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HARDY INEQUALITIES WITH REMAINDER TERMS FOR THE GENERALIZED BAOUENDI–GRUSHIN VECTOR FIELDS

JINGBO DOU, QIANQIAO GUO AND PENGCHENG NIU

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Abstract. Based on the properties of vector fields and the generalized divergence formula, we prove the Hardy inequalities with remainder terms for the generalized Baouendi-Grushin vector fields and determine the best constants in these Hardy inequalities.

1. Introduction

Consider the generalized Baouendi-Grushin (B-G) vector fields (see [12], [15])

$$Z_i = \frac{\partial}{\partial x_i}, \quad Z_{n+j} = |x|^{\alpha} \frac{\partial}{\partial y_j}, \quad (1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$$

where $\alpha > 0$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. $\nabla_L = (Z_1, \ldots, Z_n, Z_{n+1} \ldots, Z_{n+m})$ denotes the horizontal gradient and $\operatorname{div}_L(u_1, u_2, \ldots, u_n, u_{n+1} \ldots, u_{n+m}) = \sum_{i=1}^{n+m} Z_i u_i$ denotes the generalized divergence. Thus, the second order degenerate elliptic operator and p-degenerate sub-elliptic operator can be defined as

$$\mathscr{L}_\alpha = \Delta_x + |x|^{2\alpha}\Delta_y = \sum_{i=1}^{n+m} Z_i = \nabla_L \cdot \nabla_L, \text{ and}$$

$$\mathscr{L}_{p,\alpha} u = \operatorname{div}_L \left(|\nabla_L u|^{p-2} \nabla_L u \right) = \nabla_L \left(|\nabla_L u|^{p-2} \nabla_L u \right), \quad p > 1,$$

respectively, where Δ_x and Δ_y are Laplace operators on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

In general, we call $\mathscr{L}_\alpha$ the generalized B-G operator. When $\alpha = 0$, the operator $\mathscr{L}_\alpha$ is just reduced to the standard Laplacian in $\mathbb{R}^N$. If α is a positive even integer, the Z_j's satisfy Hörmander's finite rank condition. But in the general case Hörmander's condition is meaningless since the vector fields are not sufficiently smooth and $\mathscr{L}_\alpha$ does not belong to the class of sub-Laplacians since it is not left-invariant. We note that $\mathscr{L}_\alpha$ belongs to the wide class of sub-elliptic operators introduced and studied by Franchi and Lanconelli in [7, 8, 9]. Clearly, $\mathscr{L}_\alpha$ is elliptic if $x \neq 0$ and becomes degenerate on the manifold $\{0\} \times \mathbb{R}^m$.

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We start with some basic facts related to the generalized B-G vector fields. Recall that the distance function is defined as

$$d(x,y) = (|x|^{2(\alpha+1)} + (\alpha+1)^2|y|^2)^{\frac{1}{2(\alpha+1)}}.$$

For $d = d(x,y)$, it is easy to see

$$\begin{aligned} \nabla_L d &= \frac{|x|^\alpha}{d^{2\alpha+1}}(|x|^\alpha x_1, |x|^\alpha x_2, \ldots, |x|^\alpha x_n, (\alpha+1)y_1, (\alpha+1)y_2, \ldots, (\alpha+1)y_m), \\ |\nabla_L d|^2 &= \frac{|x|^{2\alpha}}{d^{2\alpha}} = \psi_{2\alpha}, \quad |\nabla_L d|^p = \frac{|x|^{p\alpha}}{d^{p\alpha}} = \psi_{p\alpha}, \quad \mathscr{L}_{p,\alpha} d = \frac{Q-1}{d}\psi_{p\alpha}. \end{aligned} \tag{1.1}$$

Let $\Omega = B_L(R_2) \setminus \overline{B_L(R_1)}$ for $0 \leqslant R_1 < \rho < R_2 \leqslant \infty$, where $B_L(R) = B_L(0,R) = \{(x,y) \in \mathbb{R}^{n+m} | d(x,y) < R\}$ denotes the ball of radius R with center in 0. If $u : \Omega \to \mathbb{R}$ is radial and satisfies $u(x,y) = \psi_{p\alpha} v(d)$, then the change of polar coordinates in [14] $(x_1, \ldots, x_n, y_1, \ldots, y_m) \to (\rho, \theta, \theta_1, \ldots, \theta_{n-1}, \gamma_1, \ldots, \gamma_{m-1})$ allows the following formula to hold

$$\int_\Omega u(x,y) = s_{n,m} \int_{R_1}^{R_2} \rho^{Q-1} v(\rho) d\rho, \tag{1.2}$$

where $s_{n,m} = \left(\frac{1}{\alpha+1}\right)^m \omega_n \omega_m \int_{a_1}^{a_2} |\sin\theta|^{\frac{n+p\alpha}{\alpha+1}-1} |\cos\theta|^{m-1} d\theta$. The definitions of ω_n, ω_m and more details can be seen in [14].

Recently, the integral inequalities related to the generalized B-G vector fields have been paid much attention by many scholars. In [13] the following Sobolev embedding inequality has been proved.

$$\left(\int_\Omega |u|^{2^*} dxdy\right)^{\frac{1}{2^*}} \leqslant S \left(\int_\Omega |\nabla_L u|^2 dxdy\right)^{\frac{1}{2}}, \quad u \in D_0^{1,2}(\Omega) \tag{1.3}$$

with some constant S, where $2^* = \frac{2Q}{Q-2}$ is the critical Sobolev exponent, $Q = n + (\alpha+1)m$ is the homogeneous dimension of $\mathbb{R}^{n+m}$, $\Omega \subset \mathbb{R}^{n+m}$, $D_0^{1,p}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{D^{1,p}} = (\int_\Omega |\nabla_L u|^p dxdy)^{\frac{1}{p}}$. When Ω is a bounded open domain, the Sobolev inequality with remainder terms was also proved. The extremal function of (1.3) in the case $\alpha = 1$ and $n = m = 1$ has been discussed by Beckner in [3].

D'Ambrosio in [4, 5] obtained the Hardy inequalities with respect to the generalized B-G vector fields on a bounded domain $\Omega \subset \mathbb{R}^{n+m}$: If $p \neq Q$, then it holds

$$\int_\Omega |\nabla_L u|^p dxdy \geqslant \left|\frac{Q-p}{p}\right|^p \int_\Omega \left(\frac{|x|}{d}\right)^{p\alpha} \frac{|u|^p}{d^p} dxdy, \quad u \in D_0^{1,p}(\Omega). \tag{1.4}$$

If $p = Q$, let $R > 0$ and set $\Omega := \{(x,y) \in \mathbb{R}^{n+m} \,|\, d(x,y) < R\}$, then one has

$$\left(\frac{p-1}{p}\right)^p \int_\Omega \left(\frac{|x|}{d}\right)^{p\alpha} \frac{|u|^p}{\left(d \ln\left(\frac{R}{d}\right)\right)^p} dxdy \leqslant \int_\Omega |\nabla_L u|^p dxdy, \quad u \in D_0^{1,p}(\Omega), \tag{1.5}$$

where the distance function $d = d(x,y)$ can be seen above. Furthermore, he has extended Hardy inequalities to more generic vector fields. For $\Omega = \mathbb{R}^{n+m}$, Garofalo in [12] obtained inequality (1.4) with $p = 2$, and Zhang, Niu in [15] established the Picone identity and proved the inequality (1.4) with $1 < p < Q$. For any bounded domain $\Omega \subset \mathbb{R}^{n+m}$ containing 0, it is known that the best constant $\left|\frac{Q-p}{p}\right|^p$ in the inequality (1.4) is never achieved for any function $u \in D_0^{1,p}(\Omega)$. So one looks forward to have an estimate of the error term on the right hand side of the inequality (1.4). However, up to now, we find nothing about improvement of the inequality (1.4).

