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SOME NEW INEQUALITIES SIMILAR TO HARDY–HILBERT'S INEQUALITY

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Abstract. In this paper we have studied some new inequalities similar to Hardy-Hilbert's inequality. As applications, we have considered the associated integral inequalities.

1. Introduction

If $a_n,\ b_n \geqslant 0,\ p > 1,\ \frac{1}{p}+\frac{1}{q} = 1$, and $0 < \sum_{n=1}^{\infty} a_n{}^p < \infty,\ 0 < \sum_{n=1}^{\infty} b_n{}^q < \infty$, then the famous Hardy-Hilbert inequality (see Hardy et al. [6]) is given by

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}, \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.1) led to many papers dealing with alternative proofs, generalizations, numerous variants, and applications in analysis (Hardy et al. [6]). Recently, Yang, Gao and Debnath [1, 2, 8] gave some strengthened version of (1.1). By introducing a parameter Yang [3] gave an extension of (1.1) as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda} < \frac{\pi}{\lambda\sin(\pi/p)}\left\{\sum_{n=1}^{\infty}n^{(p-1)(1-\lambda)}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{(q-1)(1-\lambda)}b_n^q\right\}^{1/q},$$

where the constant factor $\frac{\pi}{\lambda\sin(\pi/p)}$, $(0<\lambda\leqslant \min p,q)$ is the best possible.

In 2003 yang [4] gave an extension of (1.1) as

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=2}^{\infty}n^{p-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q-1}b_n^q\right\}^{1/q}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

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In this paper, we have generalized the results of [4], which is related to the double series of the form

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln mn}. \tag{1.3}$$

For this series $\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln(mn)}$, we have estimated the weight co-efficient of the form

$$w_{r_k}(n)=\sum_{m=2}^{\infty}\frac{1}{m\ln(mn)}\left(\frac{\ln n}{\ln m}\right)^{1/r_k} \qquad (r_k=(p_k,q_k)>1,\ n\in\mathbf{N}\setminus\{1\}). \tag{1.4}$$

2. Some preliminary results

The sequence space l_p has been generalized to $l(p)$, in the following manner (see Maddox [7] and Simmons [9]).

DEFINITION 2.1. Let $p=(p_k)$ be a bounded sequence of strictly positive numbers, with $0<p_k\leqslant \sup p_k=H<\infty$. Then

$$l(p)=\{x=(x_k):\sum|x_k|^{p_k}<\infty\}. \tag{2.1}$$

A natural metric on $l(p)$ is

$$d(x,y)=\Big(\sum_{k=0}^{\infty}|x_k-y_k|^{p_k}\Big)^{1/M},$$

where $M=\max(1,H)$.

DEFINITION 2.2. Let $1\leqslant p<\infty$, then the function space L_p is given by

$$L_p=\{f:\int_0^{\infty}|f(x)|^p dx<\infty\}. \tag{2.2}$$

The function space L_p has been generalized to $L(p)$ in the following manner.

DEFINITION 2.3. *[10]* Let p be a bounded measurable function, with $0<p(x)\leqslant \sup p(x)=H<\infty$. Then

$$L(p)=\{f:\int_0^{\infty}|f(x)|^{p(x)}dx<\infty\}. \tag{2.3}$$

Note that $L(p)$ is a linear topological space paranormed by $d(f)$

$$d(f)=\Big(\int_0^{\infty}|f(x)|^{p(x)}dx\Big)^{1/M},$$

where $M=\max(1,H)$.

Das and Nanda [5] have generalized Holder's inequality in $l(p)$ space, which is given below.

