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THE DARBOUX PROBLEM FOR HYPERBOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS AND INEQUALITIES IN THE SENSE OF CARATHÉODORY

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Abstract. We consider the linear and nonlinear problem for partial functional differential equations

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = C(x,y)u(x,y) + P(x,y)u_{(x,y)} \text{ a.e. in } [0,a]\times[0,b]$$

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = f(x,y,u_{(x,y)},u(x,y)) \text{ a.e. in } [0,a]\times[0,b]$$

with Darboux condition

$$u(x,y) = \psi(x,y) \text{ on } [-a_0,a]\times[-b_0,b]\setminus(0,a]\times(0,b]$$

where the Hale operator $u_{(x,y)} : [-a_0,0]\times[-b_0,0] \to R^n$ is defined by $u_{(x,y)}(s,t) = u(s+x,t+y)$ for $(s,t)\in[-a_0,0]\times[-b_0,0]$. We give a few theorems about weak and strong inequalities and the existence theorem for the nonlinear problem.

1. Introduction

Define $I=[0,a]\times[0,b]$, $D=[-a_0,0]\times[-b_0,0]$, $I^*=[-a_0,a]\times[-b_0,b]$, $I_0=I^*\setminus I$. We always assume that $a,b>0$ and $a_0,b_0\geqslant 0$. We denote by $C(D,R^n)$, $L^1(D,R^n)$ the space of continuous functions, Lebesgue integrable functions from D into R^n, respectively. The norm $|\cdot|$ in R^n denotes the maximum norm. For vector-functions the norm $||\cdot||$ is the maximum of the values $|u(x,y)|$ taken over the interval of definition of this function. Moreover $||u||_0 = max\{|u(x,y)| : (x,y)\in D\}$. Inequality $x<y$ in R^n means that $x_i<y_i$ for each $i\in\{1,\dots,n\}$. Similarly for "$\geqslant$","$>$" and "$\leqslant$". The function $f: I\times C(D,R^n)\times R^n \to R^n$, $f=f(x,y,\omega,\eta)$ is said to be quasimonotonically nondecreasing with respect to η if each element $f_i(x,y,\omega,\eta)$ is nondecreasing with respect to η_j for $i\neq j$. This function is said to be nondecreasing with respect to functional argument ω if the inequality $\omega_1(s,t)\leqslant\omega_2(s,t)$ for $(s,t)\in D$ implies $f(x,y,\omega_1,\eta)\leqslant f(x,y,\omega_2,\eta)$. We note that in the case $n=1$ the condition quasimonotonicity is an empty condition.

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We consider problems

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = C(x,y)u(x,y) + P(x,y)u_{(x,y)} \text{ a.e. in } I \tag{1}$$

$$u(x,y) = \psi(x,y) \text{ on } I_0 \tag{2}$$

where $P(x,y)$ is a linear operator for every $(x,y) \in I$ and $C(x,y)$ is a square $n \times n$ matrix.

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = f(x,y,u_{(x,y)},u(x,y)) \text{ a.e. in } I \tag{3}$$

$$u(x,y) = \psi(x,y) \text{ on } I_0 \tag{4}$$

where $f : I \times C(D,R^n) \times R^n \to R^n$.

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = f(x,y,u_{(x,y)}) \text{ a.e. in } I \tag{5}$$

$$u(x,y) = \psi(x,y) \quad \text{ on } I_0 \tag{6}$$

where $f : I \times C(D,R^n) \to R^n$.

Of course, the equation (3) can be written in the form (5) but the form of equation (3) sometimes is more useful, because we will assume different conditions for arguments f with respect to $u(x,y)$ and $u_{(x,y)}$. For all problems the Hale operator $u_{(x,y)} : D \to R^n$ is defined by the formula $u_{(x,y)}(s,t) = u(s+x,t+y)$ for $(s,t) \in D$ and $\psi : D \to R^n$ is a given continuous function. A solution of the problem is an absolutely continuous function on I and continuous on I_0 which satisfies equation almost everywhere and initial condition everywhere. In theorems about inequalities it will be assumed that a first order derivative in x is continuous with respect to y and a derivative in y is continuous with respect to x.

In this paper the Darboux problem for system of the hyperbolic partial functional differential equations will be considered. In the book of W. Walter [10] we can find explanation why hyperbolic equations are similar to first-order ordinary differential equations. In this book we can also find differences between hyperbolic and ordinary equations . A main difference is it that for theorems about inequalities for the system of ordinary differential equations we may assume only quasimonotonicity for the function f with respect to suitable argument while for system of the hyperbolic equations quasimonotonicity isn't enough and monotonicity is assumed even if the function f satisfies the Lipschitz condition. In this paper in some theorems only quasimonotonicity is assumed but then we will need an extra condition.

Theorems about classical solutions for ordinary differential scalar inequalities or systems of inequalities can be found in [3], [4], [7], [10] while for hyperbolic scalar inequalities or systems of inequalities in [5], [10]. Theorems about classical solutions for functional differential ordinary scalar inequalities or systems of inequalities are considered in [5] while hyperbolic scaler inequalities or systems of inequalities in [1], [2]. Solutions in the sense of Carathéodory for ordinary differential scalar inequalities or systems of inequalities can be found in [4], [8] and for ordinary functional differential systems of equations in [9]. Theorems of existence for the ordinary functional differential equations can be found in [9] and for the hyperbolic functional differential equations in [6].

Theorems about differential inequalities usually are proved in the following way: at first a theorem about strong inequalities is proved by to take the smallest point where there is equality next to use assumptions we get a contradiction. Next a theorem about weak inequalities is proved in the following way. We take some function which is a solution suitable problem. We multiply this function by ε. We subtract this function from the first function in question and we add this function to the second function in question. Using the theorem about strong inequalities and puting $\varepsilon \to 0$ we get weak inequalities. In the theorem about weak inequalities the Lipschitz condition is assumed or a more general condition with the Perron function.

In this paper we will prove theorems for the hyperbolic functional differential equations and inequalities similarly to the proof for the ordinary functional differential equations and inequalities in the paper of W. Walter [9]. The theorem about weak inequalities will be proved by to multiplying the function in question by choosing suitable function and for this modified function we will prove the weak inequalities whence we will get the weak inequalities for the function in question. We will use the Schauder fixed point theorem in the theorem of existence. We will also study strongly monotone flows. M. Hirsch considered in [3] strongly monotone flows generated by autonomous systems $u'(t) = f(u(t))$. W. Walter generalized this theory in [8] to the case where f depends also on t, satisfied Carathédory hypotheses and is only locally Lipschitz continuous with respect to u and in [9] to the case functional differential equations. We will develop this theory for hyperbolic equations.

2. Weak inequalities

In this section we will consider weak inequalities for problems (1), (2) and (3), (4). We will assume only quasimonotonicity with respect to a suitable variable.

THEOREM 1. (nonnegativity) *Suppose that $P(x,y)$ is a linear map from $C(D,R^n)$ into R^n which is positive in the sense that $u(s,t) \geqslant 0$ on $[x-a_0,x]\times[y-b_0,y]$ implies $P(x,y)u_{(x,y)} \geqslant 0$. $C(x,y) = (c_{ij}(x,y))$ is an essentially nonnegative $n\times n$ matrix i.e. $c_{ij}(x,y) \geqslant 0$ a.e. in I for $i \neq j$. There exists a function $l(x,y) \in L^1(I,R)$ such that*

$$l(x,y) \geqslant \int_0^x l(z,y)dz \int_0^y l(x,z)dz$$

$$|P(x,y)u_{(x,y)}| \leqslant l(x,y)||u||_0 \text{ and } |c_{ij}(x,y)| \leqslant l(x,y) \text{ a.e. in } I$$

$$c_{ii}(x,y) \geqslant -l(x,y) + \int_0^x l(z,y)dz \int_0^y l(x,z)dz \text{ a.e. in } I$$

A function $u(x,y)$ is absolutely continuous, $\frac{\partial u}{\partial x}(x,y)$ is continuous with respect to y, $\frac{\partial u}{\partial y}(x,y)$ is continuous with respect to x. Furthermore

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) \geqslant C(x,y)u(x,y) + P(x,y)u_{(x,y)} \text{ a.e. in } I \tag{7}$$

$$u(x,y) \geqslant 0 \text{ on } I_0,\ \frac{\partial u}{\partial x}(x,0) \geqslant 0 \text{ and } \frac{\partial u}{\partial y}(0,y) \geqslant 0$$

Then

$$u(x,y) \geqslant 0 \text{ on } I$$

Proof

Let $v(x,y) = e^{H(x,y)}u(x,y)$, where

$$H(x,y) = \int_{-a_0}^{x}\int_{-b_0}^{y} h(z_1,z_2)dz_2dz_1 \text{ for } (x,y) \in I^*$$

$$h(x,y) = \begin{cases} l(x,y) & \text{for } (x,y) \in I \\ 0 & \text{for } (x,y) \in I_0 \end{cases}$$