The purpose of this article is to prove some Hardy inequalities with remainder terms on the generalized B-G vector fields. Our methold is based on the properties of vector fields and the generalized divergence formula. We prove such Hardy inequalities and compact embedding in weighted Sobolev spaces. The best constants in Hardy inequalities are also determined .

Hardy inequalities and its generalizations in the Euclidean space have been extensively studied and applied to various interesting problems in PDE. We here refer to [1, 2, 10]. But the research on Hardy inequalities in vector fields here are more complicated and challenged than the research in Euclidean space.

This paper is organized as follows. In next section, we prove Hardy inequalities with remainder terms. In section 3, we discuss the best constants.

In the sequel, for convenience of presentation we will use c, c_1, C, etc. for a suitable positive constants usually except special narrating.

2. Hardy inequalities with remainder terms

In this section, let a bounded domain $0 \in \Omega \subset \mathbb{R}^{n+m}$.

Let us recall that

$$\Gamma(d(x,y)) = \begin{cases} d^{\frac{p-Q}{p-1}}, & \text{if} \quad p \neq Q, \\ -\ln d, & \text{if} \quad p = Q \end{cases}$$

is the solution of $\mathscr{L}_{p,\alpha}$ at the origin (see [14]), that is, $\mathscr{L}_{p,\alpha}\Gamma(d(x,y)) = 0$ on $\Omega \setminus \{0\}$. This implies

$$(Q-p)[d\mathscr{L}_{p,\alpha}d - (Q-1)|\nabla_H d|^p] \geqslant 0, \quad \text{on } \Omega \setminus \{0\}. \tag{C}$$

It is a pivotal geometric hypothesis and a guarantee for the choice of vector fields.

For simplicity, write $\mathscr{B}(s) = -\frac{1}{\ln(s)}$, $s \in (0,1)$ and $A = \frac{Q-p}{p}$. For $R > \sup_{(x,y)\in\Omega} d(x,y)$, there exists a constant $M > 0$ such that

$$0 \leqslant \mathscr{B}\left(\frac{d(x,y)}{R}\right) \leqslant M, \quad (x,y) \in \Omega.$$

Furthermore,

$$\nabla_L \mathscr{B}^\gamma\left(\frac{d}{R}\right) = \gamma \frac{\mathscr{B}^{\gamma+1}\left(\frac{d}{R}\right)\nabla_L d}{d}, \quad \frac{d\mathscr{B}^\gamma\left(\frac{\rho}{R}\right)}{d\rho} = \gamma \frac{\mathscr{B}^{\gamma+1}\left(\frac{\rho}{R}\right)}{\rho}, \ \text{for all } \gamma \in \mathbb{R}, \tag{2.1}$$

and

$$\int_a^b \frac{\mathscr{B}^{\gamma+1}(s)}{s} ds = \frac{1}{\gamma}\left[\mathscr{B}^{\gamma}(b) - \mathscr{B}^{\gamma}(a)\right]. \tag{2.2}$$

One of the main results is

THEOREM 2.1. *Let $0 \in \Omega$ be a bounded domain in $\mathbb{R}^{n+m}$ and $1 < p < \infty$.*

(1) *If $p \neq Q$, then there exists a positive constant $R_0 \geqslant \sup_{(x,y)\in\Omega} d(x,y)$ such that for any $R > R_0$ and all $u \in D_0^{1,p}(\Omega \setminus \{0\})$,*

$$\int_\Omega |\nabla_L u|^p dxdy \geqslant \left|\frac{Q-p}{p}\right|^p \int_\Omega \psi_{p\alpha} \frac{|u|^p}{d^p} dxdy + \frac{p-1}{2p}\left|\frac{Q-p}{p}\right|^{p-2} \int_\Omega \psi_{p\alpha} \frac{|u|^p}{d^p} \left(\ln\left(\frac{R}{d}\right)\right)^{-2} dxdy. \tag{2.3}$$

In particular, if $2 \leqslant p < Q$, one can take $\sup_{(x,y)\in\Omega} d(x,y) = R_0$.

(2) *If $p = Q$, then there exists $R > \sup_{(x,y)\in\Omega} d(x,y)$ such that for all $u \in D_0^{1,p}(\Omega \setminus \{0\})$*

$$\int_\Omega |\nabla_L u|^p dxdy \geqslant \left(\frac{p-1}{p}\right)^p \int_\Omega \psi_{p\alpha} \frac{|u|^p}{\left(d \ln\left(\frac{R}{d}\right)\right)^p} dxdy. \tag{2.4}$$

Proof. Let T be a C^1 vector field on Ω which is specified later. For any $u \in C_0^\infty(\Omega \setminus \{0\})$, using Hölder's inequality and Young's inequality we obtain

$$\begin{aligned}
\int_\Omega (\operatorname{div}_L \mathrm{T})|u|^p dxdy &= -p\int_\Omega \langle \mathrm{T}, \nabla_L u\rangle |u|^{p-2} u dxdy \\
&\leqslant p\left(\int_\Omega |\nabla_L u|^p dxdy\right)^{\frac{1}{p}} \left(\int_\Omega |\mathrm{T}|^{\frac{p}{p-1}} |u|^p dxdy\right)^{\frac{p-1}{p}} \\
&\leqslant \int_\Omega |\nabla_L u|^p dxdy + (p-1)\int_\Omega |\mathrm{T}|^{\frac{p}{p-1}} |u|^p dxdy.
\end{aligned}$$

So,

$$\int_\Omega |\nabla_L u|^p dxdy \geqslant \int_\Omega [\operatorname{div}_L \mathrm{T} - (p-1)|\mathrm{T}|^{\frac{p}{p-1}}]|u|^p dxdy. \tag{2.5}$$

(1) Let a be a parameter to be chosen later. Write

$$I_1(\mathscr{B}) = 1 + \frac{p-1}{pA}\mathscr{B}\left(\frac{d}{R}\right) + a\mathscr{B}^2\left(\frac{d}{R}\right), \quad I_2(\mathscr{B}) = \frac{p-1}{pA}\mathscr{B}^2\left(\frac{d}{R}\right) + 2a\mathscr{B}^3\left(\frac{d}{R}\right),$$

and take $\mathrm{T}(d) = A|A|^{p-2}\frac{|\nabla_L d|^{p-2}\nabla_L d}{d^{p-1}} I_1$. We immediately compute

$$\begin{aligned}
\operatorname{div}_L\left(A|A|^{p-2}\frac{|\nabla_L d|^{p-2}\nabla_L d}{d^{p-1}}\right) &= A|A|^{p-2}\frac{d\mathscr{L}_{p,\alpha} d - (p-1)|\nabla_L d|^p}{d^p} \\
&= A|A|^{p-2}\frac{(Q-1-p+1)|\nabla_L d|^p}{d^p} \\
&= p|A|^p \frac{|\nabla_L d|^p}{d^p},
\end{aligned} \tag{2.6}$$

