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ON A GENERALIZATION OF THE LIZORKIN THEOREM ON FOURIER MULTIPLIERS

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Abstract. A generalization of the Lizorkin theorem on Fourier multipliers is proved. The proofs are based on using the so-called net spaces and interpolation theorems. An example is given of a Fourier multiplier which satisfies the assumptions of the generalized theorem but does not satisfy the assumptions of the Lizorkin theorem.

Let F and F^{-1} be the direct and the inverse Fourier transforms respectively, namely,

$$(Ff)(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} f(x)dx,$$

$$(F^{-1}g)(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi} g(\xi)d\xi.$$

Let $1 \leqslant p \leqslant q \leqslant \infty$. It is said that φ is a Fourier multiplier from L_p to L_q, briefly, $\varphi \in m_p^q$, if there exists $c_1 > 0$ such that for every function f in Schwartz space S the following inequality holds

$$\left\|T_\varphi(f)\right\|_{L_q} \leqslant c_1 \left\|f\right\|_{L_p},$$

where $T_\varphi(f) = F^{-1}\varphi Ff$. This inequality allows defining by continuity $T_\varphi(f)$ for every f in L_p.

The set m_p^q of all Fourier multipliers from L_p to L_q is a normed space with the norm

$$\|\varphi\|_{m_p^q} = \left\|T_\varphi\right\|_{L_p \to L_q}.$$

Our aim is to find smoothness and metric characteristics for a function φ to be a Fourier multiplier from L_p to L_q.

We recall the following properties of m_p^q spaces for $1 < p \leqslant q < \infty$ (see, for example, [1]):

1. $m_p^q = m_{q'}^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$;

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2. $m_p^p \hookrightarrow m_r^r$, if $\left|\frac{1}{r}-\frac{1}{2}\right| < \left|\frac{1}{p}-\frac{1}{2}\right|$; here the symbol $\hookrightarrow$ denotes continuous embedding, i.e. $X \hookrightarrow Y$, where X and Y are normed spaces, means that $X \subset Y$ and there exists $c_2 > 0$ such that for every $x \in X$

$$\|x\|_Y \leqslant c_2 \|x\|_X ;$$

3. $m_2^2 = L_\infty$.

A function f is absolutely continuous on $[a,b]$, briefly $f \in AC([a,b])$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite system of disjoint intervals $(a_k,b_k), k = 1,...,n$, such that $\sum_{k=1}^n (b_k - a_k) < \delta$

$$\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon.$$

If $\Omega \subset \mathbb{R}$ is an open set, then $AC^{loc}(\Omega)$ is the set of functions f, defined on Ω, such that $f \in AC([a,b])$ for every $[a,b] \subset \Omega$.

THEOREM. (Lizorkin [3]). *Let* $1 < p \leqslant q < \infty$, $A > 0$ *and a function* $\varphi \in AC^{loc}(\mathbb{R} \setminus \{0\})$ *satisfy the following conditions*

$$\sup_{y \in \mathbb{R}\setminus\{0\}} |y|^{\frac{1}{p}-\frac{1}{q}} |\varphi(y)| \leqslant A$$

and

$$\operatorname*{ess\,sup}_{y \in \mathbb{R}\setminus\{0\}} |y|^{1+\frac{1}{p}-\frac{1}{q}} \left|\varphi'(y)\right| \leqslant A.$$

Then the function $\varphi \in m_p^q$ *and*

$$\|\varphi\|_{m_p^q} \leqslant c_3 A,$$

where $c_3 > 0$ *depends only on* p *and* q.

Let $0 \leqslant \alpha < \infty, 0 < \beta < \infty, a,b \in \mathbb{R}$, $0 < q,r \leqslant \infty$, $G_t = \{[a,b] : b - a \geqslant t\}$.
The net space $N^{\alpha,\beta,r}(L_q)$ is the set of all functions f in L_1 for which

$$\|f\|_{N^{\alpha,\beta,r}(L_q)} = \left(\int_0^\infty \left(t^\alpha \sup_{Q \in G_t} \frac{1}{|Q|^\beta} \|F^{-1}\chi_Q F f\|_{L_q} \right)^r \frac{dt}{t} \right)^{\frac{1}{r}} < \infty,$$

if $0 < r < \infty$, and if $r = \infty$

$$\|f\|_{N^{\alpha,\beta,\infty}(L_q)} = \sup_{t>0} t^\alpha \sup_{Q \in G_t} \frac{1}{|Q|^\beta} \|F^{-1}\chi_Q F f\|_{L_q} < \infty,$$

where χ_Q is the characteristic function of Q, $|Q|$ is the length of Q.
Similar spaces were considered in [4].