LEMMA 2.1. (Das and Nanda, [5]) *Let $(p_n)_{n=1}^{\infty}$ be a real sequence where $p_n > 1$ for all n. $(q_n)_{n=1}^{\infty}$ is defined by $\frac{1}{p_n} + \frac{1}{q_n} = 1$, for all n. Let $a_n, b_n \geqslant 0$. We write*

$$A_m = \sum_{n=1}^{m} a_n{}^{p_n}, \qquad B_m = \sum_{n=1}^{m} b_n{}^{q_n},$$
$$A = \sum_{n=1}^{\infty} a_n{}^{p_n}, \qquad B = \sum_{n=1}^{\infty} b_n{}^{q_n}$$

and whenever the series on the right converge. Then

(i)

$$\sum_{k=1}^{m} a_k b_k \leqslant \alpha_m \beta_m, \tag{2.4}$$

where $\alpha_m = \sup_{1\leqslant n\leqslant m} \frac{1}{p_n} + \sup_{1\leqslant n\leqslant m} \frac{1}{q_n}$, $\beta_m = \sup_{1\leqslant n\leqslant m} A_m^{1/p_n} B_m^{1/q_n}$.

(ii) *If $a \in l(p)$, $b \in l(q)$, then $ab \in l$ and*

$$\sum_{k=1}^{\infty} a_k b_k \leqslant \alpha\beta, \tag{2.5}$$

where $\alpha = \sup_{n\geqslant 1} \frac{1}{p_n} + \sup_{n\geqslant 1} \frac{1}{q_n}$, $\beta = \sup_{n\geqslant 1} \ (A^{1/p_n} B^{1/q_n})$.

If p_k and q_k are constants then (2.4) and (2.5) reduce to standard Holder's inequality in l_p space.

Sunanda et. al. [10] generalized the Holder's inequality in $L(p)$ space which is given below.

LEMMA 2.2. *(a) Let p and q be bounded measurable functions such that $p(x)^{-1} + q(x)^{-1} = 1$ for all x. Let $f \in L(p)$, $g \in L(q)$. Let*

$$A = \int_0^{\infty} \mid f(x) \mid^{p(x)} dx, \ \ B = \int_0^{\infty} \mid g(x) \mid^{q(x)} .dx$$

Then for $p(x) > 1$, $fg \in L_1$ and

$$\int_0^{\infty} \mid f(x) g(x) \mid dx \ \leqslant \ \alpha\beta, \tag{2.6}$$

where $\alpha = \sup_x \ 1/p(x) + \sup_x \ 1/q(x)$, $\quad \beta = \sup_x \ A^{1/p(x)} B^{1/q(x)}$.

(b) Let $0 < p(x) < 1$ and $p(x)^{-1} + q(x)^{-1} = 1$. If $f \in L(p)$ and $g \in L(q)$, then

$$\int_0^{\infty} \mid f(x) \mid^{p(x)} dx \ \leqslant \ \alpha \, [\sup_x \ p(x) + \sup_x \ (1 - p(x))], \tag{2.7}$$

where $\alpha = \sup_x \ \left[\left(\int \mid g(x) \mid^{q(x)} dx \right)^{1-p(x)} \left(\int \mid f(x) g(x) \mid dx \right)^{p(x)} \right]$.

Proof. To prove (a) For $a,\ b\ >\ 0$, we have

$$ab \leqslant \frac{a^{p(x)}}{p(x)} + \frac{b^{q(x)}}{q(x)} \quad \text{for all } x. \tag{2.8}$$

Using the above inequality we have

$$\frac{|f(x)|}{A^{1/p(x)}}\ \frac{|g(x)|}{B^{1/q(x)}} \leqslant \frac{1}{p(x)}\ \frac{|f(x)|^{p(x)}}{A} + \frac{1}{q(x)}\ \frac{|g(x)|^{q(x)}}{B}.$$

Therefore,

$$\begin{aligned}\int_0^\infty \frac{|f(x)|\,|g(x)|}{A^{1/p(x)}\ B^{1/q(x)}}\,dx &\leqslant \frac{1}{A}\int_0^\infty \frac{|f(x)|^{p(x)}}{p(x)}\,dx + \frac{1}{B}\int_0^\infty \frac{|g(x)|^{q(x)}}{q(x)}\,dx\\ &\leqslant \sup_x \frac{1}{p(x)} + \sup_x \frac{1}{q(x)}. \end{aligned} \tag{2.9}$$

Also we have

$$\frac{1}{\sup_x A^{1/p(x)}\ B^{1/q(x)}} \int_0^\infty |f(x)\ g(x)|\,dx \leqslant \int_0^\infty \frac{|f(x)g(x)|}{A^{1/p(x)}\ B^{1/q(x)}}\,dx. \tag{2.10}$$

From (2.9) and (2.10), we get (2.6), that is

$$\int_0^\infty |f(x)\ g(x)|\,dx \leqslant \alpha\,\beta.$$

We now prove (b).