Then

$$\frac{\partial^2 v}{\partial x \partial y}(x,y) = e^{H(x,y)}\Big\{ l(x,y)u(x,y) + \int_0^x l(z,y)dz \int_0^y l(x,z)dz\, u(x,y) + \int_0^x l(z,y)dz\, \frac{\partial u}{\partial x}(x,y) + \int_0^y l(x,z)dz\, \frac{\partial u}{\partial y}(x,y) + \frac{\partial^2 u}{\partial x \partial y}(x,y)\Big\} \tag{8}$$

Since

$$\frac{\partial u}{\partial x}(x,y) = e^{-H(x,y)}\left\{\frac{\partial v}{\partial x}(x,y) - \int_0^y l(x,z)dz\, v(x,y)\right\} \tag{9}$$

$$\frac{\partial u}{\partial y}(x,y) = e^{-H(x,y)}\left\{\frac{\partial v}{\partial y}(x,y) - \int_0^x l(z,y)dz\, v(x,y)\right\} \tag{10}$$

we see that

$$\frac{\partial^2 v}{\partial x \partial y}(x,y) = \left[l(x,y) - \int_0^x l(z,y)dz \int_0^y l(x,z)dz\right]v(x,y) + \int_0^x l(z,y)dz\, \frac{\partial v}{\partial x}(x,y) + \int_0^y l(x,z)dz\, \frac{\partial v}{\partial y}(x,y) + e^{H(x,y)}\frac{\partial^2 u}{\partial x \partial y}(x,y) \tag{11}$$

From (7) we get

$$\frac{\partial^2 v}{\partial x \partial y}(x,y) \geqslant \tilde{C}(x,y)v(x,y) + \int_0^x l(z,y)dz\, \frac{\partial v}{\partial x}(x,y) + \int_0^y l(x,z)dz\, \frac{\partial v}{\partial y}(x,y) + Q(x,y)(e^{-H}v)_{(x,y)} \tag{12}$$

where

$$\tilde{C}(x,y) = C(x,y) + \left[l(x,y) - \int_0^x l(z,y)dz \int_0^y l(x,z)dz\right] I_n$$

$$Q(x,y)w = e^{H(x,y)}P(x,y)w$$

$$(I_n \text{ is the unit matrix})$$

We note that $\tilde{c}(x,y) \geqslant 0$ and the operator Q is linear and positive.

Define the function $\rho : I^* \to R$ by the formula $\rho(x,y) = e^{(n+2)H(x,y)+x+y}$ and the function $r : I^* \to R^n$, $r(x,y) = (\rho(x,y),...,\rho(x,y))$. Then

$$\frac{\partial^2 r}{\partial x \partial y}(x,y) = (n+2)l(x,y)r(x,y) + (n+2)^2 \int_0^x l(z,x)dz \int_0^y l(x,z)dz\, r(x,y)$$
$$+(n+2)\int_0^x l(z,y)dz\, r(x,y) + (n+2)\int_0^y l(x,z)dz\, r(x,y) + r(x,y)$$
$$Q(x,y)(e^{-H}r)_{(x,y)} = e^{H(x,y)}P(x,y)(e^{-H}r)_{(x,y)}$$

Using a suitable estimate for matrix elements $\tilde{C}(x,y)$ and estimate for $P(x,y)$ we get

$$0 \leqslant \tilde{C}(x,y)r(x,y) \leqslant (n+1)l(x,y)r(x,y)$$

$$\int_0^x l(z,y)dz\, \frac{\partial r}{\partial x}(x,y) = (n+2)\int_0^x l(z,y)dz \int_0^y l(x,z)dz\, r(x,y) + \int_0^x l(z,y)dz\, r(x,y)$$
$$\int_0^y l(x,z)dz\, \frac{\partial r}{\partial y}(x,y) = (n+2)\int_0^x l(z,y)dz \int_0^y l(x,z)dz\, r(x,y) + \int_0^y l(x,z)dz\, r(x,y)$$
$$\left|Q(x,y)(e^{-H}r)_{(x,y)}\right| \leqslant e^{H(x,y)}l(x,y)||e^{(n+1)H(x,y)}||_0 \leqslant l(x,y)r(x,y)$$

Thus

$$\tilde{C}(x,y)r(x,y) + \int_0^x l(z,y)dz\, \frac{\partial r}{\partial x}(x,y) + \int_0^y l(x,z)dz\, \frac{\partial r}{\partial y}(x,y) + Q(x,y)(e^{-H}r)_{(x,y)}$$
$$\leqslant (n+2)l(x,y)r(x,y) + 2(n+2)\int_0^x l(z,y)dz \int_0^y l(x,z)dz\, r(x,y)$$
$$+\int_0^x l(z,y)dz\, r(x,y) + \int_0^y l(x,z)dz\, r(x,y) \leqslant \frac{\partial^2 r}{\partial x \partial y}(x,y)$$

Define $v_\varepsilon(x,y) = v(x,y) + \varepsilon r(x,y)$. From the linearity of operator Q we have

$$\frac{\partial^2 v_\varepsilon}{\partial x \partial y}(x,y) \geqslant \tilde{C}(x,y)v_\varepsilon(x,y) + \int_0^x l(z,y)dz\, \frac{\partial v_\varepsilon}{\partial x}(x,y)$$
$$+\int_0^y l(x,z)dz\, \frac{\partial v_\varepsilon}{\partial y}(x,y) + Q(x,y)(e^{-H}v_\varepsilon)_{(x,y)} \tag{13}$$

Since $v_\varepsilon(x,y) = v(x,y) + \varepsilon r(x,y)$, we have

$$\frac{\partial v_\varepsilon}{\partial x}(x,y) = e^{H(x,y)}\frac{\partial u}{\partial x}(x,y) + \int_0^y l(x,z)dz\, e^{H(x,y)}u(x,y)$$
$$+\varepsilon\left[(n+2)\int_0^y l(x,z)dz + 1\right]r(x,y)$$
$$\frac{\partial v_\varepsilon}{\partial y}(x,y) = e^{H(x,y)}\frac{\partial u}{\partial y}(x,y) + \int_0^x l(z,y)dz\, e^{H(x,y)}u(x,y)$$
$$+\varepsilon\left[(n+2)\int_0^x l(z,y)dz + 1\right]r(x,y)$$

We notice that

$$v_\varepsilon(x,y) > 0 \text{ for } (x,y) \in I_0 \text{ and } \frac{\partial v_\varepsilon}{\partial x}(x,0) > 0,\ \frac{\partial v_\varepsilon}{\partial y}(0,y) > 0 \text{ for } (x,y) \in I$$

Since $v_\varepsilon(x,y)$ is continuous and $\frac{\partial v_\varepsilon}{\partial x}(x,y)$ is continuous with respect to y and $\frac{\partial v_\varepsilon}{\partial y}(x,y)$ is continuous with respect to x thus there exists $c > 0$ and the set $I_c = I^* \setminus (c,a] \times (c,b]$ such that

$$v_\varepsilon(x,y) > 0 \text{ for } (x,y) \in I_c$$

$$\frac{\partial v_\varepsilon}{\partial x}(x,y) > 0 \text{ for } (x,y) \in [0,a] \times [0,c]$$

$$\frac{\partial v_\varepsilon}{\partial y}(x,y) > 0 \text{ for } (x,y) \in [0,c] \times [0,a]$$

Consequently,

$$\frac{\partial v_\varepsilon}{\partial x}(x,y) > 0 \text{ and } \frac{\partial v_\varepsilon}{\partial y}(x,y) > 0 \text{ for } (x,y) \in [0,c] \times [0,c]$$

Thus all elements on the right side (13) are nonnegative on $[0,c] \times [0,c]$ therefore $\frac{\partial^2 v_\varepsilon}{\partial x \partial y}(x,y) \geqslant 0$ on $[0,c] \times [0,c]$. Integrating this inequality with respect to y we get that the function $\frac{\partial v_\varepsilon}{\partial x}(x,y)$ is nondecreasing with respect to y. Similarly, we get that the function $\frac{\partial v_\varepsilon}{\partial y}(x,y)$ is nondecreasing with respect to x on $[0,c] \times [0,c]$. Furthermore $\frac{\partial v_\varepsilon}{\partial x}(x,y)$ is continuous with respect to y and $\frac{\partial v_\varepsilon}{\partial y}(x,y)$ is continuous with respect to x. Therefore $\frac{\partial v_\varepsilon}{\partial x}(x,y)$ and $\frac{\partial v_\varepsilon}{\partial y}(x,y)$ continue positive on the set $I_c \setminus (I^* \setminus I)$. Repeating this reasoning for the set I_c we get that $v_\varepsilon(x,y)$, $\frac{\partial v_\varepsilon}{\partial x}(x,y)$, $\frac{\partial v_\varepsilon}{\partial y}(x,y)$ are positive on all I thus $\frac{\partial^2 v_\varepsilon}{\partial x \partial y}(x,y) \geqslant 0$ on I whence we get that the function $\frac{\partial v_\varepsilon}{\partial x}(x,y)$ is nondecreasing with respect to y and the function $\frac{\partial v_\varepsilon}{\partial y}(x,y)$ is nondecreasing with respect to x. Therefore $v_\varepsilon(x,y)$ is nondecreasing with respect to x and y. Since $v_\varepsilon(x,y) > 0$ on I_0, we have $v_\varepsilon(x,y) > 0$ on I. Letting $\varepsilon \to 0^+$ we get $v(x,y) \geqslant 0$ and consequently $u(x,y) = e^{-H(x,y)} v(x,y) \geqslant 0$ on I.

