

## INTEGRAL INEQUALITIES AND THEIR APPLICATIONS TO THE CALCULUS OF VARIATIONS ON TIME SCALES

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*Abstract.* We discuss the use of inequalities to obtain the solution of certain variational problems on time scales.

### 1. Introduction

A time scale, denoted by  $\mathbb{T}$ , is a nonempty closed subset of the real numbers. The calculus on time scales is a relatively new area that unifies the difference and differential calculus, which are obtained by choosing  $\mathbb{T} = \mathbb{Z}$  or  $\mathbb{T} = \mathbb{R}$ , respectively. The subject was initiated by S. Hilger in the nineties of the XX century [17, 18], and is now under strong current research in many different fields in which dynamic processes can be described with discrete, continuous, or hybrid models. For concepts and preliminary results on time scales, we refer the reader to [6, 7].

In this paper we start by proving some integral inequalities on time scales involving convex functions (see Section 2). These are then applied in Section 3 to solve some classes of variational problems on time scales. A simple illustrative example is given in Section 4. The method proposed here is direct, in the sense that it permits to find directly the optimal solution instead of using variational arguments and go through the usual procedure of solving the associated delta or nabla Euler–Lagrange equations [2, 4, 11, 19]. This is particularly useful since even simple classes of problems of the calculus of variations on time scales lead to dynamic Euler–Lagrange equations for which methods to compute explicit solutions are not known. A second advantage of the method is that it provides directly an optimal solution, while the variational method on time scales initiated in [4] and further developed in [3, 12, 14, 20] is based on necessary optimality conditions, being necessary further analysis in order to conclude if the candidate is a local minimizer, a local maximizer, or just a saddle (see [4] for second order necessary and sufficient conditions). Finally, while all the previous methods of

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the calculus of variations on time scales only establish local optimality, here we provide global solutions.

The use of inequalities to solve certain classes of optimal control problems is an old idea with a rich history [8, 9, 10, 15, 16, 21]. We hope that the present study will be the beginning of a class of direct methods for optimal control problems on time scales, to be investigated with the help of dynamic inequalities — see [1, 5, 10, 13, 22, 23, 24] and references therein.

### 2. Integral Inequalities on Time Scales

The first theorem is a generalization to time scales of the well-known *Jensen inequality*. It can be found in [22, 24].

**THEOREM 1.** (Generalized Jensen’s inequality [22, 24]) *Let  $a, b \in \mathbb{T}$  and  $c, d \in \mathbb{R}$ . Suppose  $f : [a, b]_{\mathbb{T}}^{\kappa} \rightarrow (c, d)$  is rd-continuous and  $F : (c, d) \rightarrow \mathbb{R}$  is convex. Moreover, let  $h : [a, b]_{\mathbb{T}}^{\kappa} \rightarrow \mathbb{R}$  be rd-continuous with*

$$\int_a^b |h(t)|\Delta t > 0.$$

Then,

$$\frac{\int_a^b |h(t)|F(f(t))\Delta t}{\int_a^b |h(t)|\Delta t} \geq F\left(\frac{\int_a^b |h(t)|f(t)\Delta t}{\int_a^b |h(t)|\Delta t}\right). \tag{1}$$

**PROPOSITION 2.** *If  $F$  in Theorem 1 is strictly convex and  $h(t) \neq 0$  for all  $t \in [a, b]_{\mathbb{T}}^{\kappa}$ , then equality in (1) holds if and only if  $f$  is constant.*

*Proof.* Consider  $x_0 \in (c, d)$  defined by

$$x_0 = \frac{\int_a^b |h(t)|f(t)\Delta t}{\int_a^b |h(t)|\Delta t}.$$

From the definition of strict convexity, there exists  $m \in \mathbb{R}$  such that

$$F(x) - F(x_0) > m(x - x_0)$$

for all  $x \in (c, d) \setminus \{x_0\}$ . Assume  $f$  is not constant. Then,  $f(t_0) \neq x_0$  for some  $t_0 \in [a, b]_{\mathbb{T}}^{\kappa}$ . We split the proof in two cases. (i) Assume that  $t_0$  is right-dense. Then, since  $f$  is rd-continuous, we have that  $f(t) \neq x_0$  on  $[t_0, t_0 + \delta)_{\mathbb{T}}$  for some  $\delta > 0$ . Hence,

$$\begin{aligned} \int_a^b |h(t)|F(f(t))\Delta t - \int_a^b |h(t)|\Delta tF(x_0) &= \int_a^b |h(t)|[F(f(t)) - F(x_0)]\Delta t \\ &> m \int_a^b |h(t)|[f(t) - x_0]\Delta t \\ &= 0. \end{aligned}$$