where we use $(\mathbf{C})$ and (1.1). From (2.1) and (2.6) it holds

$$\begin{aligned}\operatorname{div}_L \mathrm{T} &= p|A|^p\frac{|\nabla_L d|^p}{d^p}I_1 + A|A|^{p-2}\frac{|\nabla_L d|^{p-2}\nabla_L d}{d^{p-1}}\frac{\nabla_L d}{d}\left[\frac{p-1}{pA}\mathscr{B}^2\left(\frac{d}{R}\right)+2a\mathscr{B}^3\left(\frac{d}{R}\right)\right]\\ &= p|A|^p\frac{|\nabla_L d|^p}{d^p}I_1 + A|A|^{p-2}\frac{|\nabla_L d|^p}{d^p}I_2.\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{div}_L \mathrm{T} - (p-1)|\mathrm{T}|^{\frac{p}{p-1}} &= p|A|^p\frac{|\nabla_L d|^p}{d^p}I_1 + A|A|^{p-2}\frac{|\nabla_L d|^p}{d^p}I_2 - (p-1)|A|^p\frac{|\nabla_L d|^p}{d^p}I_1^{\frac{p}{p-1}}\\ &= |A|^p\frac{|\nabla_L d|^p}{d^p}\left(pI_1+\frac{1}{A}I_2-(p-1)I_1^{\frac{p}{p-1}}\right). \qquad (2.7)\end{aligned}$$

Define $f(s) := pI_1(s)+\frac{1}{A}I_2(s)-(p-1)I_1^{\frac{p}{p-1}}(s)$. We need the following estimate which will be proved later,

$$f(s) \geqslant 1+\frac{p-1}{2pA^2}s^2, \quad s\in(0,M). \tag{2.8}$$

Hence,

$$\operatorname{div}_L \mathrm{T} - (p-1)|\mathrm{T}|^{\frac{p}{p-1}} \geqslant |A|^p\frac{|\nabla_L d|^p}{d^p}\left(1+\frac{p-1}{2pA^2}\mathscr{B}^2\left(\frac{d}{R}\right)\right), \quad 0<\mathscr{B}\left(\frac{d}{R}\right)\leqslant M, \tag{2.9}$$

where $M = M(R) := \sup_{(x,y)\in\Omega}\mathscr{B}\left(\frac{d(x,y)}{R}\right)$.

Now we check (2.8) as follows. Arguing as in Theorem 4.1 of [1], from Taylor's formula we get

$$f(s) = f(0)+f'(0)s+\frac{1}{2}f''(\eta_s)s^2, \quad 0\leqslant \eta_s\leqslant s\leqslant M. \tag{2.10}$$

Note that $f(0)=1$ and

$$\begin{aligned} f'(0) &= \frac{p-1}{A}-p\frac{p-1}{pA}=0,\\ f''(0) &= 2ap+\frac{2(p-1)}{pA^2}-2ap-\frac{p}{p-1}\left(\frac{p-1}{pA}\right)^2=\frac{p-1}{pA^2},\\ f'''(0) &= \frac{12a}{A}-\frac{6ap}{p-1}\frac{p-1}{pA}-\frac{p(2-p)}{(p-1)^2}\left(\frac{p-1}{pA}\right)^3=\frac{6a}{A}-\frac{(2-p)(p-1)}{p^2A^3}.\end{aligned}$$

Let's distinguish three cases.

(i) $1<p<2<Q$. Now $A>0$. We can choose $a>\frac{(2-p)(p-1)}{6p^2A^2}>0$ such that $f'''(0)>0$. Hence f'' is an increasing function in the interval $(0,M_0)$ for some $M_0>0$. So, for $s\in(0,M_0)$,

$$f''(\eta_s)\geqslant f''(0)=\frac{p-1}{pA^2}.$$

It then follows from (2.10) that

$$f(s)=f(0)+\frac{1}{2}f''(\eta_s)s^2\geqslant 1+\frac{p-1}{2pA^2}s^2,\quad s\in(0,M_0).$$

Hence (2.8) holds in this case.

(ii) $2\leqslant p<Q$. We still have $A>0$. Let's take $a=0$ and then $f'''(0)>0$. For $s>0$ one has

$$f'''(s)=\frac{(p-2)(p-1)}{p^2A^3}\left(1+\frac{p-1}{pA}s\right)^{\frac{3-2p}{p-1}}>0.$$

Hence, we can get (2.8) repeating the argument as the case (i) by picking $M_0=\infty$.

(iii) $p>Q$. Since $A<0$, we choose a such that $a<\frac{(2-p)(p-1)}{6p^2A^2}<0$, so that $f'''(0)>0$ and proceed as before.

It is not difficult to choose M_0 (small enough) in all cases such that for $0<\mathscr{B}<M_0$, $1+\frac{p-1}{pA}\mathscr{B}+a\mathscr{B}^2>0$. Since $\mathscr{B}(s)=-\frac{1}{\ln(s)}$, the condition $\mathscr{B}\leqslant M_0$ is equivalent to $R\geqslant R_0:=e^{\frac{1}{M_0}}\sup_{(x,y)\in\Omega}d(x,y)$. The inequality (2.3) is proved.

(2) Suppose that $p=Q$. By taking $\mathrm{T}(d)=\left(\frac{p-1}{p}\right)^{p-1}\frac{|\nabla_L d|^{p-2}\nabla_L d}{d^{p-1}}\mathscr{B}^{p-1}\left(\frac{d}{R}\right)$, we have

$$\begin{aligned}\mathrm{div}_L\mathrm{T}&=\left(\frac{p-1}{p}\right)^{p-1}\left\{\frac{[Q-1-(p-1)]|\nabla_L d|^p}{d^p}\mathscr{B}^{p-1}\left(\frac{d}{R}\right)+(p-1)\mathscr{B}^p\left(\frac{d}{R}\right)\frac{|\nabla_L d|^p}{d^p}\right\}\\&=p\left(\frac{p-1}{p}\right)^p\mathscr{B}^p\left(\frac{d}{R}\right)\frac{|\nabla_L d|^p}{d^p},\end{aligned}$$

and hence

$$\mathrm{div}_L\mathrm{T}-(p-1)|\mathrm{T}|^{\frac{p}{p-1}}=\left(\frac{p-1}{p}\right)^p\mathscr{B}^p\left(\frac{d}{R}\right)\frac{|\nabla_L d|^p}{d^p}.\tag{2.11}$$

Combining (2.11) and (2.5) follows (2.4).

REMARK 2.2. The domain Ω in (2.5) may be bounded or unbounded. In addition, if we select $\mathrm{T}(d)=A|A|^{p-2}\frac{|\nabla_L d|^{p-2}\nabla_L d}{d^{p-1}}$, then

$$\mathrm{div}_L\mathrm{T}-(p-1)|\mathrm{T}|^{\frac{p}{p-1}}=p|A|^p\frac{|\nabla_L d|^p}{d^p}-(p-1)|A|^p\frac{|\nabla_L d|^p}{d^p}=|A|^p\frac{|\nabla_L d|^p}{d^p}.$$

Therefore, from (2.5) we conclude (1.4) on a bounded domain Ω and on $\mathbb{R}^{n+m}$ (see [15]), respectively. Moreover, the constant $|A|^p=\left|\frac{p-Q}{p}\right|^p=C_{Q,p}$ is optimal (see [5]). On the other hand, we know the conclusion of Theorem 2.1 is also true in the Heisenberg group, see [6].