REMARK 1. Since $v_\varepsilon(x,y) > 0$, $\frac{\partial v_\varepsilon}{\partial x}(x,y) > 0$ and $\frac{\partial v_\varepsilon}{\partial y}(x,y) > 0$ on I, we have $v(x,y) \geqslant 0$ on I and $\frac{\partial v}{\partial x}(x,y) \geqslant 0$, $\frac{\partial v}{\partial y}(x,y) \geqslant 0$ a.e. on I. Furthermore $v(x,y)$ is absolutely continuous therefore it is nondecreasing with respect to x and y. Since $u(x,y) = e^{-H(x,y)} v(x,y)$, $v(x,y) \geqslant 0$ on I_0 and $v(x,y)$ is nondecreasing with respect to x and y hence there are two disjoint index sets α and β such that $\alpha \cup \beta = \{1,\dots,n\}$ and $u_i(x,y) > 0$ on $I^* \setminus I_0$ for $i \in \alpha$ and $u_j(x,y) = 0$ on A_j, $u_j(x,y) > 0$ on B_j for $j \in \beta$ where A_j and B_j are disjoint sets such that $A_j \cup B_j = I$ and the set A_j is such that if for a point belongs to boundary different to $I \setminus I^\circ$, where I° is interior of a set I, we lead two half line which the first half line is perpendicular to $0X$ and the second half line is perpendicular to $0Y$ in the positive direction then the area belongs to I between this two half line on the right hand is contained in the B_j.

REMARK 2. From the proof of theorem 1 we obtain

$$\frac{\partial u}{\partial x}(x,y)+\int_0^y l(x,z)dz\, u(x,y)\geqslant 0 \text{ a.e. in } I$$

$$\frac{\partial u}{\partial y}(x,y)+\int_0^x l(z,y)dz\, u(x,y)\geqslant 0 \text{ a.e. in } I$$

We note that the function $l(x,y)\in L^1(I,R_+)$ from the theorem 1 satisfying

$$l(x,y)\geqslant \int_0^x l(z,y)dz\int_0^y l(x,z)dz \tag{14}$$

Suppose that $l(x,y)=f(x)g(y)$, where $f\in L^1([0,a],R_+)$, $g\in L^1([0,b],R_+)$. Then

$$\int_0^x l(z,y)dz\int_0^y l(x,z)dz = l(x,y)\int_0^x f(z)dz\int_0^y g(z)dz$$

In order that condition (14) could be fulfilled sufficient that

$$\int_0^a f(z)dz\int_0^b g(z)dz\leqslant 1 \tag{15}$$

We notice that we can choose such fast decreasing functions f and g in order that the condition (15) could be fulfilled for every a and b for example it is sufficient that $f(x)\leqslant \alpha e^{-\alpha x}$ and $f(y)\leqslant \beta e^{-\beta y}$ for $\alpha,\beta>0$. If we want in order that functions f,g to have big values then we must choose small enough a and b for example if $f(x)=\alpha e^x$ and $g(y)=\beta e^y$ then be enough $a+b\leqslant ln\left(1+\frac{1}{\alpha\beta}\right)$ for $\alpha,\beta>0$.

For arbitrary function $\tilde{l}\in L^1(I,R_+)$ we can choose

$$p = ess\ \inf\left\{\frac{\tilde{l}(x,y)}{\int_0^x \tilde{l}(z,y)dz\int_0^y \tilde{l}(x,z)dz} : (x,y)\in(0,a]\times(0,b]\right\}>0$$

so that the function $l(x,y)=p\tilde{l}(x,y)$ fulfil condition (14).

Now we give examples of functions satisfying (14).

EXAMPLE 1.

1. $l(x,y)=e^{-x-y}$
2. $l(x,y)=\frac{1}{e^{a+b}-e^a-e^b+1}e^{x+y}$
3. $l(x,y)=\frac{(\alpha+1)(\beta+1)}{a^{\alpha+1}b^{\beta+1}}x^\alpha y^\beta$ for $\alpha,\beta>-1$
4. $l(x,y)=\frac{4(a+b)}{ab(a+2b)(2a+b)}(x+y)$

Of course, we assume that $a,b>0$. We notice that for an arbitrary big rectangle $[0,a]\times[0,b]$ we can always choose such a small function $l(x,y)$ in order to satisfy condition

(14) and for an arbitrary big function $l(x,y)$ we can choose such a small rectangle $[0,a]\times[0,b]$ in order that condition (14) is satisfied.

Now we give example which demonstrates that in general we can't get nonnegativity first order partial derivatives of solution by assumption of theorem 1. Furthermore weak inequalities with remark 2 will be changed to equalities.

EXAMPLE 2.

$$\frac{\partial^2 u}{\partial x\partial y}(x,y)=4\left[(xy)^3-xy\right]u(x,y)\text{ in }[0,1]\times[0,1] \tag{16}$$

$$u(x,0)=1\quad u(0,y)=1 \tag{17}$$

The solution of the problem (16), (17) is the function $u(x,y)=e^{-x^2y^2}$. We show that the solution of our problem fulfil assumption theorem 1 with $l(x,y)=4xy$. We note that

$$\int_0^x l(z,y)dz\int_0^y l(x,y)dz=4(xy)^3$$

thus

$$l(x,y)\geqslant\int_0^x l(z,y)dz\int_0^y l(x,y)dz\text{ for }(x,y)\in[0,1]\times[0,1]$$

From the problem (16), (17) we have $c_{11}=4\left[(xy)^3-xy\right]$. We notice

$$|c_{11}|=4|xy||(xy)^2-1|\leqslant 4|xy|=l(x,y)$$

since

$$|(xy)^2-1|\leqslant 1\text{ for }0\leqslant x,y\leqslant 1$$

and

$$c_{11}=-4xy+4(xy)^3\geqslant -l(x,y)+\int_0^x l(z,y)dz\int_0^y l(x,y)dz$$

Furthermore

$$\frac{\partial u}{\partial x}(x,y)=-2xy^2e^{-x^2y^2}\,,\qquad \frac{\partial u}{\partial y}(x,y)=-2x^2ye^{-x^2y^2}$$

thus

$$\frac{\partial u}{\partial x}(0,y)=\frac{\partial u}{\partial x}(x,0)=\frac{\partial u}{\partial y}(0,y)=\frac{\partial u}{\partial y}(x,0)=0$$

and

$$\frac{\partial u}{\partial x}(x,y)<0\,,\qquad \frac{\partial u}{\partial y}(x,y)<0\text{ on }\tilde{I}$$

where

$$\tilde{I}=[0,1]\times[0,1]\setminus(\{0\}\times[0,1]\cup[0,1]\times\{0\})$$

Moreover

$$\frac{\partial u}{\partial x}(x,y)+\int_0^y l(x,z)dz\,u(x,y)=0$$
$$\frac{\partial u}{\partial y}(x,y)+\int_0^x l(z,y)dz\,u(x,y)=0$$

Now we give a theorem about inequalities for the problem (3), (4).

THEOREM 2. (weak inequalities) *Suppose that for all* $A>0$ *there exists a function* $l(x,y)\in L^1(I,R_+)$ *such that*

$$l(x,y)\geqslant\int_0^x l(z,y)dz\int_0^y l(x,z)dz$$

$$|f(x,y,\overline{\omega},\overline{\eta})-f(x,y,\omega,\eta)|\leqslant l(x,y)(||\overline{\omega}-\omega||_0+|\overline{\eta}-\eta|) \tag{18}$$

$$f(x,y,\omega,\overline{\eta})-f(x,y,\omega,\eta)\geqslant\left[-l(x,y)+\int_0^x l(z,y)dz\int_0^y l(x,z)dz\right](\overline{\eta}-\eta) \tag{19}$$

for $||\omega||_0,||\overline{\omega}||_0,|\eta|,|\overline{\eta}|\leqslant A$ *and* $\eta\leqslant\overline{\eta}$. $f(x,y,\omega,\eta)$ *is nondecreasing with respect to* ω *and quasimonotone nondecreasing with respect to* η. $w(x,y),v(x,y)$ *are absolute continuous and* $\frac{\partial w}{\partial x}(x,y)$, $\frac{\partial v}{\partial x}(x,y)$ *are continuous with respect to* y *and* $\frac{\partial w}{\partial y}(x,y)$, $\frac{\partial v}{\partial y}(x,y)$ *are continuous* x. *Furthermore*

$$\frac{\partial^2 v}{\partial x\partial y}(x,y)\leqslant f(x,y,v_{(x,y)},v(x,y)) \ \textit{and}$$

$$\frac{\partial^2 w}{\partial x\partial y}(x,y)\geqslant f(x,y,w_{(x,y)},w(x,y)) \ \textit{a.e. in } I \tag{20}$$

$$v(x,y)\leqslant w(x,y) \textit{ on } I_0,\ \frac{\partial v}{\partial x}(x,0)\leqslant\frac{\partial w}{\partial x}(x,0),\ \frac{\partial v}{\partial y}(0,y)\leqslant\frac{\partial w}{\partial y}(0,y)$$