(ii) Assume now that  $t_0$  is right-scattered. Then (note that  $\int_{t_0}^{\sigma(t_0)} f(t)\Delta t = \mu(t_0)f(t_0)$ ),

$$\begin{aligned} & \int_a^b |h(t)|F(f(t))\Delta t - \int_a^b |h(t)|\Delta t F(x_0) \\ &= \int_a^b |h(t)|[F(f(t)) - F(x_0)]\Delta t \\ &= \int_a^{t_0} |h(t)|[F(f(t)) - F(x_0)]\Delta t + \int_{t_0}^{\sigma(t_0)} |h(t)|[F(f(t)) - F(x_0)]\Delta t \\ &\quad + \int_{\sigma(t_0)}^b |h(t)|[F(f(t)) - F(x_0)]\Delta t \\ &> \int_a^{t_0} |h(t)|[F(f(t)) - F(x_0)]\Delta t + m \int_{t_0}^{\sigma(t_0)} |h(t)|[f(t) - x_0]\Delta t \\ &\quad + \int_{\sigma(t_0)}^b |h(t)|[F(f(t)) - F(x_0)]\Delta t \\ &\geq m \left\{ \int_a^{t_0} |h(t)|[f(t) - x_0]\Delta t + \int_{t_0}^{\sigma(t_0)} |h(t)|[f(t) - x_0]\Delta t \right. \\ &\quad \left. + \int_{\sigma(t_0)}^b |h(t)|[f(t) - x_0]\Delta t \right\} \\ &= m \int_a^b |h(t)|[f(t) - x_0]\Delta t = 0. \end{aligned}$$

Finally, if  $f$  is constant, it is obvious that the equality in (1) holds.  $\square$

REMARK 3. If  $F$  in Theorem 1 is a concave function, then the inequality sign in (1) must be reversed. Obviously, Proposition 2 remains true if we let  $F$  be strictly concave.

Before proceeding, we state two particular cases of Theorem 1.

COROLLARY 4. Let  $a, b, c, d \in \mathbb{R}$ . Suppose  $f : [a, b] \rightarrow (c, d)$  is continuous and  $F : (c, d) \rightarrow \mathbb{R}$  is convex. Moreover, let  $h : [a, b] \rightarrow \mathbb{R}$  be continuous with

$$\int_a^b |h(t)|dt > 0.$$

Then,

$$\frac{\int_a^b |h(t)|F(f(t))dt}{\int_a^b |h(t)|dt} \geq F \left( \frac{\int_a^b |h(t)|f(t)dt}{\int_a^b |h(t)|dt} \right).$$

Proof. Choose  $\mathbb{T} = \mathbb{R}$  in Theorem 1.  $\square$

COROLLARY 5. Let  $a = q^n$  and  $b = q^m$  for some  $n, m \in \mathbb{N}_0$  with  $n < m$ . Define  $f$  and  $h$  on  $[q^n, q^{m-1}]_{q^{\mathbb{N}_0}}$  and assume  $F : (c, d) \rightarrow \mathbb{R}$  is convex, where  $(c, d) \supset$

$[f(q^n), f(q^{m-1})]_{q^{\mathbb{N}_0}}$ . If

$$\sum_{k=m}^{n-1} q^k (q-1) |h(q^k)| > 0,$$

then

$$\frac{\sum_{k=m}^{n-1} q^k |h(q^k)| F(f(q^k))}{\sum_{k=m}^{n-1} q^k |h(q^k)|} \geq F \left( \frac{\sum_{k=m}^{n-1} q^k |h(q^k)| f(q^k)}{\sum_{k=m}^{n-1} q^k |h(q^k)|} \right).$$

*Proof.* Choose  $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$ ,  $q > 1$ , in Theorem 1.  $\square$

Jensen's inequality (2) is proved in [1, Theorem 4.1].

