Mathematical
Inequalities
& Applications
Volume 13, Number 2 (2010), 427–435

NEW BOUNDS FOR SOLUTIONS OF A SINGULAR INTEGRO–DIFFERENTIAL INEQUALITY

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(*Communicated by J. Pečarić*)

Abstract. New bounds for solutions of an integro-differential inequality with weakly singular kernel are established using a weighted version of the Hardy-Littlewood-Sobolev inequality. Besides, some applications to real world problems are presented.

1. Introduction and Preliminaries

We would like to give in this work one more application of the famous Hardy-Littlewood-Sobolev inequality. We first establish bounds for the solutions of some integro-differential inequality with weakly singular kernel and then give some immediate applications to real world problems. The inequality we intend to look at is the following

$$\varphi'(t)+a(t)\,\varphi(t)\leqslant b(t)+c(t)\int_0^t (t-s)^{-\alpha}F(s)\varphi^m(s)ds, \tag{1}$$

where $\varphi(t)$ is a nonnegative continuously differentiable function and $a(t)$, $b(t)$, $c(t)$ and $F(t)$ are continuous functions on $(0,\infty)$.

We recall that the reader can encounter in the literature a great deal of results coping with the following inequality

$$\psi(t)\leqslant a(t)+b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}F(s)\psi^m(s)ds,\ \beta>0,\gamma>0, \tag{2}$$

where $m>1$, the linear case ($m=1$) can be found, for instance, in [6]. To overcome the problem of singularity Medved' [11, 12] used the decomposition

$$\begin{aligned}&\int_0^t (t-s)^{\beta-1}s^{\gamma-1}F(s)\Psi(s)^m ds\\ &\leqslant\left(\int_0^t (t-s)^{2(\beta-1)}e^{2\varepsilon s}ds\right)^{1/2}\left(\int_0^t s^{2(\gamma-1)}F(s)^2e^{-2\varepsilon s}\Psi(s)^{2m}ds\right)^{1/2}\end{aligned} \tag{3}$$

Mathematics subject classification (2010): 42B20, 26D07, 26D15.
Keywords and phrases: Hardy-Littlewood-Sobolev inequality, weakly singular kernel, integro-differential inequality, analytic semigroup.

Paper MIA-13-32

and Lemma 1.2 below. On the other hand, Kirane and Tatar improved the previous results in [7] by using the decomposition

$$\int_0^t (t-s)^{\beta-1}s^{\gamma-1}F(s)\Psi(s)^m ds$$
$$\leqslant \left(\int_0^t (t-s)^{2(\beta-1)}s^{2(\gamma-1)}e^{-2s}ds\right)^{1/2}\left(\int_0^t F(s)^2e^{2s}\Psi(s)^{2m}ds\right)^{1/2} \tag{4}$$

and Lemma 1.1 (see also [17, 18]).

Furthermore, we can cite the work done by the present authors in [10] dealing with the integro-differential problem

$$\begin{cases} \dfrac{du}{dt}+Au=f(t,u(t))+\int_0^t g\left(t,s,u(s),\int_0^s K(s,\tau,u(\tau))\,d\tau,\ t\in I=[0,T]\right. \\ u(0)=u_0\in X, \end{cases}$$

and where again an exponential decay result was proved using in a crucial manner the integral inequality given in Lemma 1.1. The global existence is proved, in a more general setting in [9] for a problem with non-local conditions of the form

$$u(0)+h(t_1,...,t_p,u)=u_0$$

and with delays in the arguments of the solution u. Namely, the problem treated there was

$$\begin{cases} \dfrac{du}{dt}+Au=F\left(t,u(\sigma_1(t)),\int_0^t g\left(t,s,u(\sigma_2(s)),\int_0^s K(s,\tau,u(\sigma_3(\tau)))\,d\tau\right)ds\right) \\ u(0)+h(t_1,...,t_p,u(.))=u_0\in X. \end{cases}$$

LEMMA 1. *If $\lambda,\ \nu,\ \omega>0$, then for any $t>0$ we have*

$$t^{1-\nu}\int_0^t (t-s)^{\nu-1}s^{\lambda-1}e^{-\omega s}ds\leqslant C,$$

for some positive constant C independent of t given by

$$C=\max\left\{1,2^{1-\nu}\right\}\Gamma(\lambda)(1+\lambda/\nu)\omega^{-\lambda}.$$

(see [13]).