Then

$$v(x,y)\leqslant w(x,y) \textit{ on } I$$

Proof. Let $|v(x,y)|,|w(x,y)|\leqslant A-1$ and $l(x,y)$ be the function in (18) corresponding to A. Let

$$V(x,y)=e^{H(x,y)}v(x,y),\ W(x,y)=e^{H(x,y)}w(x,y)$$

Then

$$\frac{\partial^2 V}{\partial x\partial y}(x,y)\leqslant e^{H(x,y)}f(x,y,v_{(x,y)},v(x,y))+e^{H(x,y)}\left[l(x,y)+\int_0^x l(z,y)dz\int_0^y l(x,z)dz\right]v(x,y)$$
$$+e^{H(x,y)}\int_0^x l(z,y)dz\,\frac{\partial v}{\partial x}(x,y)+e^{H(x,y)}\int_0^y l(x,z)dz\,\frac{\partial v}{\partial y}(x,y)$$

Note that

$$\frac{\partial v}{\partial x}(x,y)=e^{-H(x,y)}\left(\frac{\partial V}{\partial x}(x,y)-\int_0^y l(x,z)dz\,V(x,y)\right)$$

$$\frac{\partial v}{\partial y}(x,y)=e^{-H(x,y)}\left(\frac{\partial V}{\partial y}(x,y)-\int_0^x l(z,y)dz\,V(x,y)\right)$$

Thus

$$\frac{\partial^2 V}{\partial x\partial y}(x,y)\leqslant e^{H(x,y)}(Gv)(x,y)$$

where

$$(Gu)(x,y) = g(x,y,u_{(x,y)},u(x,y),\frac{\partial U}{\partial x}(x,y),\frac{\partial U}{\partial y}(x,y))$$

$$g(x,y,\omega,\eta,\sigma,\delta) = f(x,y,\omega,\eta) + l_1(x,y)\eta + e^{-H(x,y)}l_2(x,y)\sigma + e^{-H(x,y)}l_3(x,y)\delta$$

$$l_1(x,y) = l(x,y) - \int_0^x l(z,y)dz \int_0^y l(x,z)dz$$

$$l_2(x,y) = \int_0^x l(z,y)dz,\ l_3(x,y) = \int_0^y l(x,z)dz$$

In much the same way as above we get

$$\frac{\partial^2 W}{\partial x \partial y}(x,y) \geqslant e^{H(x,y)}(Gw)(x,y)$$

We claim that $g(x,y,\omega,\eta,\sigma,\delta)$ is nondecreasing with respect to ω,η,σ and δ. Indeed, if $\omega \leqslant \overline{\omega}$, $\eta \leqslant \overline{\eta}$, $\sigma \leqslant \overline{\sigma}$, $\delta \leqslant \overline{\delta}$ and $\eta = (\eta_i,\eta^*)$, $\overline{\eta} = (\overline{\eta}_i,\overline{\eta}^*)$, then using assumptions about monotonicity, quasimonotonicity and the condition (19) we get

$$\begin{aligned} &f_i(x,y,\omega,\eta) + l_1(x,y)\eta_i + l_2(x,y)\sigma_i + l_3(x,y)\delta_i \\ &\quad \leqslant f_i(x,y,\overline{\omega},\eta_i,\overline{\eta}^*) + l_1(x,y)\eta_i + l_2(x,y)\overline{\sigma}_i + l_3(x,y)\overline{\delta}_i \\ &\quad \leqslant f_i(x,y,\overline{\omega},\overline{\eta}_i,\overline{\eta}^*) + l_1(x,y)\overline{\eta}_i + l_2(x,y)\overline{\sigma}_i + l_3(x,y)\overline{\delta}_i \end{aligned}$$

Define the function $\rho : I^* \to R$ by the formula $\rho(x,y) = e^{3H(x,y)+x+y}$ and functions $r,R : I^* \to R^n$ by $r(x,y) = (\rho(x,y),...,\rho(x,y))$, $R(x,y) = e^{H(x,y)}r(x,y)$. Then functions $w_\varepsilon(x,y) = w(x,y) + \varepsilon r(x,y)$, $W_\varepsilon(x,y) = W(x,y) + \varepsilon R(x,y)$ fulfil

$$\begin{aligned} \frac{\partial^2 W_\varepsilon}{\partial x \partial y}(x,y) \geqslant\ & e^{H(x,y)}(Gw)(x,y) + \varepsilon e^{H(x,y)}\Big\{4l(x,y)r(x,y) \\ & +16\int_0^x l(z,y)dz \int_0^y l(x,z)dz\, r(x,y) + 4\int_0^x l(z,y)dz\, r(x,y) \\ & +4\int_0^y l(x,z)dz\, r(x,y) + r(x,y)\Big\} \end{aligned}$$

Furthermore we can choose $\varepsilon > 0$ in order that $|w_\varepsilon(x,y)| \leqslant A$ and therefore

$$\begin{aligned} |Gw_\varepsilon(x,y)\ -\ & Gw(x,y)| \\ &= \Big| f(x,y,(w_\varepsilon)_{(x,y)},w_\varepsilon(x,y)) - f(x,y,w_{(x,y)},w(x,y)) + \varepsilon l_1(x,y)r(x,y) \\ &\quad + \varepsilon e^{-H(x,y)}l_2(x,y)\frac{\partial R}{\partial x}(x,y) + \varepsilon e^{-H(x,y)}l_3(x,y)\frac{\partial R}{\partial y}(x,y)\Big| \\ &\leqslant l(x,y)\big(||\varepsilon r||_0 + |\varepsilon r|\big) + \varepsilon l_1(x,y)\rho(x,y) \\ &\quad + e^{-H(x,y)}\varepsilon l_2(x,y)\left|\frac{\partial R}{\partial x}(x,y)\right| + e^{-H(x,y)}\varepsilon l_3(x,y)\left|\frac{\partial R}{\partial y}(x,y)\right| \end{aligned}$$

$$\begin{aligned}
&\leqslant 2\varepsilon l(x,y)\rho(x,y) + \varepsilon l_1(x,y)\rho(x,y) \\
&\quad + \varepsilon l_2(x,y)(4l_3(x,y)+1)\rho(x,y) + \varepsilon l_3(x,y)(l_2(x,y)+1)\rho(x,y) \\
&= 3\varepsilon l(x,y)\rho(x,y) + 7\varepsilon \int_0^x l(z,y)dz \int_0^y l(x,z)dz\, \rho(x,y) \\
&\quad + \varepsilon \int_0^x l(z,y)dz\, \rho(x,y) + \varepsilon \int_0^y l(x,z)dz\, \rho(x,y)
\end{aligned}$$

Thus

$$\begin{aligned}
&e^{H(x,y)}(Gw_\varepsilon)(x,y) \\
&\quad \leqslant e^{H(x,y)}(Gw)(x,y) + \varepsilon e^{H(x,y)} \Big\{ 3l(x,y)r(x,y) + 7 \int_0^x l(z,y)dz \int_0^y l(x,z)dz\, r(x,y) \\
&\qquad + \int_0^x l(z,y)dz\, r(x,y) + \int_0^y l(x,z)dz\, r(x,y) \Big\} \\
&\quad \leqslant \varepsilon e^{H(x,y)} \Big\{ 4l(x,y)r(x,y) + 16 \int_0^x l(z,y)dz \int_0^y l(x,z)dz\, r(x,y) \\
&\qquad + 4\int_0^x l(z,y)dz\, r(x,y) + 4\int_0^y l(x,z)dz\, r(x,y) + r(x,y) \Big\} \\
&\quad \leqslant \frac{\partial^2 W_\varepsilon}{\partial x \partial y}(x,y)
\end{aligned}$$

Consequently

$$\frac{\partial^2 W_\varepsilon}{\partial x \partial y}(x,y) \geqslant e^{H(x,y)}(Gw_\varepsilon)(x,y)$$

We notice that

$$w_\varepsilon(x,y) > w(x,y) \geqslant v(x,y) \text{ on } I_0$$

$$\frac{\partial w_\varepsilon}{\partial x}(x,0) > \frac{\partial w}{\partial x}(x,0) \geqslant \frac{\partial v}{\partial x}(x,0)$$

$$\frac{\partial w_\varepsilon}{\partial y}(0,y) > \frac{\partial w}{\partial y}(0,y) \geqslant \frac{\partial v}{\partial y}(0,y)$$

Therefore there exists $c > 0$ and the set $I_c = I^* \setminus (c,a] \times (c,b]$ such that

$$w_\varepsilon(x,y) > v(x,y) \text{ on } I_c \tag{21}$$

$$\frac{\partial w_\varepsilon}{\partial x}(x,y) > \frac{\partial v}{\partial x}(x,y) \text{ on } [0,a] \times [0,c] \tag{22}$$

$$\frac{\partial w_\varepsilon}{\partial y}(x,y) > \frac{\partial v}{\partial y}(x,y) \text{ on } [0,c] \times [0,b] \tag{23}$$