Let $p_1(x) = 1/p(x)$, so that $p_1(x) > 1$ for all x. Let

$$A(x) = |g(x)|^{-1/p_1(x)},\ B(x) = |f(x)g(x)|^{1/p_1(x)}.$$

So by (2.6)

$$\begin{aligned}\int_0^\infty |f(x)|^{p(x)}dx &= \int_0^\infty |A(x)B(x)|dx\\ &\leqslant \sup_x \left[\left(\int_0^\infty |A(x)|^{q_1(x)}dx\right)^{1/q_1(x)}\left(\int_0^\infty |B(x)|^{p_1(x)}dx\right)^{1/p_1(x)}\right]\\ &\quad\times\left(\sup_x \frac{1}{p_1(x)} + \sup_x \frac{1}{q_1(x)}\right),\end{aligned} \tag{2.11}$$

since $\frac{1}{p_1(x)} + \frac{1}{q_1(x)} = 1$, $\frac{1}{q_1(x)} = 1 - p(x)$ and $q_1(x) = \frac{1}{1-p(x)}$.
Substituting the values in (2.11), we get (2.7)

$$\begin{aligned}\int_0^\infty |f(x)|^{p(x)}dx &= \sup_x \left[\left(\int_0^\infty |g(x)|^{q(x)}dx\right)^{1-p(x)}\left(\int_0^\infty |f(x)\ g(x)|dx\right)^{p(x)}\right]\\ &\quad\times\left[\sup_x p(x) + \sup_x (1-p(x))\right]\\ &= \alpha\,[\sup_x\ p(x) + \sup_x\ (1-p(x))].\end{aligned}$$

This completes the proof of the lemma. □

Note. Equality occurs in (2.6) when

(1) $\alpha = 1$.

(2) $\beta | f(x) |^{p(x)} = \gamma | g(x) |^{q(x)}$, where β and γ are constants not all zero.

But $\alpha = 1$ holds only when $p(x)$ and $q(x)$ are constants.

If $p(x)$ and $q(x)$ are not constants and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, then $\alpha = \sup \frac{1}{p(x)} + \sup \frac{1}{q(x)} > 1$ e.g.

For $p(x) = 1 + \frac{1}{x} > 1$, $x \in [2,3]$ and $q(x) = x + 1 > 1, x \in [2,3]$ not constants then $\alpha = \sup \frac{1}{p(x)} + \sup \frac{1}{q(x)} > 1$.

3. Main results

LEMMA 3.1. *Let $(p_k)_{k=1}^{\infty}$ and $(q_k)_{k=1}^{\infty}$ be real bounded sequences defined by $\frac{1}{p_k} + \frac{1}{q_k} = 1$ where $p_k > 1, 0 < \epsilon < q_k - 1$ for all $k \in \mathbf{N}$, then we have for any $k \geqslant 1$*

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{1}{\ln(mn)} \frac{1}{m(\ln m)^{(1+\epsilon)/p_k}} \frac{1}{n(\ln n)^{(1+\epsilon)/q_k}} > \frac{1}{\epsilon}\left[\frac{\pi}{\sin(\pi/p_k)} + o(1)\right] \quad (as \quad \epsilon \to 0^+). \tag{3.1}$$

Proof. Obviously, we have

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{1}{\ln(mn)} \frac{1}{m(\ln m)^{(1+\epsilon)/p_k}} \frac{1}{n(\ln n)^{(1+\epsilon)/q_k}} > \int_e^{\infty} \int_e^{\infty} \frac{1}{\ln(xy)} \frac{1}{x(\ln x)^{(1+\epsilon)/p_k}} \frac{1}{y(\ln y)^{(1+\epsilon)/q_k}} dydx. \tag{3.2}$$

