(u) = u^2$ equipped with the norms $\|\cdot\|_2$ and $\|\cdot\|_{\varphi}$.

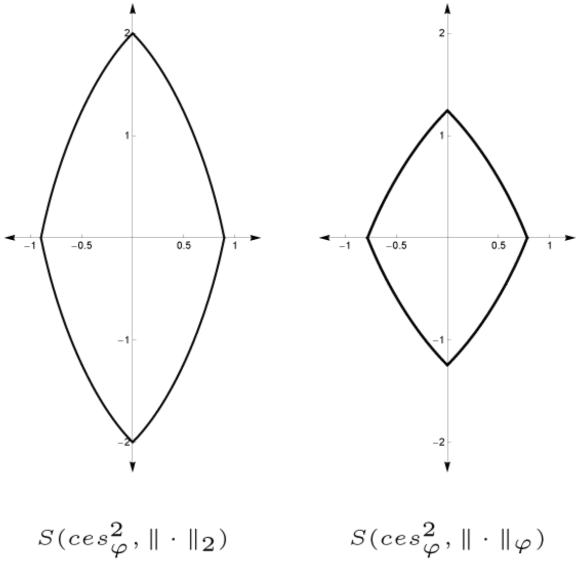

Figure 1:

In the further part of the paper the lemma presented below (proof of which is omitted) will be helpful.

LEMMA 6. *The following assertions are true:*

(i) If $\rho_n(x) = 1$, *then* $\|x\|_n = 1$ *for any* $x \in ces^n_\varphi$.

(ii) For any $x \in ces^n_\varphi$ *the equality* $\|x\|_n = 1$ *implies that* $\rho_n(x) = 1$ *if and only if* $\varphi(b_\varphi) \geqslant 1$.

Now we will present criteria for rotundity of the spaces $(ces^n_\varphi, \|\cdot\|_\varphi)$ and $(ces^n_\varphi, \|\cdot\|_n)$. The case of $n = 2$ needs a separate treatment. We start with the following

THEOREM 4. *The space* $(ces^2_\varphi, \|\cdot\|_\varphi)$ *is rotund if and only if*

(i) $\sum_{i=1}^{\infty} \varphi\left(\frac{b_\varphi}{i}\right) \geqslant 1$,

(ii) φ *vanishes only at zero.*

Proof. Sufficiency. Assume that $a_\varphi = 0$, $\sum_{i=1}^{\infty} \varphi\left(\frac{b_\varphi}{i}\right) \geqslant 1$ and take two sequences $x = (x_1, x_2, 0, 0, \ldots)$ and $y = (y_1, y_2, 0, 0, \ldots)$ such that $x \neq y$ and $\|x\|_\varphi = 1 = \|y\|_\varphi$. We can assume without loss of generality that $x, y \geqslant 0$ (see Theorem 2 in [14]). Since $ces^2_\varphi \subset (ces_\varphi)_a$, by Lemma 1, we have $\rho_\varphi(x) = 1 = \rho_\varphi(y)$.

First suppose that $x_1 + x_2 = y_1 + y_2$. Since $x \neq y$, we can assume that $x_1 < y_1$. Hence

$$1 = \rho_\varphi(x) = \varphi(x_1) + \sum_{n=2}^{\infty} \varphi\left(\frac{x_1 + x_2}{n}\right) = \varphi(x_1) + \sum_{n=2}^{\infty} \varphi\left(\frac{y_1 + y_2}{n}\right)$$
$$< \varphi(y_1) + \sum_{n=2}^{\infty} \varphi\left(\frac{y_1 + y_2}{n}\right) = \rho_\varphi(y) = 1,$$

which is a contradiction. Therefore, we may assume without loss of generality that $x_1 + x_2 < y_1 + y_2$.