From (21) and (22) we have that

$$\frac{\partial W_\varepsilon}{\partial x}(x,y) > \frac{\partial V}{\partial x}(x,y) \text{ on } [0,a] \times [0,c]$$

From (21) and (23) we get that

$$\frac{\partial W_\varepsilon}{\partial y}(x,y) > \frac{\partial V}{\partial y}(x,y) \text{ on } [0,c]\times[0,b]$$

From monotonicity G we have

$$\frac{\partial^2 W_\varepsilon}{\partial x\partial y}(x,y) \geqslant e^{H(x,y)}(Gw_\varepsilon)(x,y) \geqslant e^{H(x,y)}(Gv)(x,y) \geqslant \frac{\partial^2 V}{\partial x\partial y}(x,y)$$

on $[0,c]\times[0,c]$. Thus

$$\frac{\partial^2}{\partial x\partial y}(W_\varepsilon - V)(x,y) \geqslant 0 \text{ on } [0,c]\times[0,c] \tag{24}$$

Similarly to the proof of theorem 1 from continuous functions $\frac{\partial w}{\partial x}(x,y)$, $\frac{\partial v}{\partial x}(x,y)$, $\frac{\partial r}{\partial x}(x,y)$ with respect to y and $\frac{\partial w}{\partial y}(x,y)$, $\frac{\partial v}{\partial y}(x,y)$, $\frac{\partial r}{\partial y}(x,y)$ with respect to x and from the inequality (24) we get that $\frac{\partial}{\partial x}(W_\varepsilon - V)(x,y)$ is nondecreasing with respect to y and $\frac{\partial}{\partial y}(W_\varepsilon - V)(x,y)$ is nondecreasing with respect to x on $[0,c]\times[0,c]$. Therefore $\frac{\partial}{\partial x}(W_\varepsilon - V)(x,y)$ and $\frac{\partial}{\partial y}(W_\varepsilon - V)(x,y)$ continue positive on $I_c\setminus(I^*\setminus I)$. We may repeat this reasoning for the set I_c and we get that $(W_\varepsilon - V)(x,y)$, $\frac{\partial}{\partial x}(W_\varepsilon - V)(x,y)$ and $\frac{\partial}{\partial y}(W_\varepsilon - V)(x,y)$ continue positive on all I. Therefore $W_\varepsilon(x,y) > V(x,y)$. Letting $\varepsilon \to 0$ we get $W(x,y) \geqslant V(x,y)$ on I. This proves the theorem. $\square$

REMARK 3. Analysis similar to that in the remark 1 shows that there are two disjoint index sets α and β such that $\alpha\cup\beta = \{1,\dots,n\}$ and for $i\in\alpha$ $w_i(x,y) > v_i(x,y)$ on $I^*\setminus I_0$ and for $j\in\beta$ $w_j(x,y) = v_j(x,y)$ on A_j and $w_j(x,y) > v_j(x,y)$ on B_j, where A_j and B_j are the same sets as in the remark 1.

REMARK 4. From the proof of theorem 2 we obtain

$$\frac{\partial v}{\partial x}(x,y) + \int_0^y l(x,z)dz\, v(x,y) \leqslant \frac{\partial w}{\partial x}(x,y) + \int_0^y l(x,z)dz\, w(x,y) \text{ a.e. on } I$$

$$\frac{\partial v}{\partial y}(x,y) + \int_0^x l(z,y)dz\, v(x,y) \leqslant \frac{\partial w}{\partial y}(x,y) + \int_0^x l(z,y)dz\, w(x,y) \text{ a.e. on } I$$

For the Darboux problem for hyperbolic partial differential equations it is possible that thesis of theorem 2 isn't true in spite of inequality (20) and assumption (18) are fulfilled. It can be showed by two problems from the book of W. Walter [10].

EXAMPLE 3. The solution of the problem

$$\frac{\partial^2 v}{\partial x\partial y}(x,y) = -v(x,y),\ v(x,0) = 0,\ v(0,y) = 0$$

is the function $v(x,y) = 0$.

And the solution of the problem

$$\frac{\partial^2 w}{\partial x \partial y}(x,y) = -w(x,y) + \delta,\ w(x,0) = \sigma(x), w(0,y) = \tau(y)$$

is the function

$$\begin{aligned} w(x,y) &= \sigma(0)E_2(-xy) + \int_0^x \sigma'(\xi)E_2(-(x-\xi)y)d\xi \\ &\quad + \int_0^y \tau'(\xi)E_2(-x(y-\xi))d\xi + \delta(1 - E_2(-xy)) \end{aligned}$$

where $\sigma(0) = \tau(0),\ \sigma'(x) > 0,\ \tau'(y) > 0,\ \delta > 0$ and $E_2(t) = \sum_{i=0}^n \frac{t^i}{(i!)^2}$. Moreover we know that $E_2(t) > 0$ for $t \geqslant -1,\ |E_2(t)| \leqslant 1$ for $t \leqslant 0$ and $E_2(t) < -0,1$ for $-6,8 < t < -1,8$. Hence, it follows easily that there are $\sigma(x) > 0,\ \tau(y) > 0$ and $\delta > 0$ such that there is a point (x_0, y_0) that $w(x_0, y_0) < 0$. We also note that $w(x,y) > 0$ for $(x,y) \in [0,1] \times [0,1]$. If in the theorem 2 we put $f(x,y,\omega,\eta) = -\eta + \frac{1}{2}\delta$ then

$$\frac{\partial^2 v}{\partial x \partial y}(x,y) < f(x,y,v(x,y)) \ \text{ and } \ \frac{\partial^2 w}{\partial x \partial y}(x,y) > f(x,y,w(x,y))$$

$$v(x,y) < w(x,y) \text{ on } I_0,\ \frac{\partial v}{\partial x}(x,0) < \frac{\partial w}{\partial x}(x,0),\ \frac{\partial v}{\partial y}(0,y) < \frac{\partial w}{\partial y}(0,y)$$

Let $(x,y) \in [0,\frac{1}{2}] \times [0,\frac{1}{2}]$ and $l(x,y) = 2$ then it is easily seen that assumptions of theorem 2 are satisfied therefore we get that $v(x,y) \leqslant w(x,y)$ on $[0,\frac{1}{2}] \times [0,\frac{1}{2}]$.

3. Existence

Now, we deal with an existence theorem for Darboux problem (5), (6). In this section we suppose that $c = max\{3||\psi(x,y)||, 1\}$. We first prove a lemma.

LEMMA 1. *If* $r(x,y) = ce^{2H(x,y)}$ *then*

$$r(x,y) \geqslant c + \int_0^x \int_0^y \left\{ l(s,t) + \left[l(s,t) + 2\int_0^s l(z,t)dz \int_0^t l(s,z)dz \right] r(s,t) \right\} dtds \quad (25)$$

Proof. Integrating by parts we get

$$\begin{aligned} &2\int_0^x \int_0^y \int_0^s l(z,t)dz \int_0^t l(s,z)dz\, r(s,t)dtds \\ &= \int_0^x \int_0^y l(s,z)dz\, r(s,y)ds - \int_0^x \int_0^y l(s,t)r(s,t)dtds \end{aligned} \quad (26)$$

From (26) and integrating by substitution we get

$$\int_0^x \int_0^y \left[l(s,t) + 2\int_0^s l(z,t)dz \int_0^t l(s,z)dz \right] r(s,t)dtds = \frac{1}{2}r(x,y) - \frac{1}{2}c$$

Therefore an equivalent formulation of (25) is:

$$r(x,y) \geqslant c + 2H(x,y) \tag{27}$$

From $e^x \geqslant 1+x$ and $c \geqslant 1$ it follows (27) which completes the proof. □

THEOREM 3. (existence) *Suppose that* $f(x,y,\omega)$ *is such that if* $u_{(x,y)} \in C(D,R^n)$ *then* $f(x,y,u_{(x,y)}) \in L^1(I,R^n)$. *Furthermore uniform convergence* $\omega_n \to \omega$ *in* $C(I^*,R^n)$ *implies* $f(x,y,(\omega_n)_{(x,y)}) \to f(x,y,(\omega)_{(x,y)})$ *a.e. in* I. *There is a function* $l(x,y) \in L^1(I,R)$ *such that*

$$|f(x,y,\omega)| \leqslant l(x,y)\{1+||\omega||_0\} \tag{28}$$

Then the problem (5), (6) has a solution existing in I. *Furthermore every solution satisfies the estimate* $|u(x,y)| \leqslant ce^{2H(x,y)}$.

Proof. The problem (5), (6) is equivalent to the fixed point equation $u = Su$, where the operator S is defined by

$$\big(Su\big)(x,y) = \psi(x,y) \text{ on } I_0$$

$$\big(Su\big)(x,y) = \psi(x,0)+\psi(0,y)-\psi(0,0)+\int_0^x\int_0^y f(s,t,u_{(s,t)})dtds \text{ on } I$$

Let u be a solution of problem (5), (6) and $\rho(x,y) = max\{|u(x+s,y+t)| : (s,t) \in D\}$. Then $\rho(x,y) \leqslant c$ on I_0 and

$$\rho(x,y) \leqslant c+\int_0^x\int_0^y l(s,t)\big\{1+\rho(s,t)\big\}dtds \text{ on } I \tag{29}$$

Let $r(x,y) = ce^{2H(x,y)}$. Then from the lemma 1 we have

$$r(x,y) \geqslant c+\int_0^x\int_0^y l(s,t)\big\{1+r(s,t)\big\}dtds \text{ on } I \tag{30}$$

(29) and (30) imply $\rho(x,y) \leqslant r(x,y)$ on I^*.