Now, we give a Poincaré inequality using (2.5).

THEOREM 2.3. *Let Ω be an open subset of $\mathbb{R}^{n+m}$, and set $(x_1,x_2,\cdots,x_n,y_1,\cdots,y_m)\in\Omega$. If there is a constant $R>0$ such that $0<r=|x_1|\leqslant R$, then for $u\in D_0^{1,p}(\Omega)$, we have*

$$c\int_\Omega |u|^p dxdy \leqslant \int_\Omega |\nabla_L u|^p dxdy, \tag{2.12}$$

where $c=\left(\frac{1}{pR}\right)^p$.

In fact, by choosing $\mathrm{T}=\frac{1}{p^{p-1}}\frac{\nabla_L r}{r^{p-1}}$ in (2.5), we immediately obtain (2.12).

Now, we describe a compactness result from (1.4) and (2.12). Define

$$\mathscr{F}_p := \left\{ f:\Omega\to\mathbb{R}^+ \mid \lim_{d(x,y)\to 0}\frac{d^p(x,y)}{\psi_{p\alpha}}f(x,y)=0,\, f(x,y)\in L^\infty_{loc}(\Omega\setminus\{0\})\right\}.$$

THEOREM 2.4. *Suppose $p\neq Q$ and $f(x,y)\in\mathscr{F}_p$. Then there exists a positive constant $C_{f,Q,p}$ such that*

$$C_{f,Q,p}\int_\Omega f|u|^p dxdy \leqslant \int_\Omega |\nabla_L u|^p dxdy, \tag{2.13}$$

for any $u\in D_0^{1,p}(\Omega\setminus\{0\})$. Moreover, the embedding $D_0^{1,p}(\Omega)\hookrightarrow L^p(\Omega, fdxdy)$ is compact.

Proof. Since $f(x,y)\in\mathscr{F}_p$, for any $\varepsilon>0$, there exist $\delta>0$ and C_δ such that

$$\sup_{B_L(\delta)\subseteq\Omega}\frac{d^p}{\psi_{p\alpha}}f(x,y)\leqslant\varepsilon \quad \text{and} \quad f|_{\Omega\setminus B_L(\delta)}\leqslant C_\delta .$$

Combining (1.4) and (2.12) yields

$$\begin{aligned}\int_\Omega f|u|^p dxdy &= \int_{B_L(\delta)} f|u|^p dxdy + \int_{\Omega\setminus B_L(\delta)} f|u|^p dxdy\\ &\leqslant \varepsilon\int_{B_L(\delta)}\psi_{p\alpha}\frac{|u|^p}{d^p}dxdy + C\int_{\Omega\setminus B_L(\delta)}|u|^p dxdy\\ &\leqslant C_{f,Q,p}^{-1}\int_\Omega|\nabla_L u|^p dxdy\end{aligned}$$

for some suitable constant $C>0$. It follows (2.13).

Next, we discuss the compactness. Let $\{u_m\}\subset D_0^{1,p}(\Omega)$ be a bounded sequence. By reflexivity of the space $D_0^{1,p}(\Omega)$ and the Sobolev embedding theorem for vector fields (see [11]) it gets

$$\begin{cases} u_{m_j}\rightharpoonup u & \text{weakly in } \ D_0^{1,p}(\Omega),\\ u_{m_j}\to u & \text{strongly in } \ L^p(\Omega)\end{cases} \tag{2.14}$$

for a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ as $j\to\infty$. Write $C_\delta=\|f\|_{L^\infty(\Omega\setminus B_L(\delta))}$. By (1.4),

$$\begin{aligned}\int_\Omega f|u_{m_j}-u|^p dxdy &= \int_{B_L(\delta)} f|u_{m_j}-u|^p dxdy+\int_{\Omega\setminus B_L(\delta)} f|u_{m_j}-u|^p dxdy\\ &\leqslant \varepsilon\int_{B_L(\delta)}\psi_{p\alpha}\frac{|u_{m_j}-u|^p}{d^p}dxdy+C_\delta\int_{\Omega\setminus B_L(\delta)}|u_{m_j}-u|^p dxdy\\ &\leqslant \varepsilon C_{Q,p}^{-1}\int_\Omega|\nabla_L(u_{m_j}-u)|^p dxdy+C_\delta\int_\Omega|u_{m_j}-u|^p dxdy.\end{aligned}$$

Since $\{u_m\}\subseteq D_0^{1,p}(\Omega)$ is bounded, we deduce

$$\int_\Omega f|u_{m_j}-u|^p dxdy\leqslant \varepsilon M+C_\delta\int_\Omega|u_{m_j}-u|^p dxdy,$$

where $M>0$ is a constant depending on Q and p. By (2.14),

$$\lim_{j\to\infty}\int_\Omega f|u_{m_j}-u|^p dxdy\leqslant\varepsilon M.$$

As ε is arbitrary, $\lim_{j\to\infty}\int_\Omega f|u_{m_j}-u|^p dxdy=0$. So the embedding $D_0^{1,p}(\Omega)\hookrightarrow L^p(\Omega, fdxdy)$ is compact.

REMARK 2.5. The class of the functions $f(\xi)\in\mathscr{F}_p$ has lower order singularity than d^{-p} at the origin. The examples of such functions include any bounded function or in a small neighbourhood of 0, $f(\xi)=\frac{\psi_{p\alpha}(\xi)}{d^\beta(\xi)}$ with $0<\beta<p$.

3. Proof of best constants

In this section, we discuss that the constants appearing in Theorem 2.1 are the best. To do this, we introduce the function $v_\varepsilon(d)\in D_0^{1,p}(B_L(1))$ satisfying

$$v_\varepsilon(d)=\begin{cases}0, & \text{for } d\leqslant\varepsilon^2,\\ \dfrac{\ln\frac{d}{\varepsilon^2}}{d^A\ln\frac{1}{\varepsilon}}, & \text{for } \varepsilon^2\leqslant d\leqslant\varepsilon,\\ \dfrac{\ln\frac{1}{d}}{d^A\ln\frac{1}{\varepsilon}}, & \text{for } \varepsilon\leqslant d\leqslant 1,\end{cases}$$

with sufficiently small $\varepsilon>0$ and $A=\frac{Q-p}{p}$. In evidence, v_ε is continuous and differentiable a.e. and its derivative is given by

$$v_\varepsilon'(d)=\begin{cases}0, & \text{for } d\leqslant\varepsilon^2,\\ \dfrac{1}{d^{\frac{Q}{p}}\ln\frac{1}{\varepsilon}}\left(1-A\ln\frac{d}{\varepsilon^2}\right), & \text{for } \varepsilon^2\leqslant d\leqslant\varepsilon,\\ -\dfrac{1}{d^{\frac{Q}{p}}\ln\frac{1}{\varepsilon}}\left(1+A\ln\frac{1}{d}\right), & \text{for } \varepsilon\leqslant d\leqslant 1.\end{cases}$$