Since φ is convex, we have

$$\rho_\varphi\left(\frac{x+y}{2}\right) = \varphi\left(\frac{x_1 + y_1}{2}\right) + \sum_{n=2}^{\infty} \varphi\left(\frac{\frac{x_1+y_1}{2} + \frac{x_2+y_2}{2}}{n}\right)$$
$$\leqslant \frac{1}{2}(\varphi(x_1) + \varphi(y_1)) + \frac{1}{2}\sum_{n=2}^{\infty}\left(\varphi\left(\frac{x_1 + x_2}{n}\right) + \varphi\left(\frac{y_1 + y_2}{n}\right)\right)$$
$$= \frac{1}{2}\rho_\varphi(x) + \frac{1}{2}\rho_\varphi(y) = 1.$$

If $\rho_\varphi\left(\frac{x+y}{2}\right) = 1$, then we get

$$\varphi\left(\frac{x_1 + y_1}{2}\right) = \frac{1}{2}(\varphi(x_1) + \varphi(y_1))$$

and

$$\varphi\left(\frac{\frac{x_1+x_2}{n} + \frac{y_1+y_2}{n}}{2}\right) = \frac{1}{2}\left\{\varphi\left(\frac{x_1 + x_2}{n}\right) + \varphi\left(\frac{y_1 + y_2}{n}\right)\right\}$$

for $n \geqslant 2$. Hence we get that φ is affine on the following intervals

$$\ldots, \left[\frac{x_1 + x_2}{n+1}, \frac{y_1 + y_2}{n+1}\right], \left[\frac{x_1 + x_2}{n}, \frac{y_1 + y_2}{n}\right], \ldots, \left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right].$$

Since $x_1 + x_2 < y_1 + y_2$ and $\frac{n}{n+1} \to 1$ as $n \to \infty$, so $x_1 + x_2 < \frac{n}{n+1}(y_1 + y_2)$ for $n \in \mathbb{N}$ large enough. Hence $\frac{x_1+x_2}{n} < \frac{y_1+y_2}{n+1}$ for the same n. It means that the intervals of affinity of φ overlap each other essentially, whence we get that φ is affine on the interval $[0, \alpha]$ for some $\alpha > 0$. From this fact and the assumption that $a_\varphi = 0$, we conclude that the space $(ces_\varphi^2, \|\cdot\|_\varphi)$ is trivial, a contradiction (see Remark 1).

Therefore $\rho_\varphi\left(\frac{x+y}{2}\right) < 1$ and, by Lemma 1, we have $\|\frac{x+y}{2}\|_\varphi < 1$.

Necessity. First we assume that $\sum_{i=1}^{\infty} \varphi\left(\frac{b_\varphi}{i}\right) < 1$. We have $\varphi(b_\varphi) < \infty$ and, by Remark 1, $\sum_{i=2}^{\infty} \varphi\left(\frac{2b_\varphi}{i}\right) < \infty$. So, we can find $\varepsilon \in (0, b_\varphi]$ such that

$$\varphi(b_\varphi) + \sum_{i=2}^{\infty} \varphi\left(\frac{b_\varphi + \varepsilon}{i}\right) \leqslant 1.$$

For $x = (b_\varphi, \varepsilon, 0, 0, \ldots)$ and $y = \left(b_\varphi, 0, 0, 0, \ldots\right)$, we have $\rho_\varphi(x) \leqslant 1$, $\rho_\varphi(y) < 1$, $\rho_\varphi\left(\frac{x+y}{2}\right) < 1$ and

$$\rho_\varphi\left(\frac{x}{\lambda}\right) = \rho_\varphi\left(\frac{y}{\lambda}\right) = \rho_\varphi\left(\frac{x+y}{2\lambda}\right) = \infty.$$

for any $\lambda \in (0,1)$. Therefore $\|x\|_\varphi = \|y\|_\varphi = \|\frac{x+y}{2}\|_\varphi = 1$, so $(ces_\varphi^2, \|\cdot\|_\varphi)$ is not rotund.