Let $M = \{\phi(x,y) \in C(I^*,R^n) : |\phi(x,y)| \leqslant r(x,y)\}$. Then M is the convex subset Banach space. It is obvious that the image $S\phi$ is bounded by $r(x,y)$ for $\phi \in M$. Let $\phi_n \to \phi$ uniformly on I^* then

$$|f(x,y,(\phi_n)_{(x,y)}) - f(x,y,\phi_{(x,y)})| \leqslant 2l(x,y)\{1+r(x,y)\}$$

$$|f(x,y,(\phi_n)_{(x,y)}) - f(x,y,\phi_{(x,y)})| \to 0 \text{ a.e. in } I$$

By Lebesgue's theorem on dominated convergence

$$\big|\big(S\phi_n\big)(x,y) - \big(S\phi\big)(x,y)\big| \to 0$$

and that is mean that the operator S is continuous. Furthermore the set $S(M)$ is uniformly bounded and equicontinuous, because if $\zeta(x,y) = \big(S\phi\big)(x,y)$, where $\phi \in M$ then $\left|\frac{\partial\zeta}{\partial x}(x,y)\right| \leqslant d+\int_0^y l(x,z)\{1+r(x,z)\}dz$ and $\left|\frac{\partial\zeta}{\partial y}(x,y)\right| \leqslant d+\int_0^x l(z,y)\{1+r(z,y)\}dz$ for $d = max\big\{\big|\big|\frac{\partial\psi}{\partial x}(x,0)\big|\big|,\big|\big|\frac{\partial\psi}{\partial y}(0,y)\big|\big|\big\}$. Therefore the set $S(M)$ is relative compact. Using the Schauder's fixed point theorem we get existence of solution of the problem (5), (6). □

4. Weak inequalities for first order partial derivatives

From example 2 we know that in the thesis of theorem 1 we can not get $\frac{\partial u}{\partial x}(x,y) \geqslant 0$ and $\frac{\partial u}{\partial y}(x,y) \geqslant 0$ on I. In order to get $\frac{\partial u}{\partial x}(x,y) \geqslant 0$ and $\frac{\partial u}{\partial y}(x,y) \geqslant 0$ on I it is necessary to put some restrictions on $C(x,y)$ and $f(x,y,u_{(x,y)},u(x,y))$.

THEOREM 4. *Suppose that $P(x,y)$ is a linear map from $C(D,R^n)$ into R^n which is positive in the sense that $u(s,t) \geqslant 0$ on $[x-a_0,x]\times[y-b_0,y]$ implies $P(x,y)u_{(x,y)} \geqslant 0$. $C(x,y) = (c_{ij}(x,y))$ is an $n\times n$ matrix which elements are nonnegative i.e. $c_{ij}(x,y) \geqslant 0$ a.e. in I. There exists a function $l(x,y) \in L^1(I,R)$ such that*

$$|P(x,y)u_{(x,y)}| \leqslant l(x,y)||u||_0 \text{ and } c_{ij}(x,y) \leqslant l(x,y) \text{ a.e. in } I$$

$u(x,y)$ is absolute continuous, $\frac{\partial u}{\partial x}(x,y)$ is continuous with respect to y, $\frac{\partial u}{\partial x}(x,y)$ is continuous with respect to x. Furthermore

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) \geqslant C(x,y)u(x,y) + P(x,y)u_{(x,y)} \text{ a.e. in } I$$

$$u(x,y) \geqslant 0 \text{ on } I_0,\ \frac{\partial u}{\partial x}(x,0) \geqslant 0,\ \frac{\partial u}{\partial y}(0,y) \geqslant 0$$

Then

$$u(x,y) \geqslant 0,\ \frac{\partial u}{\partial x}(x,y) \geqslant 0,\ \frac{\partial u}{\partial y}(x,y) \geqslant 0 \text{ on } I$$

Proof. Define $v(x,y) = e^{H(x,y)}\frac{\partial u}{\partial x}(x,y)$, where $H(x,y)$ as in the first proof. Then

$$\frac{\partial v}{\partial y}(x,y) \geqslant \int_0^x l(z,y)dz\, v(x,y) + e^{H(x,y)}\left\{C(x,y)u(x,y) + P(x,y)u_{(x,y)}\right\} \tag{31}$$

Define $\rho(x,y) = e^{(n+2)H(x,y)+x}$, $r(x,y) = (\rho(x,y),...,\rho(x,y))$ and $p(x,y) = e^{H(x,y)}\frac{\partial r}{\partial x}(x,y)$. Then

$$\begin{aligned} p(x,y) &= e^{H(x,y)}\left\{(n+2)\int_0^y l(x,z)dz + 1\right\}r(x,y) \\ \frac{\partial p}{\partial y}(x,y) &= e^{H(x,y)}\Big\{(n+2)l(x,y) + (n+2)(n+3)\int_0^x l(z,y)dz\int_0^y l(x,z)dz \\ &\quad +(n+3)\int_0^x l(z,y)dz\Big\}r(x,y) \end{aligned}$$

We note that

$$0 \leqslant C(x,y)r(x,y) \leqslant nl(x,y)e^{(n+2)H(x,y)} \text{ and } \left|P(x,y)r_{(x,y)}\right| \leqslant l(x,y)e^{(n+2)H(x,y)}$$

Hence

$$\frac{\partial p}{\partial y}(x,y) \geqslant \int_0^x l(z,y)p(x,y) + e^{H(x,y)}\left\{C(x,y)r(x,y) + P(x,y)r_{(x,y)}\right\} \tag{32}$$

Define $q_\varepsilon(x,y)=u(x,y)+\varepsilon r(x,y)$ and $v_\varepsilon(x,y)=e^{H(x,y)}\frac{\partial q_\varepsilon}{\partial x}(x,y)=v(x,y)+\varepsilon p(x,y)$. Then from (31), (32) and from linearity of $P(x,y)$ we have

$$\frac{\partial v_\varepsilon}{\partial y}(x,y)\geqslant\int_0^x l(z,y)dz\, v_\varepsilon(x,y)+e^{H(x,y)}\{C(x,y)q_\varepsilon(x,y)+P(x,y)(q_\varepsilon)_{(x,y)}\} \qquad (33)$$

We note that $q_\varepsilon(x,y)>0$ on I_0 and $v_\varepsilon(x,0)>0$. Therefore there exists $c>0$ and the set $I_c=[0,a]\times[0,c]$ such that $q_\varepsilon(x,y)>0$ and $v_\varepsilon(x,y)>0$ on I_c. From (33) we get $\frac{\partial v_\varepsilon}{\partial y}(x,y)\geqslant 0$ on I_c. Therefore $v_\varepsilon(x,y)$ is nondecreasing with respect to y on I_c and $q_\varepsilon(x,y)$ is nondecreasing with respect to x on this set. Therefore $\frac{\partial v_\varepsilon}{\partial y}(x,y)\geqslant 0$ on I. This gives $v_\varepsilon(x,y)>0$ on I. Letting $\varepsilon\to 0$ we get $v(x,y)\geqslant 0$ it implies $\frac{\partial u}{\partial x}(x,y)\geqslant 0$ on I. The same reasoning applies to the function $v(x,y)=e^{H(x,y)}\frac{\partial u}{\partial y}(x,y)$ gives $\frac{\partial u}{\partial y}(x,y)\geqslant 0$ on I. Of course $\frac{\partial u}{\partial x}(x,y)\geqslant 0$, $\frac{\partial u}{\partial y}(x,y)\geqslant 0$ on I and $u(x,y)\geqslant 0$ on I_0 implies $u(x,y)\geqslant 0$ on I. □

THEOREM 5. *Suppose that for all $A>0$ there exists a function $l(x,y)\in L^1(I,R)$ such that*

$$|f(x,y,\omega)-f(x,y,\overline{\omega})|\leqslant l(x,y)||\omega-\overline{\omega}||_0 \qquad (34)$$

for $||\omega||_0,||\overline{\omega}||_0\leqslant A$. $f(x,y,\omega)$ is nondecreasing with respect to ω. $w(x,y)$, $v(x,y)$ are absolutely continuous, $\frac{\partial v}{\partial x}(x,y)$, $\frac{\partial w}{\partial x}(x,y)$ are continuous with respect to y and $\frac{\partial v}{\partial y}(x,y)$, $\frac{\partial w}{\partial y}(x,y)$ are continuous with respect to x. Furthermore

$$\frac{\partial^2 v}{\partial x\partial y}(x,y)\leqslant f(x,y,v_{(x,y)}) \text{ and } \frac{\partial^2 w}{\partial x\partial y}(x,y)\geqslant f(x,y,w_{(x,y)}) \text{ a.e. in } I$$

$$v(x,y)\leqslant w(x,y) \text{ on } I_0,\ \frac{\partial v}{\partial x}(x,0)\leqslant\frac{\partial w}{\partial x}(x,0),\ \frac{\partial v}{\partial y}(0,y)\leqslant\frac{\partial w}{\partial y}(0,y)$$

