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# FUNCTIONAL CHARACTERIZATION OF A SHARPENING OF THE TRIANGLE INEQUALITY

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*Abstract.* Motivated by recent refinements of the classical triangle inequality in normed spaces proved by Lech Maligranda we deal with related functional equation and inequality and with the corresponding stability problem.

## 1. Introduction

Alfred Tarski in [11] posed as a question (or an exercise) and then proved the following identity for real numbers $x, y$:

$$\big|\,|x| - |y|\,\big| = |x+y| + |x-y| - |x| - |y|. \tag{1}$$

One may check that this equality fails to hold in a higher dimension. Recently, Lech Maligranda in [9] presented the following multi-dimensional analogue of identity (1):

$$\big|\,\|x\| - \|y\|\,\big| \leqslant \|x+y\| + \|x-y\| - \|x\| - \|y\| \leqslant \min\{\|x+y\|, \|x-y\|\} \tag{2}$$

for any elements $x, y$ from an arbitrary normed space. Clearly, (2) strengthen both inequalities

$$\big|\|x\| - \|y\|\big| \leqslant \|x-y\|$$

and

$$\|x+y\| \leqslant \|x\| + \|y\|.$$

Some interesting related estimations were obtained by Lech Maligranda in [8].

Motivated by foregoing results we discuss the following problems:

**A)** the functional inequality:

$$|f(x)-f(y)| \leqslant f(x+y)+f(x-y)-f(x)-f(y) \leqslant \min\{f(x+y), f(x-y)\}; \tag{3}$$

**B)** the functional equation:

$$|f(x)-f(y)| = f(x+y)+f(x-y)-f(x)-f(y); \tag{4}$$

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**C)** stability of functional equation (4):

$$\big| |F(x) - F(y)| - F(x+y) - F(x-y) + F(x) + F(y) \big| \leqslant \varepsilon \left( \|x\|^p + \|y\|^p \right). \quad (5)$$

It is clear that the mapping $\mathbb{R} \ni x \mapsto |x| \in \mathbb{R}$ satisfies (3), (4) and (5) and for every normed space $(X, \|\cdot\|)$ the mapping $X \ni x \mapsto \|x\| \in \mathbb{R}$ satisfies (3). Our purpose is to determine all solutions of these problems under some reasonable assumptions.

## 2. Results

We begin with solving (3) for mappings acting on Abelian groups with the aid of the following theorem from paper [6] of Roman Ger, which is a generalization of a result obtained earlier by Edgar Berz [3].

THEOREM A. *Assume that $(G,+)$ is an Abelian group and $p\colon G \to \mathbb{R}$ has the following properties:*
*$(p1)$ $p(x+y) \leqslant p(x) + p(y)$ for each $x, y \in G$;*
*$(p2)$ $p(2x) = 2p(x)$ for each $x \in G$;*
*$(p3)$ $p$ is even.*
*Then there exist a normed space $X$ and an additive mapping $A\colon G \to X$ such that*

$$p(x) = \|A(x)\|, \quad x \in G.$$

*Moreover, the space $X$ may be taken as the space $B(T, \mathbb{R})$ of all bounded functions on $T$ with the supremum norm, where $T$ is the set of all additive real functionals on $G$ majorized by $p$, the mapping $A$ is defined by*

$$A(x)(a) := a(x), \quad a \in T, x \in G$$

*and for each $x \in G$ the supremum*

$$\|A(x)\| = \sup\{|a(x)| : a \text{ is additive and } a \leqslant p\}$$

*is realized by some additive mapping $a_x$, i.e. $\|A(x)\| = |a_x(x)|$.*

THEOREM 1. *Assume that $(G,+)$ is an Abelian group and $f\colon G \to \mathbb{R}$ is an arbitrary mapping such that $f(0) = 0$. Then $f$ is a solution of (3) if and only if there exist a normed space $(X, \|\cdot\|)$ and an additive mapping $A\colon G \to X$ such that*

$$f(x) = \|A(x)\|, \quad x \in G.$$

*Proof.* The "only if" part follows easily from inequality (2). We will prove the "if" part.

Substitute $y = x$ in (3) to obtain

$$0 \leqslant f(2x) - 2f(x) \leqslant \min\{f(2x), f(0)\} \leqslant 0, \quad x \in G.$$

Thus $f(2x) = 2f(x)$ for each $x \in G$.

In the remaining part of the proof we will be using the first inequality from (3) only. Put $x = 0$ in (3). We get $|f(y)| \leqslant f(-y)$ for each $y \in G$, which applied twice leads to

$$|f(y)| \leqslant f(-y) \leqslant |f(-y)| \leqslant f(y) \leqslant |f(y)|, \quad x \in G.$$

Since we have obtained equality, then each estimation holds with equality. In particular, $f$ is even.