Setting $u = \frac{\ln y}{\ln x}$ in the following expression, for $x \geqslant e$ and $0 < \epsilon < q_k - 1$ we obtain for $k = 1, 2...$

$$\begin{aligned}
&\int_e^{\infty} \frac{1}{\ln(xy)} \frac{1}{y(\ln y)^{(1+\epsilon)/q_k}} dy \\
&= \frac{1}{(\ln x)^{(1+\epsilon)/q_k}} \int_{1/\ln x}^{\infty} \frac{1}{1+u} \frac{1}{u^{(1+\epsilon)/q_k}} du \\
&= \frac{1}{(\ln x)^{(1+\epsilon)/q_k}} \left[\int_0^{\infty} \frac{1}{1+u} \frac{1}{u^{(1+\epsilon)/q_k}} du - \int_0^{1/\ln x} \frac{1}{1+u} \frac{1}{u^{(1+\epsilon)/q_k}} du\right] \\
&> \frac{1}{(\ln x)^{(1+\epsilon)/q_k}} \left[\int_0^{\infty} \frac{1}{1+u} \frac{1}{u^{(1+\epsilon)/q_k}} du - \int_0^{1/\ln x} \frac{1}{u^{(1+\epsilon)/q_k}} du\right] \\
&= \frac{1}{(\ln x)^{(1+\epsilon)/q_k}} \left[\frac{\pi}{\sin(\pi/p_k)} + o(1)\right] - \frac{q_k}{(q_k - 1 - \epsilon)} \frac{1}{\ln x}.
\end{aligned}$$

Hence we find for $k \geqslant 1$

$$
\begin{aligned}
&\int_e^\infty \int_e^\infty \frac{1}{\ln(xy)} \frac{1}{x(\ln x)^{(1+\epsilon)/p_k}} \frac{1}{y(\ln y)^{(1+\epsilon)/q_k}} dydx \\
&\quad > \int_e^\infty \frac{1}{x(\ln x)^{(1+\epsilon)}} \left[\frac{\pi}{\sin(\pi/p_k)} + o(1)\right] dx - \frac{q_k}{(q_k-1-\epsilon)} \int_e^\infty \frac{1}{x(\ln x)^{1+(1+\epsilon)/p_k}} dx \\
&\quad = \left[\frac{\pi}{\sin(\pi/p_k)} + o(1)\right] \frac{1}{\epsilon} - \frac{q_k}{(q_k-1-\epsilon)(1+\epsilon)} \quad (\epsilon \to 0),
\end{aligned}
\tag{3.3}
$$

by (3.2) and (3.3), it follows that (3.1) is valid. Hence the lemma. □

THEOREM 3.1. *If $a_n, b_n \geqslant 0, (p_k)$ and (q_k) are real bounded sequences defined by $\frac{1}{p_k} + \frac{1}{q_k} = 1$, where $p_k > 1$ for all $k \in \mathbf{N}$, and $0 < \sum_{n=2}^{\infty} n^{p_k-1} a_n^{p_k} < \infty$, $0 < \sum_{n=2}^{\infty} n^{q_k-1} b_n^{q_k} < \infty$. Then*

$$
\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{a_m b_n}{\ln(mn)} < \alpha \sup_k \left\{ \frac{\pi}{\sin(\pi/p_k)} \left\{ \sum_{n=2}^{\infty} n^{p_k-1} a_n^{p_k} \right\}^{1/p_k} \left\{ \sum_{n=2}^{\infty} n^{q_k-1} b_n^{q_k} \right\}^{1/q_k} \right\}, \tag{3.4}
$$

where $\alpha = \sup_{k\geqslant 1} \frac{1}{p_k} + \sup_{k\geqslant 1} \frac{1}{q_k}$.