Then

$$v(x,y)\leqslant w(x,y),\ \frac{\partial v}{\partial x}(x,y)\leqslant\frac{\partial w}{\partial x}(x,y),\ \frac{\partial v}{\partial y}(x,y)\leqslant\frac{\partial w}{\partial y}(x,y) \text{ on } I$$

Proof. Let $|v(x,y)|,|w(x,y)|\leqslant A-1$ and $l(x,y)$ be the function in (34) corresponding to A and

$$V(x,y)=e^{H(x,y)}\frac{\partial v}{\partial x}(x,y),\ W(x,y)=e^{H(x,y)}\frac{\partial w}{\partial x}(x,y)$$

where H(x,y) as in the first proof. Then

$$\frac{\partial V}{\partial y}(x,y)\leqslant e^{H(x,y)}(Gv)(x,y)$$

$$\frac{\partial W}{\partial y}(x,y)\geqslant e^{H(x,y)}(Gw)(x,y)$$

where

$$(Gu)(x,y) = g(x,y,u_{(x,y)},U(x,y))$$

$$g(x,y,\omega,\sigma) = f(x,y,\omega) + e^{-H(x,y)}\int_0^x l(z,y)dz\,\sigma$$

It is easily seen that $g(x,y,\omega,\sigma)$ is nondecreasing with respect to ω and σ.

Define $\rho : I^* \to R$, $\rho(x,y) = e^{H(x,y)+x}$ and $r,R : I^* \to R^n$, $r(x,y) = (\rho(x,y),...,\rho(x,y))$, $R(x,y) = e^{H(x,y)}\frac{\partial r}{\partial x}(x,y)$. Then $w_\varepsilon(x,y) = w(x,y)+\varepsilon r(x,y)$, $W_\varepsilon(x,y) = W(x,y)+\varepsilon R(x,y)$ satisfy

$$\frac{\partial W_\varepsilon}{\partial y}(x,y) \geqslant e^{H(x,y)}(Gw)(x,y) + \varepsilon e^{H(x,y)}\left\{l(x,y)+2\int_0^x l(z,y)dz\int_0^y l(x,z)dz\right\}r(x,y)$$

Furthermore we can choose such small $\varepsilon > 0$ in order that $|w_\varepsilon(x,y)| \leqslant A$ so

$$\begin{aligned}|Gw_\varepsilon(x,y) - Gw(x,y)| = \Big| & f(x,y,(w+\varepsilon r)_{(x,y)}) - f(x,y,w_{(x,y)}) \\ & +\varepsilon e^{-H(x,y)}\int_0^x l(z,y)dz\,R(x,y)\Big| \\ \leqslant\ & \varepsilon l(x,y)\rho(x,y) + \varepsilon\int_0^x l(z,y)dz\int_0^y l(x,z)dz\,\rho(x,y) \\ & +\varepsilon\int_0^x l(z,y)dz\,\rho(x,y)\end{aligned}$$

Therefore

$$\begin{aligned}e^{H(x,y)}(Gw_\varepsilon)(x,y) \leqslant\ & e^{H(x,y)}(Gw)(x,y) + \varepsilon e^{H(x,y)}\Big\{l(x,y) \\ & +\int_0^x l(z,y)dz\int_0^y l(x,z)dz + \int_0^x l(z,y)dz\Big\}r(x,y) \\ \leqslant\ & \frac{\partial W_\varepsilon}{\partial y}(x,y)\end{aligned}$$

Hence

$$\frac{\partial W_\varepsilon}{\partial y}(x,y) \geqslant e^{H(x,y)}(Gw_\varepsilon)(x,y)$$

We note that

$$w_\varepsilon(x,y) > w(x,y) \geqslant v(x,y) \text{ on } I_0$$

$$\frac{\partial w_\varepsilon}{\partial x}(x,0) > \frac{\partial w}{\partial x}(x,0) \geqslant \frac{\partial v}{\partial x}(x,0)$$

There exists $c > 0$ and the set $I_c = [0,a]\times[0,c]$ such that

$$w_\varepsilon(x,y) > v(x,y) \text{ on } I_c$$

$$\frac{\partial w_\varepsilon}{\partial x}(x,y) > \frac{\partial v}{\partial x}(x,y) \text{ on } I_c$$

From monotonicity G we have

$$\frac{\partial W_\varepsilon}{\partial y}(x,y) \geqslant e^{H(x,y)}(Gw_\varepsilon)(x,y) \geqslant e^{H(x,y)}(Gv)(x,y) \geqslant \frac{\partial V}{\partial y}(x,y) \text{ on } I_c$$

Hence

$$\frac{\partial}{\partial y}(W_\varepsilon - V)(x,y) \geqslant 0 \text{ on } I_c$$

Therefore the function $(W_\varepsilon - V)(x,y)$ is nondecreasing with respect to y on I_c. It is easily seen that $(w_\varepsilon - v)(x,y)$ is nondecreasing with respect to x on I_c. Analysis similar to that in the before proofs shows $\frac{\partial}{\partial y}(W_\varepsilon - V)(x,y) \geqslant 0$ on I. Hence $\frac{\partial w_\varepsilon}{\partial x}(x,y) - \frac{\partial v}{\partial x}(x,y) > 0$. Similarly, we get $\frac{\partial w_\varepsilon}{\partial y}(x,y) - \frac{\partial v}{\partial y}(x,y) > 0$ on I. Letting $\varepsilon \to 0$ we get

$$\frac{\partial w}{\partial x}(x,y) \geqslant \frac{\partial v}{\partial x}(x,y) \text{ on } I$$

$$\frac{\partial w}{\partial y}(x,y) \geqslant \frac{\partial v}{\partial y}(x,y) \text{ on } I$$

It is easy to check that

$$w(x,y) \geqslant v(x,y) \text{ on } I \qquad \square$$

REMARK 5. We note that we did not use the condition

$$l(x,y) \geqslant \int_0^x l(z,y)dz \int_0^y l(x,z)dz \tag{35}$$

in theorems 4 and 5. However we need to assume that elements of matrix $C(x,y)$ are nonnegative and the function $f(x,y,\omega,\eta)$ is nondecreasing with respect to ω and η. Example 3 shows that the theorem 5 is false if we drop the assumption of monotonicity. We see that it is necessary the extra condition if there is no monotonicity. In this paper there is the condition (35) which in some way define area where the inequality $v(x,y) \leqslant w(x,y)$ holds.

5. Strong inequalities

In the last section we will be concerned with strong inequalities. At first we present some definitions.

A measurable set $A \subset [0,p]$ is called dense at 0^+ if the set $A \cap (0,\varepsilon)$ has positive measure for every $\varepsilon > 0$. In farthest part of this paper we are considering two kind of sets A in the sense that if it is said about $u(x,0)$ then we put $p = a$ and if it is said about $u(0,y)$ then $p = b$. A measurable set $B \subset I$ is called dense at $(0^+,0^+)$ if the set $B \cap (0,\varepsilon) \times (0,\varepsilon)$ has a positive measure for every $\varepsilon > 0$. Let $g(x,y)$ and $h(x,y)$ are functions from I into R^n. We write $g(x,0) < h(x,0)$ at 0^+ if the set $\{x \in [0,a] : g(x,0) < h(x,0)\}$ is dense at 0^+. Similarly, we define $g(0,y) < h(0,y)$ at 0^+ and $g(x,y) < h(x,y)$ at $(0^+,0^+)$.

The matrix $C(x,y)$ is called irreducible at $(0^+,0^+)$ if for every pair (α,β) of disjoint nonempty index set with $\alpha\cup\beta=\{1,...,n\}$ there are indices $i\in\alpha,\ j\in\beta$ such that $c_{ij}(x,y)>0$ at $(0^+,0^+)$.

We write $P(x,y)$ in matrix form $P_{ij}(x,y)$, where $P_{ij}(x,y)$ is a positive linear operator which acts on functions $u\in C(D,R^1)$. $P(x,y)$ is called irreducible at $(0^+,0^+)$ if for every pair (α,β) of disjoint nonempty index set with $\alpha\cup\beta=\{1,...,n\}$ there are indices $i\in\alpha,\ j\in\beta$ such that $P_{ij}(x,y)u_{(x,y)}>0$ at $(0^+,0^+)$ for all $u\in C(I^*,R^1)$ such that $u(x,y)>0$ for $(x,y)\in I^*\setminus I_0$.

Now, we present a theorem about strong inequality $u(x,y)>0$ on $I^*\setminus I_0$. Write $\tilde{I}=I^*\setminus I_0$.