Clearly, (3) implies that

$$f(x) - f(y) \leqslant f(x+y) + f(x-y) - f(x) - f(y), \quad x, y \in G,$$

i.e.

$$2f(x) \leqslant f(x+y) + f(x-y), \quad x, y \in G.$$

From this, after replacing $x$ by $x+y$ and $y$ by $x-y$, we deduce the inequality

$$2f(x+y) \leqslant f(2x) + f(2y), \quad x, y \in G,$$

which, jointly with the already proved 2-homogeneity of $f$, means that $f$ is subadditive.

We have shown that $f$ satisfies all the assumptions of Theorem A. Therefore $f$ is of the form $f(x) = \|A(x)\|$, where $A\colon G \to X$ is an additive mapping and $X$ is a normed space. □

Now, applying the foregoing theorem we deduce the form of solutions of equation (4). We will show that in this case the space $X$ can be replaced by $\mathbb{R}$.

THEOREM 2. *Assume that* $(G,+)$ *is an Abelian group and* $f\colon G \to \mathbb{R}$ *is an arbitrary mapping. Then* $f$ *is a solution of* (4) *if and only if there exist an additive mapping* $A\colon G \to \mathbb{R}$ *and a constant* $c \in \mathbb{R}$ *such that*

$$f(x) = |A(x)| + c, \quad x \in G.$$

*Proof.* Again, the "only if" part follows from equality (1). Therefore, we will confess ourselves to the "if" part.

We may assume that $f(0) = 0$. Substitution $y = x$ in (4) leads to

$$0 = f(2x) - 2f(x), \quad x \in G,$$

i.e. $f(2x) = 2f(x)$ for each $x \in G$.

Next part of the proof is a repetition of the later part of the proof of the previous theorem. We get the representation of $f$ as the norm of an additive mapping having values in a normed space $X$.

It remains to prove that $X$ may be replaced by $\mathbb{R}$. We will deduce that the image $A(G) \subset X$ is contained in a one-dimensional subspace. For arbitrarily fixed $x, y \in G$ let us denote $\xi = A(x)$, $\eta = A(y)$. We may assume that $\|\xi\| \geqslant \|\eta\|$. From the form of $f$ and from equation (4) we get

$$\|\xi\| - \|\eta\| = \|\xi + \eta\| + \|\xi - \eta\| - \|\xi\| - \|\eta\|,$$

i.e.

$$2\|\xi\| = \|\xi+\eta\| + \|\xi-\eta\|,$$

which can be equivalently written as

$$\|(\xi+\eta)+(\xi-\eta)\| = \|\xi+\eta\| + \|\xi-\eta\|.$$

In other words, we have shown that for each $s,t \in G$ either

$$\|A(s)+A(t)\| = \|A(s)\| + \|A(t)\| \tag{6}$$

or

$$\|A(s)-A(t)\| = \|A(s)\| + \|A(t)\| \tag{7}$$

(we used the fact that the set $A(G)$ forms a subgroup of $X$). On the other hand, by Theorem A for each $x \in G$ we have

$$\|A(x)\| = \sup\{|a(x)| : a \text{ is additive and } a \leqslant f\}$$

and there exists an additive mapping $a_x \leqslant f$ such that $|a_x(x)| = \|A(x)\|$. From (6) and (7) we will deduce that for arbitrary $s,t \in G$ the supremum is realized by the same additive mapping. Indeed, if for instance (6) holds then

$$|a_{s+t}(s+t)| = |a_s(s)| + |a_t(t)|,$$

where $a_s$, $a_t$ and $a_{s+t}$ are additive mappings which realize the respective suprema. Clearly, $|a_{s+t}(s)| \leqslant |a_s(s)|$ and $|a_{s+t}(t)| \leqslant |a_t(t)|$. On joining this with the above equality we obtain

$$|a_s(s)| + |a_t(t)| = |a_{s+t}(s+t)| \leqslant |a_{s+t}(s)| + |a_{s+t}(t)| \leqslant \\ \leqslant |a_s(s)| + |a_t(t)|.$$

This proves that $|a_{s+t}(s)| = |a_s(s)|$ and $|a_{s+t}(t)| = |a_t(t)|$, which means that $a_{s+t}$ realizes suprema for $s$ and $t$. Therefore, we may find an additive mapping $a^*\colon G \to \mathbb{R}$ such that for each $x \in G$ the following equality is valid:

$$\sup\{|a(x)| : a \text{ is additive and } a \leqslant f\} = |a^*(x)|.$$

Therefore, we may replace $A$ by $a^*$ and thus the space $X$ can be replaced by $\mathbb{R}$. $\square$

I am thankful to professor Peter Volkmann for an observation that functional equation (4) for mapping $f$ vanishing at zero can be transformed equivalently to the following equation:

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y),$$

which was first discussed by Alice Chaljub-Simon and Peter Volkmann in [4]. To show that this equation and (4) are equivalent it suffices to make use of the fact that $f(2x) = 2f(x)$ is true for both equations and then apply the identity

$$2\max\{s,t\} = s + t + |s-t|$$

for $s,t \in \mathbb{R}$. However, the method we use to solve (4) is quite different from the method used in [4].