LEMMA 2. *Let $\alpha\in[0,1)$ and $\beta\in\mathbf{R}$. There exists a positive constant $C=C(\alpha,\beta)$ such that*

$$\int_0^t s^{-\alpha}e^{\beta s}ds\leqslant\begin{cases} Ce^{\beta t},\ if\ \beta>0 \\ C(t+1),\ if\ \beta=0 \\ C,\ if\ \beta<0. \end{cases}$$

LEMMA 3. *Let $a(t),\ b(t),\ K(t),\ \psi(t)$ be nonnegative, continuous functions in the interval $I=(0,T)$ ($0<T\leqslant\infty$), $\Phi:(0,\infty)\to\mathbf{R}$ be a continuous, nonnegative and*

nondecreasing function, $\Phi(0)=0,\ \Phi(u)>0$ *for* $u>0$*, and let* $A(t)=\max_{0\leqslant s\leqslant t} a(s)$*,* $B(t)=\max_{0\leqslant s\leqslant t} b(s)$*. If*

$$\psi(t) \leqslant a(t)+b(t)\int_0^t K(s)\Phi(\psi(s))ds,\ t\in I,$$

then

$$\psi(t) \leqslant W^{-1}\left[W(A(t))+B(t)\int_0^t K(s)ds\right],\ t\in(0,T_1),$$

where

$$W(v)=\int_{v_0}^{v}\frac{d\sigma}{\Phi(\sigma)},$$

for all $v\geqslant v_0>0$*,* W^{-1} *being the inverse function of* W *and* $T_1>0$ *is such that*

$$W(A(t))+B(t)\int_0^t K(s)ds\in D(W^{-1}),\ \forall t\in(0,T_1).$$

The proof of this lemma may be found in [1] for instance (see also [1, 14]).

2. The main result

In this section we intend to give an appropriate bound for the solutions $\varphi(t)$ of the integro-differential inequality (1). In order to avoid the use of the standard desingularization, we need the following weighted Hardy-Littlewood-Sobolev Lemma (see [8])

LEMMA 4. *Let* $p>1,\ l>1/p-1,\ t^{-l}f(t)\in L^p(0,\infty),\ 0\leqslant k\leqslant\alpha<1/p;\ \alpha>0,$ *if* $k=0$; *and* $q=1/(1/p+k-\alpha)$*, then*

$$\left\|s^{-(k+l)}f_\alpha\right\|_q\leqslant K\left\|s^{-l}f(s)\right\|_p,$$

for some K *depending only on* $a,\ k,\ l,\ p$ *and* q*. Here*

$$f_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{-\alpha}f(s)ds.$$

Note that $f_\alpha(t)$ is the (Riemann-Liouville) fractional integral of order $1-\alpha$. Our main result is the following estimate,

THEOREM 1. *Let* $p>1,\ l>1/p-1,\ 0\leqslant k\leqslant\alpha<1/p;\ \alpha>0,$ *if* $k=0;\ q=1/(1/p+k-\alpha)$ *and* q' *its conjugate exponent. Assume that* $a(t)$ *is a continuous functions in* $(0,\infty)$ *and that* $b(t)$*,* $c(t)$ *and* $F(t)$ *are nonnegative continuous functions in* $(0,\infty)$ *such that*

$$c(t)t^{k+l}\exp\left(\int_0^t a(s)\,ds\right)\in L^{q'}(0,\infty)\ \text{and}\ t^{-l}F(t)\in L^p(0,\infty).$$

If $\varphi(t)$ is a nonnegative continuously differentiable function in $(0,\infty)$ satisfying the integro-differential inequality (1), then

$$\varphi(t) \leqslant \left(C_0(t)^{1-pm} + (1-pm) D_0(t) \int_0^t s^{-pl} F^p(s)\, ds \right)^{1/(1-pm)}, \tag{5}$$

for all $t \in (0,T_1)$ provided that

$$C_0(t)^{1-pm} + (1-pm) D_0(t) \int_0^t s^{-pl} F^p(s)\, ds > 0,$$

for all $t \in (0,T_1)$, where

$$C_0(t) = \max_{0\leqslant\sigma\leqslant t} \left\{ E^-(\sigma) \left(\varphi(0) + \int_0^\sigma b(s) E^+(s)\, ds \right) + \frac{1}{q'} K\Gamma(\alpha) E^-(\sigma) \left(\int_0^\sigma \left| s^{k+l} c(s) E^+(s) \right|^{q'} ds \right)^{1/q'} \right\}$$

and

$$D_0(t) = \max_{0\leqslant\sigma\leqslant t} \left\{ \frac{1}{p} K\Gamma(\alpha) E^-(\sigma) \left(\int_0^\sigma \left| s^{k+l} c(s) E^+(s) \right|^{q'} ds \right)^{1/q'} \right\},$$

with

$$E^+(t) = \exp\left(\int_0^t a(s)\, ds \right),\ E^-(t) = \exp\left(-\int_0^t a(s)\, ds \right)$$

and K a positive constant.