LEMMA 3.1. *Suppose that there exists a constant $R > 0$ such that $\sup_{(x,y)\in B_L(1)} d(x,y) \leqslant R$ and $\varepsilon > 0$ small enough. Then*

$$(a)\quad \int_{B_L(1)} |\nabla_L v_\varepsilon(d)|^p dxdy = \frac{|A|^p s_{n,m}}{p+1} \ln\frac{1}{\varepsilon}\left[2+p(p+1)\left(\frac{1}{|A|\ln\frac{1}{\varepsilon}}\right)^2 + O\left(\frac{1}{\ln\frac{1}{\varepsilon}}\right)\right].$$

$$(b)\quad \int_{B_L(1)} \psi_{p\alpha}\frac{|v_\varepsilon(d)|^p}{d^p} dxdy = \frac{2s_{n,m}}{p+1}\ln\frac{1}{\varepsilon}.$$

$$(c)\quad \int_\Omega \psi_{p\alpha}\frac{|v_\varepsilon(d)|^p}{d^p}\mathscr{B}^\iota\left(\frac{d}{R}\right) dxdy \geqslant \frac{s_{n,m}}{p+1}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{1-\iota}, \quad \iota > 0.$$

Proof. (a) By (1.2),

$$\begin{aligned}
&\int_{B_L(1)} |\nabla_L v_\varepsilon(d)|^p dxdy = \int_{B_L(1)} |v'_\varepsilon(d)\nabla_L d|^p dxdy = s_{n,m}\int_0^1 |v'_\varepsilon(\rho)|^p \rho^{Q-1} d\rho \\
&= \frac{s_{n,m}}{(\ln\frac{1}{\varepsilon})^p}\left[\int_{\varepsilon^2}^{\varepsilon}\left|A\ln\frac{\rho}{\varepsilon^2}-1\right|^p \frac{1}{\rho}d\rho + \int_\varepsilon^1 \left|1+A\ln\frac{1}{\rho}\right|^p\frac{1}{\rho}d\rho\right] \\
&= \begin{cases} \dfrac{s_{n,m}}{(\ln\frac{1}{\varepsilon})^p}\left[\displaystyle\int_{\varepsilon^2}^{\varepsilon}\left|A\ln\frac{\rho}{\varepsilon^2}-1\right|^p d\left(\ln\frac{\rho}{\varepsilon^2}\right) + \int_\varepsilon^1\left|1+A\ln\frac{1}{\rho}\right|^p(-1)d\left(\ln\frac{1}{\rho}\right)\right], \; p<Q \\ \dfrac{s_{n,m}}{(\ln\frac{1}{\varepsilon})^p}\left[\displaystyle\int_{\varepsilon^2}^{\varepsilon}\left||A|\ln\frac{\rho}{\varepsilon^2}+1\right|^p d\left(\ln\frac{\rho}{\varepsilon^2}\right) + \int_\varepsilon^1\left||A|\ln\frac{1}{\rho}-1\right|^p(-1)d\left(\ln\frac{1}{\rho}\right)\right], \; p>Q \end{cases} \\
&= \begin{cases} \dfrac{s_{n,m}}{A(\ln\frac{1}{\varepsilon})^p}\left[\displaystyle\int_0^1 s^p ds + \int_0^{A\ln\frac{1}{\varepsilon}-1} s^p ds + \int_1^{A\ln\frac{1}{\varepsilon}+1}\tau^p d\tau\right], \; p<Q \\ \dfrac{s_{n,m}}{|A|(\ln\frac{1}{\varepsilon})^p}\left[\displaystyle\int_1^{|A|\ln\frac{1}{\varepsilon}+1} s^p ds + \int_0^{|A|\ln\frac{1}{\varepsilon}-1}\tau^p d\tau + \int_0^1 \tau^p d\tau\right], \; p>Q \end{cases} \\
&= \frac{s_{n,m}}{(p+1)|A|(\ln\frac{1}{\varepsilon})^p}\left[\left(|A|\ln\frac{1}{\varepsilon}-1\right)^{p+1} + \left(|A|\ln\frac{1}{\varepsilon}+1\right)^{p+1}\right] \\
&= \frac{s_{n,m}}{(p+1)|A|(\ln\frac{1}{\varepsilon})^p}\left(|A|\ln\frac{1}{\varepsilon}\right)^{p+1}\left[\left(1-\frac{1}{|A|\ln\frac{1}{\varepsilon}}\right)^{p+1} + \left(1+\frac{1}{|A|\ln\frac{1}{\varepsilon}}\right)^{p+1}\right]
\end{aligned} \tag{3.1}$$

for ε small enough.

Since ε is sufficiently small, using Taylor's series shows

$$\left(1+\frac{1}{|A|\ln\frac{1}{\varepsilon}}\right)^{p+1} = 1+(p+1)\frac{1}{|A|\ln\frac{1}{\varepsilon}} + \frac{p(p+1)}{2}\left(\frac{1}{|A|\ln\frac{1}{\varepsilon}}\right)^2 + O\left(\frac{1}{\ln\frac{1}{\varepsilon}}\right), \tag{3.2}$$

and

$$\left(1-\frac{1}{|A|\ln\frac{1}{\varepsilon}}\right)^{p+1} = 1-(p+1)\frac{1}{|A|\ln\frac{1}{\varepsilon}} + \frac{p(p+1)}{2}\left(\frac{1}{|A|\ln\frac{1}{\varepsilon}}\right)^2 + O\left(\frac{1}{\ln\frac{1}{\varepsilon}}\right). \tag{3.3}$$

By (3.1), (3.2) and (3.3) we obtain

$$\int_{B_L(1)} |\nabla_L v_\varepsilon(d)|^p dxdy = \frac{|A|^p s_{n,m}}{p+1} \ln\frac{1}{\varepsilon}\left[2+p(p+1)\left(\frac{1}{|A|\ln\frac{1}{\varepsilon}}\right)^2 + O\left(\frac{1}{\ln\frac{1}{\varepsilon}}\right)\right].$$

Hence, it concludes (a).

(b) We have the following estimates after the change of coordinates (1.2)

$$\begin{aligned}
&\int_{B_L(1)} \psi_{p\alpha}\frac{|v_\varepsilon(d)|^p}{d^p} dxdy \\
&\qquad = s_{n,m}\int_0^1 \frac{|v_\varepsilon(\rho)|^p}{\rho^p}\rho^{Q-1}d\rho \\
&\qquad = \frac{s_{n,m}}{(\ln\frac{1}{\varepsilon})^p}\left[\int_{\varepsilon^2}^{\varepsilon}\left(\frac{\ln\frac{\rho}{\varepsilon^2}}{\rho^A}\right)^p \rho^{Q-p-1}d\rho + \int_\varepsilon^1 \left(\frac{\ln\frac{1}{\rho}}{\rho^A}\right)^p \rho^{Q-p-1}d\rho\right] \\
&\qquad = \frac{s_{n,m}}{(\ln\frac{1}{\varepsilon})^p}\left[\int_{\varepsilon^2}^{\varepsilon}\left(\ln\frac{\rho}{\varepsilon^2}\right)^p \frac{d\rho}{\rho} + \int_\varepsilon^1 \left(\ln\frac{1}{\rho}\right)^p \frac{d\rho}{\rho}\right] \\
&\qquad = \frac{s_{n,m}}{(p+1)(\ln\frac{1}{\varepsilon})^p}\left[\int_{\varepsilon^2}^{\varepsilon}\frac{d}{d\rho}\left(\ln\frac{\rho}{\varepsilon^2}\right)^{p+1} - \int_\varepsilon^1 \frac{d}{d\rho}\left(\ln\frac{1}{\rho}\right)^{p+1}\right] \\
&\qquad = \frac{2s_{n,m}}{p+1}\ln\frac{1}{\varepsilon},
\end{aligned}$$

which is (b).