Now, let us discuss the stability of equation (4). The stability problem (5) corresponding to this equation is called the Hyers-Ulam-Aoki stability problem (terminology is in accordance with a recent survey paper [10] of Lech Maligranda). We will show that each solution $F$ of inequality (5) is in a sense close to a unique solution $f$ of equation (4), which has been already solved in Theorem 2. We apply a standard technique known as the direct method or the Hyers method, which is discussed in [10]. Therefore, we provide only a sketch of the proof.

THEOREM 3. *Assume that* $(X, \|\cdot\|)$ *is a normed space,* $\varepsilon \geqslant 0$ *and* $p \in \mathbb{R}$ *is a real number such that* $p \neq 1$. *If* $F\colon X \to \mathbb{R}$ *satisfies* (5) *(for each* $x,y \in X$ *or each* $x,y \in X \setminus \{0\}$ *if* $p < 0$*) then there exists a unique mapping* $f\colon X \to \mathbb{R}$ *such that* $f$ *is a solution of* (4) *and: if* $p > 1$ *then*

$$|F(x) - f(x)| \leqslant \frac{2\varepsilon}{2^p - 2} \cdot \|x\|^p, \quad x \in X \tag{8}$$

*or if* $p < 1$ *then*

$$|F(x) - f(x)| \leqslant \frac{2\varepsilon}{2 - 2^p} \cdot \|x\|^p, \quad x \in X. \tag{9}$$

*Proof.* The proof differs slightly from the proof of the analogous result for the Cauchy functional equation, which is due to Donald H. Hyers [7] and Tosio Aoki [2]. We need to observe that if $F$ satisfies (5) then $F - F(0)$ satisfies it, too. Further, if $F(0) = 0$ then substitution $y = x$ in (5) leads to the estimation

$$|F(x) - \frac{1}{2}F(2x)| \leqslant \varepsilon\|x\|^p, \quad x \in X.$$

From this, by some elementary calculations, we derive that if $p < 1$ then the sequence $2^{-n}F(2^n x)$ is pointwise convergent and if $p > 1$ then the sequence $2^n F(2^{-n}x)$ is pointwise convergent. Further, a standard reasoning (which relies on replacing arguments $x$ and $y$ in (5) by $2^n x$ and $2^n y$, multiplying (5) side by side by $2^{-n}$ and tending with $n$ to $-\infty$ or $+\infty$ respectively) allow us to prove that the limit function $f$ of both sequences satisfies (4) and, after replacing $f$ by $f + F(0)$, the desired estimation (8) or (9) respectively, holds true.

It remains to show the uniqueness of solution. If $f$ and $f'$ are two solutions of (4) and both satisfy one of estimations (8) or (9) then by adding them side by side we get

$$|f(x) - f'(x)| \leqslant \frac{4\varepsilon}{2^p - 2} \cdot \|x\|^p, \quad x \in X$$

if $p > 1$ or

$$|f(x) - f'(x)| \leqslant \frac{4\varepsilon}{2 - 2^p} \cdot \|x\|^p, \quad x \in X$$

if $p < 1$. Now, fix $n \in \mathbb{N}$ arbitrarily. In the first case replace $x$ by $2^{-n}x$ and multiply the estimation side by side by $2^n$ and in the second case replace $x$ by $2^n x$ and divide it side by side by $2^n$. In both resulting inequalities the left hand side remains unchanged (by the 2-homogeneity of solutions of (4)) whereas the right hand side will always tend to zero as $n$ tends to infinity. Therefore, necessarily $f = f'$, which we wanted to show. $\square$

As a straightforward corollary we get that equation (4) is stable in a sense of Hyers-Ulam (case $p = 0$ in the previous theorem). It is worth to note that in case $p = 0$ the structure of normed space in the domain is not really needed in its full strength. An inspection of the proof allows us to state the following corollary.

COROLLARY 4. *Assume that $(G,+)$ is an Abelian group and $\varepsilon \geqslant 0$. If $F\colon G \to \mathbb{R}$ satisfies*

$$\big| |F(x) - F(y)| - F(x+y) - F(x-y) + F(x) + F(y) \big| \leqslant \varepsilon$$

*for each $x, y \in G$ then there exists a unique mapping $f\colon G \to \mathbb{R}$ such that $f$ is a solution of* (4) *and*

$$|F(x) - f(x)| \leqslant \varepsilon, \quad x \in G.$$

An example due to Zbigniew Gajda [5] shows that the Cauchy functional equation is not stable for the "critical case" $p = 1$. In what follows we provide a modification of this example in order to show that the stability effect for (5) fails if $p = 1$.