Proof. By Holder's inequality and (1.4), we have

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{a_m b_n}{\ln mn} \\
&\quad = \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \left[\frac{a_m}{(\ln mn)^{\frac{1}{p_k}}} \left(\frac{\ln m}{\ln n}\right)^{\frac{1}{p_k q_k}} \left(\frac{m^{1/q_k}}{n^{1/p_k}}\right)\right] \left[\frac{b_n}{(\ln mn)^{1/q_k}} \left(\frac{\ln n}{\ln m}\right)^{\frac{1}{p_k q_k}} \left(\frac{n^{1/p_k}}{m^{1/q_k}}\right)\right] \\
&\quad \leqslant \alpha \sup_{k\geqslant 1} \Bigg[\Bigg(\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m^{p_k}}{\ln mn} \left(\frac{\ln m}{\ln n}\right)^{\frac{1}{q_k}} \\
&\quad \times \left(\frac{m^{\frac{p_k}{q_k}}}{n}\right)\Bigg)^{\frac{1}{p_k}} \left(\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{b_n^{q_k}}{\ln mn} \left(\frac{\ln n}{\ln m}\right)^{\frac{1}{p_k}} \left(\frac{n^{\frac{q_k}{p_k}}}{m}\right)\right)^{\frac{1}{q_k}}\Bigg] \\
&\quad = \alpha \sup_{k\geqslant 1} \Bigg[\Bigg(\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{1}{\ln mn} \left(\frac{\ln m}{\ln n}\right)^{\frac{1}{q_k}} \left(\frac{m^{p_k-1}}{n}\right) a_m^{p_k}\Bigg)^{\frac{1}{p_k}} \\
&\qquad\qquad \times \left(\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{1}{\ln mn} \left(\frac{\ln n}{\ln m}\right)^{\frac{1}{p_k}} \left(\frac{n^{q_k-1}}{m}\right) b_n^{q_k}\right)^{\frac{1}{q_k}}\Bigg] \\
&\quad = \alpha \sup_{k\geqslant 1} \left[\left\{ \sum_{m=2}^{\infty} w_{q_k}(m) m^{p_k-1} {a_m}^{p_k} \right\}^{\frac{1}{p_k}} \left\{ \sum_{n=2}^{\infty} w_{p_k}(n) n^{q_k-1} {b_n}^{q_k} \right\}^{\frac{1}{q_k}}\right].
\end{aligned}
\tag{3.5}
$$

For $r_k = p_k, q_k$ for all k, and $n \geqslant 2$ in (1.4), setting $u = \frac{\ln x}{\ln n}$ in the following integral, we find

$$w_{r_k}(n) < \int_1^\infty \frac{1}{x \ln nx}\Big(\frac{\ln n}{\ln x}\Big)^{\frac{1}{r_k}} dx = \int_0^\infty \frac{1}{(1+u)u^{\frac{1}{r_k}}} du = \frac{\pi}{\sin \pi(1-\frac{1}{r_k})}. \tag{3.6}$$

Since $\sin(\pi/p_k) = \sin(\pi/q_k)$, for all k, by (3.5) and (3.6), we have (3.4). □

THEOREM 3.2. *Let* $a_n \geqslant 0, (p_k)$ *and* (q_k) *be real bounded sequences defined by* $\frac{1}{p_k} + \frac{1}{q_k} = 1$, *where* $p_k > 1$ *for all* $k \in \mathbf{N}$, *and* $0 < \sum_{n=2}^\infty n^{p_k-1} a_n^{p_k} < \infty$. *Then*

$$\sum_{n=2}^\infty \frac{1}{n}\Big(\sum_{m=2}^\infty \frac{a_m}{\ln(mn)}\Big)^{p_k} < \alpha^{\sup p_k} \sup_{k\geqslant 1}\Big\{\Big[\frac{\pi}{\sin(\pi/p_k)}\Big]^{p_k} \sum_{n=2}^\infty n^{p_k-1} a_n^{p_k}\Big\}, \tag{3.7}$$

where $\alpha = \sup_{n\geqslant 1} \frac{1}{p_n} + \sup_{n\geqslant 1} \frac{1}{q_n}$.