**THEOREM 6.** Let  $a, b \in \mathbb{T}$  and  $c, d \in \mathbb{R}$ . Suppose  $f : [a, b]_{\mathbb{T}}^{\kappa} \rightarrow (c, d)$  is rd-continuous and  $F : (c, d) \rightarrow \mathbb{R}$  is convex (resp., concave). Then,

$$\frac{\int_a^b F(f(t)) \Delta t}{b-a} \geq F \left( \frac{\int_a^b f(t) \Delta t}{b-a} \right) \quad (2)$$

(resp., the reverse inequality). Moreover, if  $F$  is strictly convex or strictly concave, then equality in (2) holds if and only if  $f$  is constant.

*Proof.* This is a particular case of Theorem 1 and Proposition 2 with  $h(t) = 1$  for all  $t \in [a, b]_{\mathbb{T}}^{\kappa}$ .  $\square$

We now state and prove some consequences of Theorem 6.

**COROLLARY 7.** Let  $a, b \in \mathbb{T}$  and  $c, d \in \mathbb{R}$ . Suppose  $f : [a, b]_{\mathbb{T}}^{\kappa} \rightarrow (c, d)$  is rd-continuous and  $F : (c, d) \rightarrow \mathbb{R}$  is such that  $F'' \geq 0$  (resp.,  $F'' \leq 0$ ). Then,

$$\frac{\int_a^b F(f(t)) \Delta t}{b-a} \geq F \left( \frac{\int_a^b f(t) \Delta t}{b-a} \right) \quad (3)$$

(resp., the reverse inequality). Furthermore, if  $F'' > 0$  or  $F'' < 0$ , equality in (3) holds if and only if  $f$  is constant.

*Proof.* This follows immediately from Theorem 6 and the facts that a function  $F$  with  $F'' \geq 0$  (resp.,  $F'' \leq 0$ ) is convex (resp., concave) and with  $F'' > 0$  (resp.,  $F'' < 0$ ) is strictly convex (resp., strictly concave).  $\square$

**COROLLARY 8.** Let  $a, b \in \mathbb{T}$  and  $c, d \in \mathbb{R}$ . Suppose  $f : [a, b]_{\mathbb{T}}^{\kappa} \rightarrow (c, d)$  is rd-continuous and  $\varphi, \psi : (c, d) \rightarrow \mathbb{R}$  are continuous functions such that  $\varphi^{-1}$  exists,  $\psi$  is strictly increasing, and  $\psi \circ \varphi^{-1}$  is convex (resp., concave) on  $\text{Im}(\varphi)$ . Then,

$$\varphi^{-1} \left( \frac{\int_a^b \psi(f(t)) \Delta t}{b-a} \right) \geq \varphi^{-1} \left( \frac{\int_a^b \varphi(f(t)) \Delta t}{b-a} \right)$$

(resp., the reverse inequality). Furthermore, if  $\psi \circ \varphi^{-1}$  is strictly convex or strictly concave, the equality holds if and only if  $f$  is constant.

*Proof.* Since  $\varphi$  is continuous and  $\varphi \circ f$  is rd-continuous, it follows from Theorem 6 with  $f = \varphi \circ f$  and  $F = \psi \circ \varphi^{-1}$  that

$$\frac{\int_a^b (\psi \circ \varphi^{-1})((\varphi \circ f)(t))\Delta t}{b-a} \geq (\psi \circ \varphi^{-1}) \left( \frac{\int_a^b (\varphi \circ f)(t)\Delta t}{b-a} \right).$$

Since  $\psi$  is strictly increasing, we obtain

$$\psi^{-1} \left( \frac{\int_a^b \psi(f(t))\Delta t}{b-a} \right) \geq \varphi^{-1} \left( \frac{\int_a^b \varphi(f(t))\Delta t}{b-a} \right).$$

Finally, when  $\psi \circ \varphi^{-1}$  is strictly convex, the equality holds if and only if  $\varphi \circ f$  is constant, or equivalently (since  $\varphi$  is invertible),  $f$  is constant. The case when  $\psi \circ \varphi^{-1}$  is concave is treated analogously.  $\square$

**COROLLARY 9.** Assume  $f : [a, b]_{\mathbb{T}}^{\mathbb{K}} \rightarrow \mathbb{R}$  is rd-continuous and positive. If  $\alpha < 0$  or  $\alpha > 1$ , then

$$\int_a^b (f(t))^\alpha \Delta t \geq (b-a)^{1-\alpha} \left( \int_a^b f(t)\Delta t \right)^\alpha.$$

If  $0 < \alpha < 1$ , then

$$\int_a^b (f(t))^\alpha \Delta t \leq (b-a)^{1-\alpha} \left( \int_a^b f(t)\Delta t \right)^\alpha.$$

Furthermore, in both cases equality holds if and only if  $f$  is constant.