Proof. Multiplying both sides of (1) by $E^+(t) = \exp\left(\int_0^t a(s)\, ds\right)$ we obtain

$$\begin{aligned} \left[E^+(t)\varphi(t)\right]' &= a(t)E^+(t)\varphi(t) + E^+(t)\varphi'(t) \\ &\leqslant b(t)E^+(t) + c(t)E^+(t) \int_0^t (t-s)^{-\alpha} F(s)\varphi^m(s)\, ds. \end{aligned}$$

Next, integrating both sides from 0 to t, we find

$$E^+(t)\varphi(t) - \varphi(0) \leqslant \int_0^t b(s)E^+(s)\, ds + \int_0^t c(s)E^+(s) \int_0^s (s-z)^{-\alpha} F(z)\varphi^m(z)\, dzds.$$

Setting $E^-(t) = \exp\left(-\int_0^t a(s)\, ds\right)$, we see that

$$\begin{aligned} \varphi(t) \leqslant E^-(t) \left\{ \varphi(0) + \int_0^t b(s)E^+(s)\, ds \right\} + E^-(t) \left\{ \int_0^t c(s)E^+(s) \right. \\ \left. \times \int_0^s (s-z)^{-\alpha} F(z)\varphi^m(z)\, dzds \right\}, \end{aligned}$$

which we can rewrite as

$$\varphi(t) \leqslant A(t) + E^{-}(t) \int_0^t B(s) \int_0^s (s-z)^{-\alpha} F(z)\, \varphi^m(z)\, dz ds,$$

where $A(t) = E^{-}(t) \left\{ \varphi(0) + \int_0^t b(s) E^{+}(s)\, ds \right\}$ and $B(t) = c(t) E^{+}(t)$. On the other hand, thanks to Hölder Inequality we find

$$\begin{aligned}
\varphi(t) &\leqslant A(t) + E^{-}(t) \int_0^t B(s) s^{k+l} s^{-(k+l)} \int_0^s (s-z)^{-\alpha} F(z)\, \varphi^m(z)\, dz ds \\
&\leqslant A(t) + E^{-}(t) \left(\int_0^t \left| B(s) s^{k+l} \right|^{q'} ds \right)^{1/q'} \\
&\quad \times \left(\int_0^t \left| s^{-(k+l)} \int_0^s (s-z)^{-\alpha} F(z)\, \varphi^m(z)\, dz \right|^{q} ds \right)^{1/q}.
\end{aligned}$$

Now, since $s^{-l}F(s) \in L^p(0,\infty)$, it follows that $s^{-l}F(s)\varphi^m(s) \in L^p(0,t)$, for every $t > 0$, then applying Hardy-Littlewood Inequality for $f(z) = F(z)\varphi^m(z)$, and next Young's Inequality we get

$$\varphi(t) \leqslant A(t) + K\Gamma(\alpha) E^{-}(t) \left(\int_0^t \left| B(s) s^{k+l} \right|^{q'} ds \right)^{1/q'} \left(\int_0^t \left(s^{-l} F(s)\, \varphi^m(s) \right)^p ds \right)^{1/p}$$

$$\leqslant A(t) + K\Gamma(\alpha) E^{-}(t) \left(\int_0^t \left| B(s) s^{k+l} \right|^{q'} ds \right)^{1/q'} \left(\frac{1}{q'} + \frac{1}{p} \int_0^t \left(s^{-l} F(s) \right)^p \varphi^{mp}(s)\, ds \right).$$

Denoting

$$C(t) = A(t) + \frac{1}{q'} K\Gamma(\alpha) E^{-}(t) \left(\int_0^t \left| B(s) s^{k+l} \right|^{q'} ds \right)^{1/q'},$$

and

$$D(t) = \frac{1}{p} K\Gamma(\alpha) E^{-}(t) \left(\int_0^t \left| B(s) s^{k+l} \right|^{q'} ds \right)^{1/q'},$$

we obtain the Gronwall's type Inequality

$$\varphi(t) \leqslant C(t) + D(t) \int_0^t \left(s^{-l} F(s) \right)^p \varphi^{mp}(s)\, ds.$$

Finally, applying Lemma 3 with $\Phi(x) = x^{pm}$, we get at once

$$W(v) = \int_{v_0}^{v} x^{-pm} dx = \frac{1}{1-pm}\left(v^{1-pm} - v_0^{1-pm}\right)$$

with $v \geqslant v_0 > 0$ and

$$W^{-1}(z) = \left(v_0^{1-pm} + (1-pm)z\right)^{1/(1-pm)}$$

for all $z \geqslant 0$.