(c) For $\iota > 0$, arguing as (b) we get

$$\begin{aligned}
&\int_\Omega \psi_{p\alpha}\frac{|v_\varepsilon(d)|^p}{d^p}\mathscr{B}^\iota\left(\frac{d}{R}\right)dxdy = s_{n,m}\int_0^1 \frac{|v_\varepsilon(\rho)|^p}{\rho^p}\left(\ln\left(\frac{R}{\rho}\right)\right)^{-\iota}\rho^{Q-1}d\rho \\
&\quad = \frac{s_{n,m}}{\left(\ln\frac{1}{\varepsilon}\right)^p}\left[\int_{\varepsilon^2}^{\varepsilon}\frac{\left(\ln\frac{\rho}{\varepsilon^2}\right)^p}{\left(\ln\left(\frac{R}{\rho}\right)\right)^\iota\rho}d\rho + \int_\varepsilon^1 \frac{\left(\ln\frac{1}{\rho}\right)^p}{\left(\ln\left(\frac{R}{\rho}\right)\right)^\iota\rho}d\rho\right] \\
&\quad \geqslant \frac{s_{n,m}}{\left(\ln\frac{1}{\varepsilon}\right)^p}\left[\left(\ln\left(\frac{R}{\varepsilon^2}\right)\right)^{-\iota}\int_{\varepsilon^2}^{\varepsilon}\frac{\left(\ln\frac{\rho}{\varepsilon^2}\right)^p}{\rho}d\rho + \left(\ln\left(\frac{R}{\varepsilon}\right)\right)^{-\iota}\int_\varepsilon^1 \frac{\left(\ln\frac{1}{\rho}\right)^p}{\rho}d\rho\right] \\
&\quad = \frac{s_{n,m}\ln\frac{1}{\varepsilon}}{p+1}\left[\left(\ln\left(\frac{R}{\varepsilon^2}\right)\right)^{-\iota} + \left(\ln\left(\frac{R}{\varepsilon}\right)\right)^{-\iota}\right] \\
&\quad \geqslant \frac{s_{n,m}\ln\frac{1}{\varepsilon}}{p+1}\left[\left(2\ln\left(\frac{1}{\varepsilon}\right)\right)^{-\iota} + \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-\iota}\right] \\
&\quad \geqslant \frac{s_{n,m}}{p+1}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{1-\iota}.
\end{aligned}$$

THEOREM 3.2. *Let $0 \in \Omega$ be a bounded domain in $\mathbb{R}^{n+m}$ and $R > 0$ such that $\sup_{(x,y)\in\Omega} d(x,y) < R$. Suppose $p \neq Q$, and for some constants $B > 0, D \geqslant 0$, $\iota > 0$, the following inequality holds for any $u \in C_0^\infty(\Omega \setminus \{0\})$*

$$\int_\Omega |\nabla_L u|^p dxdy \geqslant B \int_\Omega \psi_{p\alpha} \frac{|u|^p}{d^p} dxdy + D \int_\Omega \psi_{p\alpha} \frac{|u|^p}{d^p} \mathscr{B}^\iota \left(\frac{d}{R}\right) dxdy. \tag{3.4}$$

Then

(*i*) $B \leqslant |A|^p$;

(*ii*) $\iota \geqslant 2$ *if* $B = |A|^p$ *and* $D > 0$.

(*iii*) $D \leqslant \frac{p-1}{2p}|A|^{p-2}$ *if* $B = |A|^p$ *and* $\iota = 2$.

Proof. Since (3.4) holds for every $u \in D_0^{1,p}(\Omega \setminus \{0\})$, we just prove the theorem on the unit ball $B_L(1)$ for $u = v_\varepsilon(d)$.

(i) By (a) and (b) in Lemma 3.1 we derive

$$B \leqslant \frac{\int_{B_L(1)} |\nabla_L v_\varepsilon(d)|^p dxdy}{\int_{B_L(1)} \psi_{p\alpha} \frac{|v_\varepsilon(d)|^p}{d^p} dxdy} \leqslant \frac{\frac{|A|^p s_{n,m}}{p+1} \ln\frac{1}{\varepsilon} \left[2 + p(p+1)\left(\frac{1}{|A|\ln\frac{1}{\varepsilon}}\right)^2 + O\left(\frac{1}{\ln\frac{1}{\varepsilon}}\right)\right]}{\frac{2s_{n,m}}{p+1}\ln\frac{1}{\varepsilon}}.$$

The result is obtained by taking $\varepsilon \to 0$.

(ii) Let $B = |A|^p, D > 0$. Assume by contradiction that $0 < \iota < 2$. Invoking (a), (b) and (c) of Lemma 3.1 yields

$$\begin{aligned}
0 < D &\leqslant \frac{\int_{B_L(1)} |\nabla_L v_\varepsilon(d)|^p dxdy - |A|^p \int_{B_L(1)} \psi_{p\alpha} \frac{|v_\varepsilon(d)|^p}{d^p} dxdy}{\int_\Omega \psi_{p\alpha} \frac{|v_\varepsilon(d)|^p}{d^p} \mathscr{B}^\iota \left(\frac{d}{R}\right) dxdy} \\
&\leqslant \frac{\frac{|A|^p s_{n,m}}{p+1} \ln\frac{1}{\varepsilon} \left[p(p+1)\left(\frac{1}{|A|\ln\frac{1}{\varepsilon}}\right)^2 + O\left(\frac{1}{\ln\frac{1}{\varepsilon}}\right)\right]}{\frac{s_{n,m}}{p+1}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{1-\iota}} \\
&\leqslant C\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\iota-2} \to 0, \quad \varepsilon \to 0,
\end{aligned}$$

which is a contradiction. Hence $\iota \geqslant 2$.

(iii) To show that $D \leqslant \frac{p-1}{2p}|A|^{p-2}$ for $B = |A|^p$ and $\iota = 2$, we redefine the cut-function on $B_L(1)$, that is, let $v_{\varepsilon,\kappa}(d) \in D_0^{1,p}(B_L(1))$ satisfy

$$v_{\varepsilon,\kappa}(d) = \begin{cases} 0, & \text{if } d \leqslant \varepsilon^2, \\ \dfrac{\left(\ln\frac{d}{\varepsilon^2}\right)^{-\kappa}}{d^A \ln\frac{1}{\varepsilon}}, & \text{if } \varepsilon^2 \leqslant d \leqslant \varepsilon, \\ \dfrac{\left(\ln\frac{1}{d}\right)^{-\kappa}}{d^A \ln\frac{1}{\varepsilon}}, & \text{if } \varepsilon \leqslant d \leqslant 1, \end{cases}$$

where $A=\frac{Q-p}{p}$, $\frac{1}{p}<\kappa<\frac{2}{p}$, $\varepsilon>0$ small. Clearly, $v_{\varepsilon,\kappa}$ is continuous and differentiable almost everywhere and its derivative is

$$v'_{\varepsilon,\kappa}(d)=\begin{cases}0, & \text{if } d\leqslant\varepsilon^2,\\ -\dfrac{\left(\ln\frac{d}{\varepsilon^2}\right)^{-\kappa}}{d^{\frac{Q}{p}}\ln\frac{1}{\varepsilon}}\left(A+\dfrac{\kappa}{\ln\frac{d}{\varepsilon^2}}\right), & \text{if } \varepsilon^2\leqslant d\leqslant\varepsilon,\\ -\dfrac{\left(\ln\frac{1}{d}\right)^{-\kappa}}{d^{\frac{Q}{p}}\ln\frac{1}{\varepsilon}}\left(A-\dfrac{\kappa}{\ln\frac{1}{d}}\right), & \text{if } \varepsilon\leqslant d\leqslant 1.\end{cases}$$