*Proof.* Define  $F(x) = x^\alpha$ ,  $x > 0$ . Then

$$F''(x) = \alpha(\alpha - 1)x^{\alpha-2}, \quad x > 0.$$

Hence, when  $\alpha < 0$  or  $\alpha > 1$ ,  $F'' > 0$ , i.e.,  $F$  is strictly convex. When  $0 < \alpha < 1$ ,  $F'' < 0$ , i.e.,  $F$  is strictly concave. Applying Corollary 7 with this function  $F$ , we obtain the above inequalities with equality if and only if  $f$  is constant.  $\square$

**COROLLARY 10.** Assume  $f : [a, b]_{\mathbb{T}}^{\mathbb{K}} \rightarrow \mathbb{R}$  is rd-continuous and positive. If  $\alpha < -1$  or  $\alpha > 0$ , then

$$\left( \int_a^b \frac{1}{f(t)} \Delta t \right)^\alpha \int_a^b (f(t))^\alpha \Delta t \geq (b-a)^{1+\alpha}.$$

If  $-1 < \alpha < 0$ , then

$$\left( \int_a^b \frac{1}{f(t)} \Delta t \right)^\alpha \int_a^b (f(t))^\alpha \Delta t \leq (b-a)^{1+\alpha}.$$

Furthermore, in both cases the equality holds if and only if  $f$  is constant.

*Proof.* This follows from Corollary 9 by replacing  $f$  by  $1/f$  and  $\alpha$  by  $-\alpha$ .  $\square$

**COROLLARY 11.** *If  $f : [a, b]_{\mathbb{T}}^{\kappa} \rightarrow \mathbb{R}$  is rd-continuous, then*

$$\int_a^b e^{f(t)} \Delta t \geq (b-a) e^{\frac{1}{b-a} \int_a^b f(t) \Delta t}. \quad (4)$$

*Moreover, equality in (4) holds if and only if  $f$  is constant.*

*Proof.* Choose  $F(x) = e^x$ ,  $x \in \mathbb{R}$ , in Corollary 7.  $\square$

**COROLLARY 12.** *If  $f : [a, b]_{\mathbb{T}}^{\kappa} \rightarrow \mathbb{R}$  is rd-continuous and positive, then*

$$\int_a^b \ln(f(t)) \Delta t \leq (b-a) \ln \left( \frac{1}{b-a} \int_a^b f(t) \Delta t \right). \quad (5)$$

*Moreover, equality in (5) holds if and only if  $f$  is constant.*

*Proof.* Let  $F(x) = \ln(x)$ ,  $x > 0$ , in Corollary 7.  $\square$

**COROLLARY 13.** *If  $f : [a, b]_{\mathbb{T}}^{\kappa} \rightarrow \mathbb{R}$  is rd-continuous and positive, then*

$$\int_a^b f(t) \ln(f(t)) \Delta t \geq \int_a^b f(t) \Delta t \ln \left( \frac{1}{b-a} \int_a^b f(t) \Delta t \right). \quad (6)$$

*Moreover, equality in (6) holds if and only if  $f$  is constant.*

*Proof.* Let  $F(x) = x \ln(x)$ ,  $x > 0$ . Then,  $F''(x) = 1/x$ , i.e.,  $F''(x) > 0$  for all  $x > 0$ . By Corollary 7, we get

$$\frac{1}{b-a} \int_a^b f(t) \ln(f(t)) \Delta t \geq \frac{1}{b-a} \int_a^b f(t) \Delta t \ln \left( \frac{1}{b-a} \int_a^b f(t) \Delta t \right),$$

and the result follows.  $\square$

### 3. Applications to the Calculus of Variations

We now show how the results of Section 2 can be applied to determine the minimum or maximum of problems of calculus of variations and optimal control on time scales.