Now assume that $\sum_{i=1}^{\infty} \varphi\left(\frac{b_\varphi}{i}\right) \geqslant 1$ and $a_\varphi > 0$. If $\varphi(b_\varphi) \geqslant 1$, we can find $u_1 > 0$ such that $\varphi(u_1) = 1$; obviously $\sum_{i=2}^{\infty} \varphi\left(\frac{2u_1}{i}\right) \geqslant 1$. If $\varphi(b_\varphi) < 1$, we define $u_1 = b_\varphi$. By

$$\begin{aligned}\sum_{i=2}^{\infty} \varphi\left(\frac{2b_\varphi}{i}\right) &= \varphi(b_\varphi) + \varphi\left(\frac{2}{3}b_\varphi\right) + \varphi\left(\frac{2}{4}b_\varphi\right) + \varphi\left(\frac{2}{5}b_\varphi\right) + \ldots \\ &\geqslant \varphi(b_\varphi) + \varphi\left(\frac{1}{2}b_\varphi\right) + \varphi\left(\frac{1}{3}b_\varphi\right) + \varphi\left(\frac{1}{4}b_\varphi\right) + \ldots = \sum_{i=1}^{\infty} \varphi\left(\frac{b_\varphi}{i}\right) \geqslant 1,\end{aligned}$$

we get again $\sum_{i=2}^{\infty} \varphi\left(\frac{2u_1}{i}\right) \geqslant 1$. Since the function

$$f(v) = \sum_{i=2}^{\infty} \varphi\left(\frac{v}{i}\right)$$

is continuous on the interval $[a_\varphi, 2u_1]$ and $f(a_\varphi) = 0$ we can find $c > 0$ such that

$$\sum_{i=2}^{\infty} \varphi\left(\frac{a_\varphi + c}{i}\right) = 1.$$

Defining $x = (a_\varphi, c, 0, 0, \ldots)$ and $y = (0, a_\varphi + c, 0, 0, \ldots)$, we have $\rho_\varphi(x) = \rho_\varphi(y) = \rho_\varphi\left(\frac{x+y}{2}\right) = 1$ whence, by Lemma 1, we get $\|x\|_\varphi = \|y\|_\varphi = \|\frac{x+y}{2}\|_\varphi = 1$. This means that $(ces_\varphi^2, \|\cdot\|_\varphi)$ is not rotund.

Before formulating the next theorem assume that $\varphi(b_\varphi) \geqslant 1$ and define $u_1 > 0$ and $u_2 > 0$ such that $\varphi(u_1) = 1$ and $\varphi(u_2) = \frac{1}{2}$.

THEOREM 5. *The space $(ces_\varphi^2, \|\cdot\|_2)$ is rotund if and only if*

(i) $\varphi(b_\varphi) \geqslant 1$,

(ii) φ *vanishes only at zero,*

(iii) if φ is affine on an interval $[c,d] \subset [0,u_2]$, then φ is strictly convex on the interval $[e,f] \subset [u_2,u_1]$, where $\varphi(c)+\varphi(f) = \varphi(d)+\varphi(e) = 1$, and conversely, if φ is affine on $[e,f]$, then φ is strictly convex on $[c,d]$, where $[c,d]$ and $[e,f]$ are as above.

Proof. *Sufficiency.* Assume that conditions (i)-(iii) are satisfied. Let

$$x=(x_1,x_2,0,0,\dots),\quad y=(y_1,y_2,0,0,\dots)\in S(ces_\varphi^2,\|\cdot\|_2),\quad x\geqslant 0,\ y\geqslant 0$$

(see Theorem 2 in [14]) and $x\neq y$. Define $u=\left(x_1,\frac{x_1+x_2}{2},0,0,\dots\right)$ and $v=\left(y_1,\frac{y_1+y_2}{2},0,0,\dots\right)$. We have $u,v\in(S(l_\varphi^2))_+$ (l_φ^2 is the 2-dimensional Orlicz space, see [15]), $u\neq v$ and

$$\sigma\frac{x+y}{2}(i)=\frac{\sigma x(i)+\sigma y(i)}{2}=\frac{u(i)+v(i)}{2}\quad\text{for}\quad i=1,2.$$

From Theorem 2.2 in [15], we get that $\rho_2\left(\frac{x+y}{2}\right)=I_\varphi\left(\frac{u+v}{2}\right)<1$. By Lemma 6, $\|\frac{x+y}{2}\|_2<1$, so $(ces_\varphi^2,\|\cdot\|_2)$ is rotund.