Similar to the proof of Lemma 3.1, we have

$$\begin{aligned} I_1 &:= \int_{B_L(1)}|\nabla_L v_{\varepsilon,\kappa}(d)|^p dxdy-|A|^p\int_{B_L(1)}\psi_{p\alpha}\frac{|v_{\varepsilon,\kappa}(d)|^p}{d^p}dxdy\\ &=\begin{cases}\dfrac{s_{n,m}}{\left(\ln\frac{1}{\varepsilon}\right)^p}\left[\displaystyle\int_{\varepsilon^2}^{\varepsilon}\left(\ln\frac{\rho}{\varepsilon^2}\right)^{-p\kappa}\left(|A+\frac{\kappa}{\ln\frac{\rho}{\varepsilon^2}}|^p-|A|^p\right)\rho^{-1}d\rho\right.\\ \qquad\left.+\displaystyle\int_{\varepsilon}^{1}\left(\ln\frac{1}{\rho}\right)^{-p\kappa}\left(|A-\frac{\kappa}{\ln\frac{1}{\rho}}|^p-|A|^p\right)\rho^{-1}d\rho\right], & p<Q,\\ \dfrac{s_{n,m}}{\left(\ln\frac{1}{\varepsilon}\right)^p}\left[\displaystyle\int_{\varepsilon^2}^{\varepsilon}\left(\ln\frac{\rho}{\varepsilon^2}\right)^{-p\kappa}\left(||A|-\frac{\kappa}{\ln\frac{\rho}{\varepsilon^2}}|^p-|A|^p\right)\rho^{-1}d\rho\right.\\ \qquad\left.+\displaystyle\int_{\varepsilon}^{1}\left(\ln\frac{1}{\rho}\right)^{-p\kappa}\left(||A|+\frac{\kappa}{\ln\frac{1}{\rho}}|^p-|A|^p\right)\rho^{-1}d\rho\right], & p>Q.\end{cases}\end{aligned}$$

Since $\varepsilon>0$ is sufficiently small, using Taylor's series concludes that

$$\begin{aligned}\left|A+\frac{\kappa}{\ln\frac{\rho}{\varepsilon^2}}\right|^p-|A|^p &\leqslant p|A|^{p-1}\kappa\left(\ln\frac{\rho}{\varepsilon^2}\right)^{-1}+\frac{p(p-1)\kappa^2}{2}|A|^{p-2}\left(\ln\frac{\rho}{\varepsilon^2}\right)^{-2}\\ &\quad+C\left(\ln\frac{\rho}{\varepsilon^2}\right)^{-3},\ p<Q,\\ \left||A|-\frac{\kappa}{\ln\frac{\rho}{\varepsilon^2}}\right|^p-|A|^p &\leqslant -p|A|^{p-1}\kappa\left(\ln\frac{\rho}{\varepsilon^2}\right)^{-1}+\frac{p(p-1)\kappa^2}{2}|A|^{p-2}\left(\ln\frac{\rho}{\varepsilon^2}\right)^{-2}\\ &\quad+C\left(\ln\frac{\rho}{\varepsilon^2}\right)^{-3},\ p>Q\end{aligned}$$

for $\rho\in(\varepsilon^2,\varepsilon)$, where C is some suitable positive constant, and for $\rho\in(\varepsilon,1)$,

$$\begin{aligned}\left|A-\frac{\kappa}{\ln\frac{1}{\rho}}\right|^p-|A|^p &\leqslant -p|A|^{p-1}\kappa\left(\ln\frac{1}{\rho}\right)^{-1}+\frac{p(p-1)\kappa^2}{2}|A|^{p-2}\left(\ln\frac{1}{\rho}\right)^{-2}\\ &\quad+C\left(\ln\frac{1}{\rho}\right)^{-3},\ p<Q,\end{aligned}$$

$$\left||A|+\frac{\kappa}{\ln\frac{1}{\rho}}\right|^p-|A|^p\leqslant p|A|^{p-1}\kappa\left(\ln\frac{1}{\rho}\right)^{-1}+\frac{p(p-1)\kappa^2}{2}|A|^{p-2}\left(\ln\frac{1}{\rho}\right)^{-2}$$
$$+C\left(\ln\frac{1}{\rho}\right)^{-3},\ p>Q.$$

Then, for $p<Q$, we obtain

$$\begin{aligned}I_1\leqslant{}&\frac{s_{n,m}}{\left(\ln\frac{1}{\varepsilon}\right)^p}\left\{\int_{\varepsilon^2}^{\varepsilon}\left[p|A|^{p-1}\kappa\left(\ln\frac{\rho}{\varepsilon^2}\right)^{-p\kappa-1}+\frac{p(p-1)\kappa^2}{2}|A|^{p-2}\left(\ln\frac{\rho}{\varepsilon^2}\right)^{-p\kappa-2}\right.\right.\\&\left.+C\left(\ln\frac{\rho}{\varepsilon^2}\right)^{-p\kappa-3}\right]\rho^{-1}d\rho\\&+\int_{\varepsilon}^{1}\left[-p|A|^{p-1}\kappa\left(\ln\frac{1}{\rho}\right)^{-p\kappa-1}+\frac{p(p-1)\kappa^2}{2}|A|^{p-2}\left(\ln\frac{1}{\rho}\right)^{-p\kappa-2}\right.\\&\left.\left.+C\left(\ln\frac{1}{\rho}\right)^{-p\kappa-3}\right]\rho^{-1}d\rho\right\}\\={}&\frac{p(p-1)\kappa^2}{-p\kappa-1}|A|^{p-2}\left(\ln\frac{1}{\varepsilon}\right)^{-p\kappa-1}-C\left(\ln\frac{1}{\varepsilon}\right)^{-p\kappa-2}.\end{aligned}$$

Similarly, for $p>Q$,

$$I_1\leqslant\frac{s_{n,m}}{\left(\ln\frac{1}{\varepsilon}\right)^p}\left[\frac{p(p-1)\kappa^2}{-p\kappa-1}|A|^{p-2}\left(\ln\frac{1}{\varepsilon}\right)^{-p\kappa-1}-C\left(\ln\frac{1}{\varepsilon}\right)^{-p\kappa-2}\right].$$

All in all, we infer that

$$I_1\leqslant-\frac{s_{n,m}}{\left(\ln\frac{1}{\varepsilon}\right)^p}\left[\frac{p(p-1)\kappa^2}{p\kappa+1}|A|^{p-2}\left(\ln\frac{1}{\varepsilon}\right)^{-p\kappa-1}+C\left(\ln\frac{1}{\varepsilon}\right)^{-p\kappa-2}\right].$$