**THEOREM 14.** *Let  $\mathbb{T}$  be a time scale,  $a, b \in \mathbb{T}$  with  $a < b$ , and  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a positive and continuous function. Consider the functional*

$$\mathcal{F}(y(\cdot)) = \int_a^b \left[ \int_0^1 \varphi(y(t) + h\mu(t)y^\Delta(t)) dh \right] y^\Delta(t)^\alpha \Delta t, \quad \alpha \in \mathbb{R} \setminus \{0, 1\},$$

defined on all  $C_{\text{id}}^1$ -functions  $y : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  satisfying  $y^\Delta(t) > 0$  on  $[a, b]_{\mathbb{T}}^{\text{K}}$ ,  $y(a) = 0$ , and  $y(b) = B$ . Let  $G(x) = \int_0^x \varphi(s) ds$ ,  $x \geq 0$ , and let  $G^{-1}$  denote its inverse. Let

$$C = \frac{\int_0^B \varphi(s) ds}{b-a}. \tag{7}$$

(i) If  $\alpha < 0$  or  $\alpha > 1$ , then the minimum of  $\mathcal{F}$  occurs when

$$y(t) = G^{-1}(C(t-a)), \quad t \in [a, b]_{\mathbb{T}},$$

and  $\mathcal{F}_{\min} = (b-a)C^\alpha$ .

(ii) If  $0 < \alpha < 1$ , then the maximum of  $\mathcal{F}$  occurs when

$$y(t) = G^{-1}(C(t-a)), \quad t \in [a, b]_{\mathbb{T}},$$

and  $\mathcal{F}_{\max} = (b-a)C^\alpha$ .

REMARK 15. Since  $\varphi$  is continuous and positive,  $G$  and  $G^{-1}$  are well defined.

REMARK 16. In cases  $\alpha = 0$  or  $\alpha = 1$  there is nothing to minimize or maximize, i.e., the problem of extremizing  $\mathcal{F}(y(\cdot))$  is trivial. Indeed, if  $\alpha = 0$ , then  $\mathcal{F}(y(\cdot)) = b-a$ ; if  $\alpha = 1$ , then it follows from [6, Theorem 1.90] that

$$\begin{aligned} \mathcal{F}(y(\cdot)) &= \int_a^b \left\{ \int_0^1 \varphi(y(t) + h\mu(t)y^\Delta(t)) dh \right\} y^\Delta(t) \Delta t \\ &= \int_a^b (G \circ y)^\Delta(t) \Delta t \\ &= G(B). \end{aligned}$$

In both cases  $\mathcal{F}$  is a constant and does not depend on the function  $y$ .

*Proof of Theorem 14.* Suppose that  $\alpha < 0$  or  $\alpha > 1$ . Using Corollary 9, we can write

$$\begin{aligned} \mathcal{F}(y(\cdot)) &\geq (b-a)^{1-\alpha} \left[ \int_a^b \left\{ \int_0^1 \varphi(y(t) + h\mu(t)y^\Delta(t)) dh \right\} y^\Delta(t) \Delta t \right]^\alpha \\ &= (b-a)^{1-\alpha} (G(y(b)) - G(y(a)))^\alpha, \end{aligned}$$

where the equality holds if and only if

$$\left\{ \int_0^1 \varphi(y(t) + h\mu(t)y^\Delta(t)) dh \right\} y^\Delta(t) = c \quad \text{for some } c \in \mathbb{R}, \quad t \in [a, b]_{\mathbb{T}}^{\text{K}}.$$

Using [6, Theorem 1.90], we arrive at

$$(G \circ y)^\Delta(t) = c.$$

Delta integrating from  $a$  to  $t$  yields (note that  $y(a) = 0$  and  $G(0) = 0$ )

$$G(y(t)) = c(t - a),$$

from which we get

$$y(t) = G^{-1}(c(t - a)).$$

The value of  $c$  is obtained using the boundary condition  $y(b) = B$ :

$$c = \frac{G(B)}{b - a} = \frac{\int_0^B \varphi(s) ds}{b - a} = C,$$

with  $C$  as in (7). Finally, in this case

$$\mathcal{F}_{\min} = \int_a^b C^\alpha \Delta t = (b - a)C^\alpha.$$

The proof of the second part of the theorem is done analogously using the second part of Corollary 9.  $\square$