EXAMPLE 5. Fix arbitrary $\varepsilon > 0$ and put $\mu := \frac{1}{12}\varepsilon$. Next, define a mapping $\phi\colon \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} \mu & \text{if} \quad x \geqslant 1, \\ \mu x & \text{if} \quad 0 \leqslant x < 1, \\ -\mu x & \text{if} \quad -1 \leqslant x < 0, \\ \mu & \text{if} \quad x \leqslant -1, \end{cases}$$

Observe that a mapping $F\colon \mathbb{R} \to \mathbb{R}$ is well defined by

$$F(x) := \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n}, \quad x \in \mathbb{R}.$$

Moreover, $F$, as a sum of an uniformly convergent series of continuous mappings, is continuous. Further, for each $x \in \mathbb{R}$ we have

$$0 \leqslant F(x) \leqslant \sum_{n=0}^{\infty} \frac{\mu}{2^n} = 2\mu. \tag{10}$$

Now, we will check that (5) is satisfied with $p = 1$. Obviously, (5) is true for $x = y = 0$. Let us assume that $0 < |x| + |y| < 1$. Take $N$ as a nonnegative integer for which we have

$$\frac{1}{2^N} \leqslant |x| + |y| \leqslant \frac{1}{2^{N-1}}.$$

From this we easily deduce that each of the values

$$2^n x,\ 2^n y,\ 2^n(x+y),\ 2^n(x-y)$$

belong to the open interval $(-1,1)$ for each $n = 0,1,\dots,N-1$. Consequently, by the definition of $\phi$ we have

$$|\phi(2^n x) - \phi(2^n y)| - \phi(2^n(x+y)) - \phi(2^n(x-y)) + \phi(2^n x) + \phi(2^n y) = 0$$

for each $n = 0,1,\dots,N-1$. Now, we may calculate that the expected estimation is valid:

$$\begin{aligned}
&\frac{\big||F(x)-F(y)| - F(x+y) - F(x-y) + F(x) + F(y)\big|}{|x|+|y|} \\
&\leqslant \sum_{n=N}^{\infty} \frac{\big||\phi(2^n x) - \phi(2^n y)| - \phi(2^n(x+y)) - \phi(2^n(x-y)) + \phi(2^n x) + \phi(2^n y)\big|}{2^n(|x|+|y|)} \\
&\leqslant \sum_{k=0}^{\infty} \frac{6\mu}{2^{k+N}(|x|+|y|)} \leqslant \varepsilon.
\end{aligned}$$

If $|x|+|y| \geqslant 1$ then from (10) we immediately get that

$$\big||F(x)-F(y)| - F(x+y) - F(x-y) + F(x) + F(y)\big| \leqslant 12\mu = \varepsilon \leqslant \varepsilon(|x|+|y|).$$

Therefore, (5) is satisfied by $F$ with $p = 1$ for each $x, y \in \mathbb{R}$.

Now, we will show that there is no solution $f$ of (4) which is close to $F$ is a sense that there exists a constant $\delta > 0$ such that

$$|F(x) - f(x)| \leqslant \delta|x|, \quad x \in \mathbb{R}. \tag{11}$$

Assume for a contrary that such a map exists. From Theorem 2 there exists an additive mapping $A\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = |A(x)|, \quad x \in \mathbb{R}.$$

Since $F$ is bounded then (11) forces that $A$ is continuous (see e.g. Janos Aczél [1, Chapter 2.1.1., Theorem 1]) which means that $f$ is of the form

$$f(x) = c|x|, \quad x \in \mathbb{R}$$

with a certain constant $c \geqslant 0$. Therefore,

$$\big|F(x) - c|x|\,\big| \leqslant \delta|x|, \quad x \in \mathbb{R},$$

i.e.

$$\left|\frac{F(x)}{x}\right| \leqslant \delta + c, \quad x \in \mathbb{R}.$$

Next, take $N := [\frac{\delta+c}{\mu}] + 1$ and $x \in (0, \frac{1}{2^{N-1}})$. We have

$$\frac{F(x)}{x} = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n x} \geqslant \sum_{n=0}^{N-1} \frac{\phi(2^n x)}{2^n x} \geqslant \sum_{n=0}^{N-1} \frac{\mu \cdot 2^n x}{2^n x} = N\mu > \delta + c,$$

which is a contradiction. Therefore, for $p = 1$ we cannot expect the stability effect for equation (4).

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