Next, it follows

$$\begin{aligned}I_2:={}&\int_\Omega\psi_{p\alpha}\frac{|v_{\varepsilon,\kappa}(d)|^p}{d^p}\mathscr{B}^2\left(\frac{d}{R}\right)dxdy\\={}&\frac{s_{n,m}}{\left(\ln\frac{1}{\varepsilon}\right)^p}\left[\int_{\varepsilon^2}^{\varepsilon}\frac{\left(\ln\frac{\rho}{\varepsilon^2}\right)^{-p\kappa}}{\left(\ln\left(\frac{R}{\rho}\right)\right)^2\rho}d\rho+\int_{\varepsilon}^{1}\frac{\left(\ln\frac{1}{\rho}\right)^{-p\kappa}}{\left(\ln\left(\frac{R}{\rho}\right)\right)^2\rho}d\rho\right]\\\geqslant{}&\frac{s_{n,m}}{\left(\ln\frac{1}{\varepsilon}\right)^p}\left[\left(\ln\left(\frac{R}{\varepsilon^2}\right)\right)^{-2}\int_{\varepsilon^2}^{\varepsilon}\frac{\left(\ln\frac{\rho}{\varepsilon^2}\right)^p}{\rho}d\rho+\left(\ln\left(\frac{R}{\varepsilon}\right)\right)^{-2}\int_{\varepsilon}^{1}\frac{\left(\ln\frac{1}{\rho}\right)^p}{\rho}d\rho\right]\end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{s_{n,m}\left(\ln\frac{1}{\varepsilon}\right)^{-p\kappa+1}}{\left(\ln\frac{1}{\varepsilon}\right)^{p}(-p\kappa+1)}\left[\left(2\ln\left(\frac{1}{\varepsilon}\right)\right)^{-2}+\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-2}\right] \\
&\geqslant \frac{s_{n,m}}{\left(\ln\frac{1}{\varepsilon}\right)^{p}(-p\kappa+1)}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-p\kappa-1}.
\end{aligned}
$$

Hence, we deduce that

$$
\begin{aligned}
D \leqslant \frac{I_1}{I_2} &\leqslant \frac{-\frac{s_{n,m}}{\left(\ln\frac{1}{\varepsilon}\right)^{p}}\left[\frac{p(p-1)\kappa^2}{p\kappa+1}|A|^{p-2}\left(\ln\frac{1}{\varepsilon}\right)^{-p\kappa-1}+C\left(\ln\frac{1}{\varepsilon}\right)^{-p\kappa-2}\right]}{\frac{s_{n,m}}{\left(\ln\frac{1}{\varepsilon}\right)^{p}(-p\kappa+1)}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-p\kappa-1}} \\
&= \frac{p(p-1)\kappa^2(p\kappa-1)}{p\kappa+1}|A|^{p-2}+C\left(\ln\frac{1}{\varepsilon}\right)^{-1} \\
&\to \frac{p-1}{2p}|A|^{p-2}, \\
&\qquad \text{as } \varepsilon\to 0, \kappa\to\frac{1}{p}\left(\frac{1}{3}+\frac{1}{6}(80-30\sqrt{6})^{\frac{1}{3}}+\frac{1}{2^{\frac{10}{3}}}(40+15\sqrt{6})^{\frac{1}{3}}\right),
\end{aligned}
$$

where $\frac{1}{3}+\frac{1}{6}(80-30\sqrt{6})^{\frac{1}{3}}+\frac{1}{2^{\frac{10}{3}}}(40+15\sqrt{6})^{\frac{1}{3}}\approx 1.53697$. Here we achieve the numerical value by using Mathematica 4. This completes the proof of the theorem.

THEOREM 3.3. *Let $0\in\Omega$ be a domain in $\mathbb{R}^{n+m}$ and $R>0$ such that* $\sup_{(x,y)\in\Omega} d(x,y)<R$. *Suppose $p=Q$, and for some constants $D\geqslant 0$, $\iota>0$, the following inequality holds for all $u\in C_0^\infty(\Omega\setminus\{0\})$*

$$
\int_\Omega |\nabla_L u|^p dxdy \geqslant D\int_\Omega \psi_{p\alpha}\frac{|u|^p}{d^p}\mathscr{B}^\iota\left(\frac{d}{R}\right)dxdy. \tag{3.5}
$$

Then, $\iota\geqslant p$ for $D>0$.

Proof. The proof uses the argument similar to Theorem 3.2. Taking the test functions $v_\varepsilon(d)\in D_0^{1,p}(B_L(1))$ in the proof of (i) in Theorem 3.2 with $p=Q$, it is easy to verify that

$$
\begin{aligned}
\int_{B_L(1)}|\nabla_L v_\varepsilon(d)|^p dxdy = s_{n,m}\int_0^1 |v_\varepsilon'(\rho)|^p\rho^{Q-1}d\rho &= s_{n,m}\int_{\varepsilon^2}^1\left(\frac{1}{\rho\ln\frac{1}{\varepsilon}}\right)^p\rho^{p-1}d\rho \\
&= \frac{s_{n,m}}{\left(\ln\frac{1}{\varepsilon}\right)^p}\ln\frac{1}{\varepsilon^2} = 2s_{n,m}\left(\ln\frac{1}{\varepsilon}\right)^{1-p},
\end{aligned} \tag{3.6}
$$

and

$$\int_\Omega \psi_{p\alpha} \frac{|v_\varepsilon(d)|^p}{d^p} \mathscr{B}^\iota \left(\frac{d}{R}\right) dxdy = \frac{s_{n,m}}{\left(\ln \frac{1}{\varepsilon}\right)^p} \left[\int_{\varepsilon^2}^{\varepsilon} \frac{\left(\ln \frac{\rho}{\varepsilon^2}\right)^p}{\left(\ln\left(\frac{R}{\rho}\right)\right)^\iota \rho} d\rho + \int_\varepsilon^1 \frac{\left(\ln \frac{1}{\rho}\right)^p}{\left(\ln\left(\frac{R}{\rho}\right)\right)^\iota \rho} d\rho \right]$$
$$\geqslant \frac{s_{n,m}}{p+1} \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{1-\iota}.$$

Assuming that $0 < \iota < p$ we have

$$0 < D \leqslant \frac{\int_{B_L(1)} |\nabla_L v_\varepsilon(d)|^p dxdy}{\int_\Omega \psi_{p\alpha} \frac{|v_\varepsilon(d)|^p}{d^p} \mathscr{B}^\iota \left(\frac{d}{R}\right) dxdy}$$
$$\leqslant \frac{2s_{n,m} \left(\ln \frac{1}{\varepsilon}\right)^{1-p}}{\frac{s_{n,m}}{p+1} \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{1-\iota}} \leqslant C \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\iota - p} \to 0, \quad \varepsilon \to 0.$$

It is a contradiction. Hence $\iota \geqslant p$.

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Jingbo Dou
School of Statistics
Xi'an Institute of Finance and Economics
Xi'an, Shaanxi, 710100
China
e-mail: djbmn@126.com

Qianqiao Guo
Department of Applied Mathematics
Northwestern Polytechnical University
Xi'an, Shaanxi,710072
China

Pengcheng Niu
Department of Applied Mathematics
Northwestern Polytechnical University
Xi'an, Shaanxi,710072
China