REMARK 17. We note that the optimal solution found in the proof of the previous theorem satisfies  $y^\Delta > 0$ . Indeed,

$$\begin{aligned} y^\Delta(t) &= (G^{-1}(C(t - a)))^\Delta \\ &= \int_0^1 (G^{-1})'[C(t - a) + h\mu(t)C] dh C \\ &> 0, \end{aligned}$$

because  $C > 0$  and  $(G^{-1})'(G(x)) = \frac{1}{\varphi(x)} > 0$  for all  $x \geq 0$ .

THEOREM 18. Let  $\varphi : [a, b]_{\mathbb{T}}^K \rightarrow \mathbb{R}$  be a positive and rd-continuous function. Then, among all  $C_{\text{rd}}^1$ -functions  $y : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  with  $y(a) = 0$  and  $y(b) = B$ , the functional

$$\mathcal{F}(y(\cdot)) = \int_a^b \varphi(t) e^{y^\Delta(t)} \Delta t$$

has minimum value  $\mathcal{F}_{\min} = (b - a)e^C$  attained when

$$y(t) = - \int_a^t \ln(\varphi(s)) \Delta s + C(t - a), \quad t \in [a, b]_{\mathbb{T}},$$

where

$$C = \frac{\int_a^b \ln(\varphi(t)) \Delta t + B}{b - a}. \quad (8)$$



*Proof.* By Corollary 11,

$$\begin{aligned} \mathcal{F}(y(\cdot)) &= \int_a^b e^{\ln(\varphi(t))+y^\Delta(t)} \Delta t \\ &\geq (b-a)e^{\frac{1}{b-a} \int_a^b [\ln(\varphi(t))+y^\Delta(t)] \Delta t} = (b-a)e^{\frac{1}{b-a} [\int_a^b \ln(\varphi(t)) \Delta t + B]}, \end{aligned}$$

with  $\mathcal{F}(y(\cdot)) = (b-a)e^{\frac{1}{b-a} [\int_a^b \ln(\varphi(t)) \Delta t + B]}$  if and only if

$$\ln(\varphi(t)) + y^\Delta(t) = c \quad \text{for some } c \in \mathbb{R}, \quad t \in [a, b]_{\mathbb{T}}^K. \tag{9}$$

Integrating (9) from  $a$  to  $t$  (note that  $y(a) = 0$ ) gives

$$y(t) = - \int_a^t \ln(\varphi(s)) \Delta s + c(t-a), \quad t \in [a, b]_{\mathbb{T}}.$$

Using the boundary condition  $y(b) = B$  we have

$$c = \frac{\int_a^b \ln(\varphi(t)) \Delta t + B}{b-a} = C,$$

with  $C$  as in (8). A simple calculation shows that  $\mathcal{F}_{\min} = (b-a)e^C$ .  $\square$

**THEOREM 19.** *Let  $\varphi : [a, b]_{\mathbb{T}}^K \rightarrow \mathbb{R}$  be a positive and rd-continuous function. Then, among all  $C_{\text{rd}}^1$ -functions  $y : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  satisfying  $y^\Delta > 0$ ,  $y(a) = 0$ , and  $y(b) = B$ , with*

$$\frac{B + \int_a^b \varphi(s) \Delta s}{b-a} > \varphi(t), \quad t \in [a, b]_{\mathbb{T}}^K, \tag{10}$$

the functional

$$\mathcal{F}(y(\cdot)) = \int_a^b [\varphi(t) + y^\Delta(t)] \ln[\varphi(t) + y^\Delta(t)] \Delta t$$

has minimum value  $\mathcal{F}_{\min} = (b-a)C \ln(C)$  attained when

$$y(t) = C(t-a) - \int_a^t \varphi(s) \Delta s, \quad t \in [a, b]_{\mathbb{T}},$$