Necessity. First we assume that $\varphi(b_\varphi)<1$. We can find $u_0\in(0,b_\varphi]$ such that $\varphi(u_0)+\varphi(b_\varphi)<1$. For $x=(u_0,2b_\varphi-u_0,0,0,\dots)$ and $y=(0,2b_\varphi,0,0,\dots)$ we have $\sigma x\chi_{\{1,2\}}=(u_0,b_\varphi)$, $\sigma y\chi_{\{1,2\}}=(0,b_\varphi)$ and $\sigma\frac{x+y}{2}\chi_{\{1,2\}}=\left(\frac{u_0}{2},b_\varphi\right)$. Therefore $\rho_2(x)<1$, $\rho_2(y)<1$, and $\rho_2\left(\frac{x+y}{2}\right)<1$. Simultaneously, for any $\lambda\in(0,1)$, we have

$$\rho_2\left(\frac{x}{\lambda}\right)=\rho_2\left(\frac{y}{\lambda}\right)=\rho_2\left(\frac{x+y}{2\lambda}\right)=\infty,$$

whence $\|x\|_2=\|y\|_2=\|\frac{x+y}{2}\|_2=1$. Since $x\neq y$, this means that $(ces_\varphi^2,\|\cdot\|_2)$ is not rotund.

Let now $\varphi(b_\varphi)\geqslant 1$ and $a_\varphi>0$. There exists $u_1\in(a_\varphi,b_\varphi]$ such that $\varphi(u_1)=1$. Take two sequences $x=(0,2u_1,0,0,\dots)$ and $y=(a_\varphi,2u_1-a_\varphi,0,0,\dots)$. Obviously, $x\neq y$ and, moreover, $\rho_2(x)=\rho_2(y)=\rho_2\left(\frac{x+y}{2}\right)=1$. Therefore, the space $(ces_\varphi^2,\|\cdot\|_2)$ is not rotund.

Finally, we assume that conditions (i) and (ii) are satisfied, the function φ is affine on an interval $[c,d]\subset[0,u_2]$ and it is not strictly convex on the interval $[e,f]\subset[u_2,u_1]$, that is, there exsists an interval $[e_1,f_1]\subset[e,f]$ such that φ is affine on $[e_1,f_1]$. We find c_1, d_1 such that $0<c\leqslant c_1\leqslant d_1\leqslant c$ and

$$\varphi(c_1)+\varphi(f_1)=\varphi(d_1)+\varphi(e_1)=1. \tag{15}$$

Defining the sequences $x=(c_1,2f_1-c_1,0,0,\dots)$ and $y=(d_1,2e_1-d_1,0,0,\dots)$, we have $\sigma x\chi_{\{1,2\}}=(c_1,f_1)$, $\sigma y\chi_{\{1,2\}}=(d_1,e_1)$ and $\sigma\frac{x+y}{2}\chi_{\{1,2\}}=\left(\frac{c_1+d_1}{2},\frac{e_1+f_1}{2}\right)$. Since φ is affine on the intervals $[c_1,d_1]$ and $[e_1,f_1]$, by equation (15), we get

$$\rho_2(x)=\rho_2(y)=\rho_2\left(\frac{x+y}{2}\right)=1,$$

whence we have again that $(ces_\varphi^2,\|\cdot\|_2)$ is not rotund.

If φ is affine on an interval $[e,f]$ and φ is not strictly convex on the interval $[c,d]$, the proof proceeds in the same way. So, the proof is finished.

REMARK 5. It is easy to show that if the space $(ces_\varphi^2,\|\cdot\|_2)$ is rotund, then the space $(ces_\varphi^2,\|\cdot\|_\varphi)$ is rotund either. Example 2 on page 377 and Example 5 show that there exists an Orlicz function for which the space $(ces_\varphi^2,\|\cdot\|_\varphi)$ is rotund, but the space $(ces_\varphi^2,\|\cdot\|_2)$ is not rotund.