Next, defining

$$C_0(t) = \max_{0 \leqslant s \leqslant t} C(s),\ D_0(t) = \max_{0 \leqslant s \leqslant t} D(s)$$

we obtain

$$\varphi(t) \leqslant \left\{ C_0(t)^{1-pm} + (1-pm) D_0(t) \int_0^t s^{-pl} F^p(s)\, ds \right\}^{1/(1-pm)},$$

for all $t \in (0, T_1)$ provided that

$$C_0(t)^{1-pm} + (1-pm) D_0(t) \int_0^t s^{-pl} F^p(s)\, ds > 0,$$

for all $t \in (0, T_1)$.

REMARK 1. Of course one can use the usual desingularization as in Medved [11, 12] or in Kirane and Tatar [7, 17, 18], then we integrate both sides. In doing so, we get crude bounds involving polynomials in t (and therefore unbounded terms) instead of constants. It is also possible to rewrite (1) in the form

$$\varphi'(t) + a^+(t)\varphi(t) \leqslant a^-(t)\varphi(t) + b(t) + c(t)\int_0^t (t-s)^{-\alpha} F(s)\varphi^m(s) ds.$$

If $\varphi'(t) \leqslant \varphi'(t) + a^+(t)\varphi(t)$, then we lose the term $a^+(t)\varphi(t)$ unless $a(t)$ is non-positive in which case the term $a(t)\varphi(t)$ can be shifted to the right hand side of the inequality.

3. Application

Let us consider the following weighted Cauchy-type problem:

$$\begin{cases} D^\alpha\left[u'(t) + a(t)u(t)\right] = f(t,u), & t > 0, \\ t^{1-\alpha}\left[u'(t) + a(t)u(t)\right]|_{t=0} = b \in \mathbf{R}, \end{cases} \tag{6}$$

where f is a continuous function in two variables satisfying

$$|f(t,u)| \leqslant F(t)u^m(t),\ m > 1$$

for some continuous function $F(t)$.

Here α is a real number such that $0 < \alpha < 1$.

DEFINITION 1. If $f(x) \in L^1(a,b)$, the integral

$$(I^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt,\ x > a$$

where $\alpha > 0$, is called the Riemann-Liouville fractional integral of order α.

DEFINITION 2. The expression

$$(D^\alpha f)(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} dt$$

is called the Riemann-Liouville fractional derivative of order α when the right hand side is pointwise defined on (a,b).

We refer the interested reader to [15, 16] for more on fractional calculus.

Let us define the space

$$C_r^0([0,h]) := \left\{ v \in C^0((0,h]) : \lim_{t \to 0^+} t^r v(t) \text{ exists and is finite} \right\}.$$

Here $C^0((0,h])$ is the usual space of continuous functions on $(0,h]$. It turns out that the space $C_r^0([0,h])$ endowed with the norm

$$\|v\|_r := \max_{0 \leqslant t \leqslant h} t^r |v(t)|$$

is a Banach space. Next, we define the space

$$C_{1-\alpha}^\alpha([0,h]) := \Big\{ v \in C_{1-\alpha}^0([0,h]) : \text{ there exist } c \in \mathbf{R} \text{ and } v^* \in C_{1-\alpha}^0([0,h]) \\ \text{such that } v(t) = ct^{\alpha-1} + I^\alpha v^*(t) \Big\}.$$

The space $(C_{1-\alpha}^\alpha[0,h], \|\cdot\|_{1-\alpha,\alpha})$, where

$$\|v\|_{1-\alpha,\alpha} := \|v\|_{1-\alpha} + \|D^\alpha v\|_{1-\alpha} \text{ and } \alpha > 1/2,$$

is also a Banach space. We refer to [3, 4, 5] for these facts and also for the question of local existence of solutions.

We can conclude by Theorem 2.1 that any local solution u satisfying

$$u' \in (C_{1-\alpha}^\alpha[0,h], \|\cdot\|_{1-\alpha,\alpha})$$

is bounded by a function like that of the right hand side of the estimate (5) within the stated interval. Indeed, this follows from the fact that we can then write

$$u'(t)+a(t)u(t)\leqslant |b|t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s)u^m(s)ds.$$

This inequality is of the form (1) with $\alpha-1$ instead of $-\alpha$ (note that both are between -1 and 0).

One can also notice that if

$$D_0(\infty)\int_0^\infty s^{-pl}F^p(s)\,ds<\frac{C_0(\infty)^{1-pm}}{pm-1}$$

then the condition

$$C_0(t)^{1-pm}+(1-pm)D_0(t)\int_0^t s^{-pl}F^p(s)\,ds>0$$

holds for all $t>0$ and therefore solutions exists globally in time.

Acknowledgement. The first author would like to express his gratitude to the "Laboratoire de Mathematiques Appliquees" of the University of Annaba for its continuous support. The second author wishes to thank King Fahd University of Petroleum and Minerals for its financial support.

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(Received February 27, 2008)

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