Proof. Since $\sum_{n=2}^\infty n^{p_k-1} a_n^{p_k} > 0$, there exists $t_0 \geqslant 2$, such that for any $t > t_0$, $\sum_{n=2}^t n^{p_k-1} a_n^{p_k} > 0$, and $b_n(t) = \frac{1}{n}\Big(\sum_{m=2}^t \frac{a_m}{\ln(mn)}\Big)^{p_k-1} > 0$. Then we have

$$0 < \sum_{n=2}^t n^{q_k-1} b_n^{q_k}(t) = \sum_{n=2}^t \frac{1}{n}\Big(\sum_{m=2}^t \frac{a_m}{\ln(mn)}\Big)^{p_k} = \sum_{n=2}^t \sum_{m=2}^t \frac{a_m b_n(t)}{\ln(mn)}. \tag{3.8}$$

If we set $\tilde{a}_n = a_n$ and $\tilde{b}_n = b_n(t)$, for $n = 2, 3, ...t$; and $\tilde{a}_n = \tilde{b}_n = 0$, for $n > t$. By using (3.4), we may have

$$\begin{aligned}
\sum_{n=2}^t \sum_{m=2}^t \frac{a_m b_n(t)}{\ln(mn)} &= \sum_{n=2}^\infty \sum_{m=2}^\infty \frac{\tilde{a}_m \tilde{b}_n}{\ln(mn)} \\
&< \alpha \sup_{k\geqslant 1}\Big[\frac{\pi}{\sin(\pi/p_k)}\Big\{\sum_{n=2}^\infty n^{p_k-1}\tilde{a}_n^{p_k}\Big\}^{\frac{1}{p_k}}\Big\{\sum_{n=2}^\infty n^{q_k-1}\tilde{b}_n^{q_k}\Big\}^{\frac{1}{q_k}}\Big] \\
&= \alpha \sup_{k\geqslant 1}\Big[\frac{\pi}{\sin(\pi/p_k)}\Big\{\sum_{n=2}^t n^{p_k-1} a_n^{p_k}\Big\}^{\frac{1}{p_k}}\Big\{\sum_{n=2}^t n^{q_k-1} b_n^{q_k}(t)\Big\}^{\frac{1}{q_k}}\Big].
\end{aligned}$$

Hence by (3.8), we have

$$\begin{aligned}
0 < \sum_{n=2}^\infty n^{q_k-1} b_n^{q_k}(t) &= \sum_{n=2}^t \frac{1}{n}\Big(\sum_{m=2}^t \frac{a_m}{\ln mn}\Big)^{p_k} \\
&< \sup_{k\geqslant 1}\Big[\frac{\alpha\pi}{\sin(\pi/p_k)}\Big]^{p_k} \sum_{n=2}^t n^{p_k-1} a_n^{p_k} \\
&= \alpha^{\sup p_k} \sup_{k\geqslant 1}\Big\{\Big[\frac{\pi}{\sin(\pi/p_k)}\Big]^{p_k} \sum_{n=2}^t n^{p_k-1} a_n^{p_k}\Big\}.
\end{aligned}$$

It follows that $0 < \sum_{n=2}^{t} n^{q_k-1} b_n^{q_k}(\infty) < \infty$, since $0 < \sum_{n=2}^{t} n^{p_k-1} a_n^{p_k} < \infty$. Hence by (3.4) we have,

$$\sum_{n=2}^{t} \frac{1}{n}\Big(\sum_{m=2}^{\infty} \frac{a_m}{\ln(mn)}\Big)^{p_k}$$
$$= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{a_m b_n(\infty)}{\ln mn}$$
$$< \sup_{k\geqslant 1}\Big[\frac{\alpha\pi}{\sin(\pi/p_k)}\Big\{\sum_{n=2}^{\infty} n^{p_k-1}a_n^{p_k}\Big\}^{\frac{1}{p_k}}\Big\{\sum_{n=2}^{\infty} n^{q_k-1}b_n^{q_k}(\infty)\Big\}^{\frac{1}{q_k}}\Big]$$
$$< \sup_{k\geqslant 1}\Big[\frac{\alpha\pi}{\sin(\pi/p_k)}\Big\{\sum_{n=2}^{\infty} n^{p_k-1}a_n^{p_k}\Big\}^{\frac{1}{p_k}}\Big\{\sum_{n=2}^{\infty} \frac{1}{n}\Big(\sum_{m=2}^{\infty}\frac{a_m}{\ln mn}\Big)^{p_k}\Big\}^{\frac{1}{q_k}}\Big]$$

by simplification, we have (3.7). □

4. Associated Integral Inequalities

Define the weight function as in [4] $w_r(x)$ as

$$w_r(x) = \int_1^{\infty} \frac{1}{y\ln(xy)}\Big(\frac{\ln x}{\ln y}\Big)^{\frac{1}{r(x)}} dy \quad (r(x) = p(x),\ q(x) \geqslant 2,\ x \geqslant 1). \tag{4.1}$$