EXAMPLE 5. Let $p(u)=1$ for $u\in[\frac{1}{2},\infty)$ and $p(u)=\frac{1}{n}$ for $u\in[\frac{1}{n+1},\frac{1}{n})$, $n\geqslant 2$ and let $\varphi(u)=\int_0^{|u|}p(v)dv$. Since

$$\varphi\left(\frac{1}{n}\right)=\int_0^{\frac{1}{n}}p(v)dv<\int_0^{\frac{1}{n}}\frac{1}{n}dv=\frac{1}{n^2},$$

the space ces_φ^2 is nontrivial. We have $a_\varphi=0$, $b_\varphi=\infty$, so the space $(ces_\varphi^2,\|\cdot\|_\varphi)$ is rotund. Since for any interval $[a,b]$ $(0\leqslant a<b<\infty)$ we can find real numbers c and d such that $a\leqslant c<d\leqslant b$ and φ is affine on $[c,d]$, we get that the space $(ces_\varphi^2,\|\cdot\|_2)$ is not rotund.

THEOREM 6. *The following conditions are equivalent:*

(i) $((ces_\varphi)_a,\|\cdot\|_\varphi)$ *is rotund,*

(ii) $(ces_\varphi^n,\|\cdot\|_\varphi)$ *is rotund for any* $n\geqslant 3$*,*

(iii) *(a)* $\sum_{i=1}^\infty\varphi\left(\frac{b_\varphi}{i}\right)\geqslant 1$*,*

(b) φ *is strictly convex on the interval* $[0,v_2]$*, where* $2\varphi(v_2)+\sum_{i=3}^\infty\varphi\left(\frac{2}{i}v_2\right)=1$*.*

Proof. The implication (i) $\Rightarrow$ (ii) is obious. The proofs of the others implications are the same as the proof of Theorem 2.7 in [7]; only in the proof of the implication (ii) $\Rightarrow$ (iii) we must take the following sequences: $x=(b,c,k,0,0,\ldots)$ and $y=(b_1,c_1,k_1+k,0,0,\ldots)$.

THEOREM 7. *The following conditions are equivalent:*

(i) $(ces_\varphi^n,\|\cdot\|_n)$ *is rotund for any (equivalently for some)* $n\geqslant 3$*,*

(ii) *(a)* $\varphi(b_\varphi)\geqslant 1$*,*

(b) φ *is strictly convex on the interval* $[0,u_2]$*, where* $\varphi(u_2)=\frac{1}{2}$*.*

Proof. (ii) $\Rightarrow$ (i). Let $x,y\in S(ces_\varphi^n,\|\cdot\|_n)$, $x\geqslant 0$, $y\geqslant 0$ (by Theorem 2 in [14] it is enough to consider only nonnegative elements) and $x\neq y$, where $n\geqslant 3$. Let us note that $u=(\sigma x(1),...,\sigma x(n),0,\ldots)$ and $v=(\sigma y(1),...,\sigma y(n),0,\ldots)$ are elements of l_φ^n (l_φ^n is the n-dimensional Orlicz space, see [15]) and $u\neq v$. Since

$$0\leqslant\sigma\frac{x+y}{2}(i)=\frac{\sigma x(i)+\sigma y(i)}{2}=\frac{u(i)+v(i)}{2}$$

for $i=1,...,n$, by Theorem 2.3 in [15], we get $\rho_n\left(\frac{x+y}{2}\right)=I_\varphi\left(\frac{u+v}{2}\right)<1$. By Lemma 6, $\|\frac{x+y}{2}\|_n<1$, so $(ces_\varphi^n,\|\cdot\|_n)$ is rotund.

(i) $\Rightarrow$ (ii). The necessity of the condition $\varphi(b_\varphi)\geqslant 1$ can be proved analogously as in Theorem 5. Suppose now that there exist a and b $(0\leqslant a<b\leqslant u_2)$ such that φ is affine on the interval $[a,b]$.