THEOREM 6. (positivity) *In theorem* 1 *the assertion* $u(x,y)>0$ *for* $(x,y)\in\tilde{I}$ *holds under each of the following conditions:*

(I) $u(x,0)>0$ *at* 0^+ *or* $u(0,y)>0$ *at* 0^+

(II) $\frac{\partial^2 u}{\partial x\partial y}(x,y)>C(x,y)u(x,y)+P(x,y)u_{(x,y)}$ *at* $(0^+,0^+)$

(III) $u(x,0)\neq 0$ *or* $u(0,y)\neq 0$ *at* 0^+ *and* $C(x,y)$ *is irreducible at* $(0^+,0^+)$

(IV) $u(x,0)\neq 0$ *or* $u(0,y)\neq 0$ *at* 0^+ *and* $P(x,y)$ *is irreducible at* $(0^+,0^+)$

Proof.

(I) We suppose that $u(x,0)>0$ at 0^+. We conclude from the proof of theorem 1 and the remark 1 that $u(x,y)=e^{-H(x,y)}v(x,y)$ where $v(x,y)$ is nondecreasing with respect to y hence that $u(x,y)>0$ in $\{0^+\}\times[0,b]$ and finally that the set β is empty. The same reasoning applies to the case $u(0,y)>0$ at 0^+.

(II) The assumption (II) and the inequality (7) imply $\frac{\partial^2 u}{\partial x\partial y}(x,y)>0$ in $(0^+,0^+)$. Therefore $u(x,y)$ is increasing in $(0^+,0^+)$ with respect to x and y. Hence $u(x,y)>0$ on some square $(0,c]\times(0,c]$. From the remark 1 we have that the set β is empty.

(III) Suppose that the set β is nonempty and we notice that the set α is also nonempty, because $u(x,0)\neq 0$ or $u(0,y)\neq 0$ at 0^+ implies that there exists k such that $u_k(x,y)>0$ on $\tilde{I}$. Due to irreducibility there exists an element $c_{jk}(x,y)$ where $j\in\beta$, $k\in\alpha$ such that $c_{jk}(x,y)>0$ at $(0^+,0^+)$. Due to $u(x,y)\geqslant 0$ on I^* and $u_k(x,y)>0$ on $\tilde{I}$ we have $\frac{\partial^2 u_j}{\partial x\partial y}(x,y)\geqslant c_{jk}(x,y)u_k(x,y)>0$ at $(0^+,0^+)$ and finally that $u_j(x,y)$ is increasing with respect to x and y at $(0^+,0^+)$. It is contradictory to $j\in\beta$.

(IV) Similarly to the proof of (IV) if there are $j\in\beta$ and $k\in\alpha$ then we get that $\frac{\partial^2 u_j}{\partial x\partial y}(x,y)\geqslant P_{jk}(x,y)u_k(x,y)>0$ in $(0^+,0^+)$. It is contradictory to $j\in\beta$. □

Now we give a example which demonstrates that even though a initial function is equal zero on the set $I\setminus\tilde{I}$ then it is possible that a solution of problem is positive on the set $\tilde{I}$. Furthermore $\frac{\partial u}{\partial x}(x,y)$ will not be continuous with respect to x and $\frac{\partial u}{\partial y}(x,y)$ will not be continuous with respect to y.

EXAMPLE 4.

$$\frac{\partial^2 u}{\partial x\partial y}(x,y)=u(x-1,y-1)\text{ a.e. on } I=[0,1]\times[0,1]$$

$$u(x,y) = u_0(x,y) = \begin{cases} x^2y^2 & \text{for } (x,y) \in [-1,0]\times[-1,0] \\ 0 & \text{for } (x,y) \in [-1,0]\times(0,1]\cup(0,1]\times[-1,0] \end{cases}$$

We note that the initial function is continuous and $\frac{\partial u_0}{\partial x}(x,y)$, $\frac{\partial u_0}{\partial y}(x,y)$ are continuous. It is easy to check that the solution of the problem is $u(x,y) = \frac{1}{9}xy(x^2-3x+3)(y^2-3y+3)$. Hence $u(x,y) > 0$ on $\tilde{I}$. Moreover the left-hand derivative $\frac{\partial u}{\partial x}(0^-,y) = 0$ and the right-hand derivative $\frac{\partial u}{\partial x}(0^+,y) = \frac{1}{3}y(y^2-3y+3)$. Therefore the derivative isn't continuous at zero with respect to x. It is easy to check that this derivative is continuous with respect to y. Similarly, the derivative $\frac{\partial u}{\partial y}(x,y)$ isn't continuous with respect to y and is continuous with respect to x.

Now we will discuss the nonlinear problem (3), (4). At first we give a few notations. Let α and β denotes sets from the remark 3 then $v_\alpha(x,y)$ is the vector function consist of this elements $v(x,y) = \big(v_1(x,y),\dots,v_n(x,y)\big)$ for which $i\in\alpha$. Analogously it is understood $w_\alpha(x,y), f_\alpha(x,y,\omega,\eta)$, $v_\beta(x,y)$, $w_\beta(x,y)$ and $f_\beta(x,y,\omega,\eta)$. $e_i = (0,\dots,0,1,0,\dots,0)$ with 1 standing on i-th place.

The function $f(x,y,\omega,\eta)$ is called irreducible if for every pair (α,β) of disjoint nonempty index sets such that $\alpha\cup\beta = \{1,...,n\}$ there are indices $i\in\alpha, j\in\beta$ such that for some $c>0$ and for $(x,y)\in(0,c]\times(0,c]$ we have

$$f_j(x,y,\omega,\eta) < f_j(x,y,\omega+\theta_{(x,y)}e_i,\eta+\theta(x,y)e_i) \tag{36}$$

where $\theta(x,y)$ is the function $C(I^*,R^1)$ equal zero on I_0 and positive on $\tilde{I}$.

THEOREM 7. (strong inequalities) *In theorem* 2 *the assertion* $v(x,y) < w(x,y)$ *for* $(x,y)\in\tilde{I}$ *holds under each of the following conditions:*

(I) $v(x,0) < w(x,0)$ *at* 0^+ *or* $v(0,y) < w(0,y)$ *at* 0^+

(II) $f(x,y,v_{(x,y)},v(x,y)) < f(x,y,w_{(x,y)},w(x,y))$ *at* $(0^+,0^+)$

(III) $v(x,0)\neq w(x,0)$ *or* $v(0,y)\neq w(0,y)$ *at* 0^+ *and the function* $f(x,y,\omega,\eta)$ *is inrreducible*

Proof. Proofs of (I) and (II) are similar to proofs of theorem 6.

(III) We note that $v(x,0)\neq w(x,0)$ or $v(0,y)\neq w(0,y)$ at 0^+ imply there is k such that $v_k(x,y) < w_k(x,y)$ on $\tilde{I}$. Therefore the set α is nonempty. We suppose that the set β also is nonempty. From quasimonotonicity and monotonicity with respect to suitable arguments we have

$$\begin{aligned} f_\beta(x,y,v_{(x,y)},v_\alpha(x,y),v_\beta(x,y)) &\leqslant f_\beta(x,y,w_{(x,y)},w_\alpha(x,y),v_\beta(x,y)) \\ &= f_\beta(x,y,w_{(x,y)},w_\alpha(x,y),w_\beta(x,y)) \end{aligned}$$

Furthermore from the inequality (20) we get

$$f_\beta(x,y,w_{(x,y)},w_\alpha(x,y),w_\beta(x,y)) \leqslant f_\beta(x,y,v_{(x,y)},v_\alpha(x,y),v_\beta(x,y))$$

Therefore

$$f_\beta(x,y,v_{(x,y)},v(x,y)) = f_\beta(x,y,w_{(x,y)},w(x,y)) \text{ at } (0^+,0^+) \tag{37}$$

From irreducible $f(x,y,\omega,\eta)$ for $\theta(x,y) = w_k(x,y) - v_k(x,y)$ we have

$$f_\beta(x,y,v_{(x,y)},v(x,y)) < f_\beta(x,y,v^*_{(x,y)},w_{k(x,y)},v^*(x,y),w_k(x,y)) \leqslant f_\beta(x,y,w_{(x,y)},w(x,y)) \text{ at } (0^+,0^+) \tag{38}$$

The equality (37) and the inequality (38) give a contradiction. □

REMARK 6. We note that in the case delay equation $\frac{\partial^2 u}{\partial x \partial y}(x,y) = f(x,y,u(x - a_0, y - b_0))$ condition (III) is useful, because the inequality $f(x,y,v(x-a_0,y-b_0)) < f(x,y,w(x-a_0,y-b_0))$ at $(0^+,0^+)$ contains only given values and may by true. Whereas condition (IV) $f_j(x,y,v(x-a_0,y-b_0)) < f_j(x,y,v(x-a_0,y-b_0)+\theta(x-a_0,y-b_0)e_i)$ is false for $(x,y) \in I \setminus (a_0,a] \times (b_0,b]$ because $\theta(x,y) = 0$ on I_0. We note that can change condition (IV). Namely, if (IV') $v(-a_0,-b_0) < w(-a_0,-b_0)$ and the function $f(x,y,\omega,\eta)$ is irreducible then for $\theta(x,y) = w(x,y) - v(x,y)$ we have $f_j(x,y,v(x-a_0,y-b_0)) < f_j(x,y,v(x-a_0,y-b_0)+\theta(x-a_0,y-b_0)e_i)$ in some square $[0,c] \times [0,c]$ if $f(x,y,\eta)$ is increasing with respect to η_i.

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