Setting $u = \frac{\ln y}{\ln x}$ in (4.1), we have

$$w_r(x) = \int_0^{\infty} \frac{1}{1+u}\Big(\frac{1}{u}\Big)^{\frac{1}{r(x)}} du = \frac{\pi}{\sin\pi(1-\frac{1}{r(x)})} \tag{4.2}$$
$$(r(x) = p(x),\ q(x) \geqslant 2,\ x \geqslant 1).$$

THEOREM 4.1. *Let $f, g \geqslant 0, p(x)$ and $q(x)$ be real bounded measurable functions defined by $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ where $p(x) > 1$ for all $x \geqslant 1$, and $0 < \int_1^{\infty} x^{p(x)-1} f^{p(x)}(x)dx < \infty$, $0 < \int_1^{\infty} x^{q(x)-1} g^{q(x)}(x)dx < \infty$, then*

$$\int_1^{\infty}\int_1^{\infty} \frac{f(x)g(y)}{\ln(xy)}\,dxdy \;<\; \alpha\sup_{x\geqslant 1}\Big[\frac{\pi}{\sin(\pi/p(x))}\Big\{\int_1^{\infty} x^{p(x)-1}f^{p(x)}(x)dx\Big\}^{\frac{1}{p(x)}}$$
$$\times\Big\{\int_1^{\infty} x^{q(x)-1}g^{q(x)}(x)dx\Big\}^{\frac{1}{q(x)}}\Big], \tag{4.3}$$

and

$$\int_1^\infty \frac{1}{y}\Big(\int_1^\infty \frac{f(x)}{\ln(xy)}dx\Big)^{p(y)}dy$$
$$< \alpha^{\sup p(x)} \sup_{x\geqslant 1}\Big[\frac{\pi}{\sin(\pi/p(x))}\Big]^{p(x)} \int_1^\infty x^{p(x)-1}f^{p(x)}(x)dx \tag{4.4}$$

where $\alpha = \sup\ p(t)^{-1} + \sup\ q(t)^{-1}$.

Proof. By generalized Holder's inequality (2.6), we have

$$\int_1^\infty\int_1^\infty \frac{f(x)g(y)}{\ln(xy)}\,dxdy = \int_1^\infty\int_1^\infty \Big[\frac{f(x)}{(\ln xy)^{1/p(x)}}\Big(\frac{\ln x}{\ln y}\Big)^{\frac{1}{p(x)q(x)}}\frac{x^{1/q(x)}}{y^{1/p(x)}}\Big]$$
$$\times\Big[\frac{g(y)}{(\ln xy)^{1/q(x)}}\Big(\frac{\ln y}{\ln x}\Big)^{\frac{1}{p(x)q(x)}}\frac{y^{1/p(x)}}{x^{1/q(x)}}\Big]dxdy$$
$$\leqslant \alpha \sup_{x\geqslant 1}\Big[\Big\{\int_1^\infty\int_1^\infty \frac{f^{p(x)}(x)}{\ln(xy)}\Big(\frac{\ln x}{\ln y}\Big)^{\frac{1}{q(x)}}\frac{x^{p(x)-1}}{y}\,dydx\Big\}^{\frac{1}{p(x)}}$$
$$\times\Big\{\int_1^\infty\int_1^\infty \frac{g^{q(y)}(y)}{\ln(xy)}\Big(\frac{\ln y}{\ln x}\Big)^{\frac{1}{p(y)}}\frac{y^{q(y)-1}}{x}\,dxdy\Big\}^{\frac{1}{q(x)}}\Big]. \tag{4.5}$$