Assume first that $2\varphi(a)+\varphi\left(\frac{n-1}{n}a\right)<1$. Then, we can find $\varepsilon>0$ such that $a+\varepsilon\leqslant\min\left(b,\frac{n-1}{n-2}a\right)$ and $\varphi(a)+\varphi(a+\varepsilon)+\varphi\left(\frac{(n-1)(a+\varepsilon)}{n}\right)\leqslant 1$. We can also find $u_s\in[0,u_1]$ such that $\varphi(a)+\varphi(a+\varepsilon)+\varphi(u_s)=1$. Obviously $u_s\geqslant\frac{(n-1)(a+\varepsilon)}{n}$. Define two sequences

$$x=(\underbrace{0,\dots,0}_{n-3\text{ times}},(n-2)a,a+(n-1)\varepsilon,nu_s-(n-1)(a+\varepsilon),0,\dots)$$

and

$$y=(\underbrace{0,\dots,0}_{n-3\text{ times}},(n-2)(a+\varepsilon),a-(n-2)\varepsilon,nu_s-(n-1)a,0,\dots).$$

We have

$$\sigma x\chi_{\{1,\dots,n\}}=(\underbrace{0,\dots,0}_{n-3\text{ times}},a,a+\varepsilon,u_s),\quad \sigma y\chi_{\{1,\dots,n\}}=(\underbrace{0,\dots,0}_{n-3\text{ times}},a+\varepsilon,a,u_s)$$

and

$$\sigma\frac{x+y}{2}\chi_{\{1,\dots,n\}}=\left(\underbrace{0,\dots,0}_{n-3\text{ times}},a+\frac{\varepsilon}{2},a+\frac{\varepsilon}{2},u_s\right).$$

Since φ is affine on $[a,b]$, we get $\rho_n(x)=\rho_n(y)=\rho_n\left(\frac{x+y}{2}\right)=1$, so $(ces_\varphi^n,\|\cdot\|_n)$ is not rotund.

Let now $2\varphi(a)+\varphi\left(\frac{n-1}{n}a\right)\geqslant 1$. We can find $\eta>0$ $(a+\eta\leqslant\min\left(b,\frac{n}{n-1}a\right))$ and v_s $(0<v_s<a)$ such that $\varphi(v_s)+\varphi(a+\eta)+\varphi(a)=1$. Defining the following two sequences

$$x=(\underbrace{0,\dots,0}_{n-3\text{ times}},(n-2)v_s,(n-1)a-(n-2)v_s,a+n\eta,0,\dots),$$

and

$$y=(\underbrace{0,\dots,0}_{n-3\text{ times}},(n-2)v_s,(n-1)(a+\eta)-(n-2)v_s,a-(n-1)\eta,0,\dots),$$

we have

$$\sigma x\chi_{\{1,\dots,n\}}=(\underbrace{0,\dots,0}_{n-3\text{ times}},v_s,a,a+\eta),\quad \sigma y\chi_{\{1,\dots,n\}}=(\underbrace{0,\dots,0}_{n-3\text{ times}},v_s,a+\eta,a)$$

and

$$\sigma\frac{x+y}{2}\chi_{\{1,\dots,n\}}=\left(\underbrace{0,\dots,0}_{n-3\text{ times}},v_s,a+\frac{\eta}{2},a+\frac{\eta}{2}\right).$$

Therefore $\rho_n(x)=\rho_n(y)=\rho_n\left(\frac{x+y}{2}\right)=1$, whence we get again that $(ces_\varphi^n,\|\cdot\|_n)$ is not rotund.

REMARK 6. For $n \geqslant 3$ there holds the dependence: if the space $(ces_{\varphi}^{n}, \|\cdot\|_{n})$ is rotund, then the space $(ces_{\varphi}^{n}, \|\cdot\|_{\varphi})$ is rotund either. Examples 2 and 3 on page 377 show that for some Orlicz functions φ the space $(ces_{\varphi}^{n}, \|\cdot\|_{\varphi})$ is strictly convex, but the space $(ces_{\varphi}^{n}, \|\cdot\|_{n})$ is not strictly convex.

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