where

$$C = \frac{B + \int_a^b \varphi(s) \Delta s}{b-a}. \tag{11}$$

*Proof.* By Corollary 13,

$$\begin{aligned} \mathcal{F}(y(\cdot)) &\geq \int_a^b [\varphi(t) + y^\Delta(t)] \Delta t \ln \left( \frac{1}{b-a} \int_a^b [\varphi(t) + y^\Delta(t)] \Delta t \right) \\ &= \left( \int_a^b \varphi(t) \Delta t + B \right) \ln \left( \frac{\int_a^b \varphi(t) \Delta t + B}{b-a} \right) \end{aligned}$$

with  $\mathcal{F}(y(\cdot)) = \left( \int_a^b \varphi(t) \Delta t + B \right) \ln \left( \frac{\int_a^b \varphi(t) \Delta t + B}{b-a} \right)$  if and only if

$$\varphi(t) + y^\Delta(t) = c \quad \text{for some } c \in \mathbb{R}, \quad t \in [a, b]_{\mathbb{T}}^{\kappa}.$$

Upon integration from  $a$  to  $t$  (note that  $y(a) = 0$ ),

$$y(t) = c(t-a) - \int_a^t \varphi(s) \Delta s, \quad t \in [a, b]_{\mathbb{T}}.$$

Using the boundary condition  $y(b) = B$ , we have

$$c = \frac{B + \int_a^b \varphi(s) \Delta s}{b-a} = C,$$

where  $C$  is as in (11). Note that with this choice of  $y$  we have, using (10), that  $y^\Delta(t) = C - \varphi(t) > 0$ ,  $t \in [a, b]_{\mathbb{T}}^{\kappa}$ . It follows that  $\mathcal{F}_{\min} = (b-a)C \ln(C)$ .  $\square$

#### 4. An Example

Let  $\mathbb{T} = \mathbb{Z}$ ,  $a = 0$ ,  $b = 5$ ,  $B = 25$  and  $\varphi(t) = 2t + 1$  in Theorem 19:

EXAMPLE 20. *The functional*

$$\mathcal{F}(y(\cdot)) = \sum_{t=0}^4 [(2t+1) + (y(t+1) - y(t))] \ln[(2t+1) + (y(t+1) - y(t))],$$

defined for all  $y : [0, 5] \cap \mathbb{Z} \rightarrow \mathbb{R}$  such that  $y(t+1) > y(t)$  for all  $t \in [0, 4] \cap \mathbb{T}$ , attains its minimum when

$$y(t) = 10t - t^2, \quad t \in [0, 5] \cap \mathbb{Z},$$

and  $\mathcal{F}_{\min} = 50 \ln(10)$ .

*Proof.* First we note that  $\max\{\varphi(t) : t \in [0, 4] \cap \mathbb{Z}\} = 9$ . Hence

$$\frac{B + \sum_{k=0}^4 \varphi(k)}{b-a} = \frac{25 + 25}{5} = 10 > 9 \geq \varphi(t).$$

Observing that since, when  $\mathbb{T} = \mathbb{Z}$ ,  $(t^2)^\Delta = 2t + 1$ , we just have to invoke Theorem 19 to get the desired result.  $\square$

REMARK 21. There is an inconsistency in [8, Theorem 3.6] due to the fact that the bound on the functional  $I$  considered there is not constant. For example, let  $a = A = 1$ ,  $\varphi(x) = x + 1$ , and  $\tilde{y}(x) = x$  for all  $x \in [0, 1]$ . Then the hypotheses of [8, Theorem 3.6] are satisfied. Moreover,

$$I(\tilde{y}(x)) = \int_0^1 \ln(\varphi(x)) \tilde{y}'(x) dx = [(x+1)(\ln(x+1) - 1)]_{x=0}^{x=1} = 2 \ln(2) - 1 \approx 0.386.$$

According to [8, Theorem 3.6], the maximum of the functional  $I$  is given by  $I_{\max} = -\ln(C)$ , where

$$C = \frac{1}{A} \int_0^1 \frac{1}{\varphi(x)} dx.$$

A simple calculation shows that  $C = \ln(2)$ . Hence  $I_{\max} = -\ln(\ln(2)) \approx 0.367$ . Therefore,  $I(\tilde{y}(x)) > I_{\max}$ .