From (2.6) equality holds in (4.5), when $p(x)$ and $q(x)$ are constants p and q, and for $p,\ q$ there exists numbers a and b, such that

$$\frac{af^p(x)}{\ln xy}\Big(\frac{\ln x}{\ln y}\Big)^{\frac{1}{q}}\frac{x^{p-1}}{y} = \frac{bg^q(y)}{\ln xy}\Big(\frac{\ln y}{\ln x}\Big)^{\frac{1}{p}}\frac{y^{q-1}}{x} \quad \text{a.e. in } (1,\infty)\times(1,\infty).$$

Then we have $ax^{p-1}f^p(x)\ln x = by^{q-1}g^q(y)\ln y$ a.e. in $(1,\infty)\times(1,\infty)$. That is

$$ax^{p-1}f^p(x)\ln x = by^{q-1}g^q(y)\ln y = \text{constant} \quad \text{a.e. in } (1,\infty)\times(1,\infty)$$

which contradicts the fact that $0<\int_1^\infty x^{p-1}f^p(x)dx<\infty$ and $0<\int_1^\infty x^{q-1}g^q(x)dx<\infty$. It follows that (4.5) takes the form of strict inequality. By eqn. (4.1)

$$\int_1^\infty\int_1^\infty \frac{f(x)g(y)}{\ln xy}\,dxdy < \alpha \sup_{x\geqslant 1}\Big[\Big\{\int_1^\infty w_q(x)x^{p(x)-1}f^{p(x)}(x)\,dx\Big\}^{\frac{1}{p(x)}}$$
$$\times\Big\{\int_1^\infty w_p(y)y^{q(y)-1}g^{q(y)}(y)\,dy\Big\}^{\frac{1}{q(x)}}\Big]. \tag{4.6}$$

Since $\int_1^\infty x^{p(x)-1}f^{p(x)}(x)dx > 0$, then there exists $T_0 \geqslant 1$ such that for any $T > T_0$, $\int_1^T x^{p(x)-1}f^{p(x)}(x)dx > 0$, and $g(y,T) = \frac{1}{y}\Big(\int_1^T \frac{f(x)}{\ln xy}dx\Big)^{p(y)-1} > 0\ (y\in(1,\infty))$, by (4.3) we have

$$
\begin{aligned}
0 < \int_1^T y^{q(y)-1} g^{q(y)}(y,T)dy &= \int_1^T \frac{1}{y}\Big(\int_1^T \frac{f(x)}{\ln xy}dx\Big)^{p(y)}dy \\
&= \int_1^T\int_1^T \frac{f(x)g(y,T)}{\ln xy}\,dxdy \\
&< \alpha \sup_{x\geqslant 1}\Big[\frac{\pi}{\sin(\pi/p(x))}\Big\{\int_1^T x^{p(x)-1}f^{p(x)}(x)dx\Big\}^{1/p(x)} \\
&\quad \times\Big\{\int_1^T x^{q(x)-1}g^{q(x)}(x,T)dx\Big\}^{1/q(x)}\Big], \qquad (4.7)
\end{aligned}
$$

Let us denote

$$
B = B(T,q,g) = \int_1^T y^{q(y)-1}g^{q(y)}(y,T)dy,\ A = A(T,p,f) = \int_1^T x^{p(x)-1}f^{p(x)}(x,T)dx.
$$

Then (4.7) is

$$
B < \sup_{x\geqslant 1}\Big[\frac{\pi\alpha}{\sin(\frac{\pi}{p(x)})}A^{1/p(x)}B^{1/q(x)}\Big]
$$

If we denote $C = A/B$, then this is

$$
1 < \sup_{x\geqslant 1}\Big[\frac{\pi\alpha}{\sin(\frac{\pi}{p(x)})}C^{1/p(x)}\Big]
$$

hence $\exists\, x_0$ such that

$$
1 < \Big[\frac{\pi\alpha}{\sin(\frac{\pi}{p(x_0)})}C^{1/p(x_0)}\Big]
$$

and hence

$$
1 < \Big[\frac{\pi\alpha}{\sin(\frac{\pi}{p(x_0)})}\Big]^{p(x_0)}C,
$$

So

$$
B < \alpha^{\sup p(x)}\sup_{x\geqslant 1}\Big[\frac{\pi}{\sin(\frac{\pi}{p(x)})}\Big]^{p(x)}A
$$

(4.4) follows. □

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