## REFERENCES

- [1] R. AGARWAL, M. BOHNER AND A. PETERSON, *Inequalities on time scales: a survey*, Math. Inequal. Appl., **4**, 4 (2001), 535–557.
- [2] F. M. ATICI, D. C. BILES AND A. LEBEDINSKY, *An application of time scales to economics*, Math. Comput. Modelling, **43**, 7–8 (2006), 718–726.
- [3] Z. BARTOSIEWICZ AND D. F. M. TORRES, *Noether's theorem on time scales*, J. Math. Anal. Appl., **342**, 2 (2008), 1220–1226.
- [4] M. BOHNER, *Calculus of variations on time scales*, Dynam. Systems Appl., **13**, 3–4 (2004), 339–349.
- [5] M. BOHNER AND T. MATTHEWS, *Ostrowski inequalities on time scales*, JIPAM. J. Inequal. Pure Appl. Math., **9**, 1 (2008), Article 6, 8 pp.
- [6] M. BOHNER AND A. PETERSON, *Dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2001.
- [7] M. BOHNER AND A. PETERSON, *Advances in dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2003.
- [8] W.-S. CHEUNG, *Integral inequalities and applications to the calculus of variations*, Int. J. Appl. Math., **9**, 1 (2002), 85–108.
- [9] O. DUNKEL, *Integral Inequalities With Applications to the Calculus of Variations*, Amer. Math. Monthly, **31**, 7 (1924), 326–337.
- [10] R. A. C. FERREIRA, M. R. SIDI AMMI AND D. F. M. TORRES, *Diamond-alpha integral inequalities on time scales*, Int. J. Math. Stat., **5**, A09 (2009), 52–59.
- [11] R. A. C. FERREIRA AND D. F. M. TORRES, *Remarks on the calculus of variations on time scales*, Int. J. Ecol. Econ. Stat., **9**, F07 (2007), 65–73.
- [12] R. A. C. FERREIRA AND D. F. M. TORRES, *Higher-order calculus of variations on time scales*, in *Mathematical Control Theory and Finance*, Springer, 2008, 149–159.
- [13] R. A. C. FERREIRA AND D. F. M. TORRES, *Generalizations of Gronwall-Bihari inequalities on time scales*, J. Difference Equ. Appl., **15**, 6 (2009), 529–539.
- [14] R. A. C. FERREIRA AND D. F. M. TORRES, *Isoperimetric problems of the calculus of variations on time scales*, Contemporary Mathematics (2010), in press.
- [15] G. H. HARDY, J. E. LITTLEWOOD, *Some integral inequalities connected with the calculus of variations*, Q. J. Math., Oxf. Ser., **3** (1932), 241–252.
- [16] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, 2nd ed., Cambridge University Press, 1952.
- [17] S. HILGER, *Analysis on measure chains—a unified approach to continuous and discrete calculus*, Results Math., **18**, 1–2 (1990), 18–56.
- [18] S. HILGER, *Differential and difference calculus—unified!*, Nonlinear Anal., **30**, 5 (1997), 2683–2694.
- [19] R. HILSCHER AND V. ZEIDAN, *Calculus of variations on time scales: weak local piecewise  $C^1_{rd}$  solutions with variable endpoints*, J. Math. Anal. Appl., **289**, 1 (2004), 143–166.
- [20] A. B. MALINOWSKA AND D. F. M. TORRES, *Necessary and sufficient conditions for local Pareto optimality on time scales*, J. Math. Sci. (N. Y.), **161**, 6 (2009), 803–810.
- [21] C. SBORDONE, *On some integral inequalities and their applications to the calculus of variations*, Boll. Un. Mat. Ital. C (6), **5**, 1 (1986), 73–94.
- [22] M. R. SIDI AMMI, R. A. C. FERREIRA AND D. F. M. TORRES, *Diamond- $\alpha$  Jensen's inequality on time scales*, J. Inequal. Appl., **2008**, Art. ID 576876, 13 pp.
- [23] M. R. SIDI AMMI AND D. F. M. TORRES, *Combined dynamic Grüss inequalities on time scales*, J. Math. Sci. (N. Y.), **161**, 6 (2009), 792–802.

- [24] F.-H. WONG, C.-C. YEH AND W.-C. LIAN, *An extension of Jensen's inequality on time scales*, Adv. Dyn. Syst. Appl., **1**, 1 (2006), 113–120.

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