LEMMA 1. *Let* $0<\alpha<1, 0<\beta<1, 0<q\leqslant\infty$.
a) If $0<r\leqslant r_1\leqslant\infty$*, then*

$$N^{\alpha,\beta,r}(L_q)\hookrightarrow N^{\alpha,\beta,r_1}(L_q).$$

b) If $0<\sigma<\min\{1-\alpha,1-\beta\}$, $0<r\leqslant\infty$*, then*

$$N^{\alpha,\beta,r}(L_q)\hookrightarrow N^{\alpha+\sigma,\beta+\sigma,r}(L_q). \tag{1}$$

Proof. First we prove that $N^{\alpha,\beta,r}(L_q)\hookrightarrow N^{\alpha,\beta,\infty}(L_q)$. Indeed,

$$\begin{aligned}
\|f\|_{N^{\alpha,\beta,\infty}(L_q)} &= \sup_{t>0} t^{\alpha}\sup_{Q\in G_t}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff\|_{L_q}\\
&= (\alpha r)^{\frac{1}{r}}\sup_{t>0}\left(\int_0^t x^{\alpha r-1}dx\right)^{\frac{1}{r}}\sup_{Q\in G_t}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff\|_{L_q}\\
&\leqslant (\alpha r)^{\frac{1}{r}}\left(\int_0^{\infty}\left(x^{\alpha}\sup_{Q\in G_x}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff\|_{L_q}\right)^r\frac{dx}{x}\right)^{\frac{1}{r}}\\
&= (\alpha r)^{\frac{1}{r}}\|f\|_{N^{\alpha,\beta,r}(L_q)}.
\end{aligned}$$

Using this inequality and the multiplicative inequality for spaces L_p, we obtain

$$\|f\|_{N^{\alpha,\beta,r_1}(L_q)}\leqslant\|f\|^{\frac{r}{r_1}}_{N^{\alpha,\beta,r}(L_q)}\|f\|^{1-\frac{r}{r_1}}_{N^{\alpha,\beta,\infty}(L_q)}\leqslant(\alpha r)^{\frac{1}{r}-\frac{1}{r_1}}\|f\|_{N^{\alpha,\beta,r}(L_q)}.$$

Moreover,

$$\begin{aligned}
\|f\|_{N^{\alpha+\sigma,\beta+\sigma,r}(L_q)} &= \left(\int_0^{\infty}\left(t^{\alpha+\sigma}\sup_{Q\in G_t}\frac{1}{|Q|^{\beta+\sigma}}\|F^{-1}\chi_Q Ff\|_{L_q}\right)^r\frac{dt}{t}\right)^{\frac{1}{r}}\\
&\leqslant\left(\int_0^{\infty}\left(t^{\alpha}\sup_{Q\in G_t}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff\|_{L_q}\right)^r\frac{dt}{t}\right)^{\frac{1}{r}}=\|f\|_{N^{\alpha,\beta,r}(L_q)}.
\end{aligned}$$

Lemma 1 is proved. □

Let (A_0,A_1) be a compatible pair of Banach spaces [1], and

$$K(\tau,a;A_0,A_1)=\inf_{a=a_0+a_1}(\|a_0\|_{A_0}+\tau\|a_1\|_{A_1}),\ a\in A_0+A_1,\quad \tau>0$$

be the Peetre functional.

Moreover, let for $0<q<\infty$, $0<\theta<1$

$$(A_0,A_1)_{\theta,q}=\left\{a\in A_0+A_1:\|a\|_{(A_0,A_1)_{\theta,q}}=\left(\int_0^{\infty}(\tau^{-\theta}K(\tau,a;A_0,A_1)^q\frac{d\tau}{\tau}\right)^{\frac{1}{q}}<\infty\right\},$$

and for $q=\infty$

$$(A_0,A_1)_{\theta,\infty}=\left\{a\in A_0+A_1:\|a\|_{(A_0,A_1)_{\theta,\infty}}=\sup_{0<\tau<\infty}\tau^{-\theta}K(\tau,a;A_0,A_1)<\infty\right\}.$$

LEMMA 2. *Let* $0<\alpha_1<1, 0<\beta<1,\ 0<r,q\leqslant\infty$. *Then*

$$(N^{\alpha_1,\beta,\infty}(L_q),N^{0,\beta,\infty}(L_q))_{\theta,r}\hookrightarrow N^{\alpha,\beta,r}(L_q)$$

for $0<\theta<1$ *and* $\alpha=(1-\theta)\alpha_1$.

Proof. Let $f=f_1+f_0,\ f_0\in N^{0,\beta,\infty}(L_q),\ f_1\in N^{\alpha_1,\beta,\infty}(L_q)$. It is clear that

$$\sup_{Q\in G_t}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff\|_{L_q}$$
$$\leqslant 2^{\left(\frac{1}{q}-1\right)_+}\left(\sup_{Q\in G_t}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff_1\|_{L_q}+\sup_{Q\in G_t}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff_0\|_{L_q}\right),$$

where $x_+=x$, if $x>0$ and $x_+=0$, if $x\leqslant 0$. Denoting $v(\tau)=\tau^{\frac{1}{\alpha_1}},\ \tau>0$, we obtain

$$\sup_{0<t\leqslant v(\tau)}t^{\alpha_1}\sup_{Q\in G_t}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff\|_{L_q}$$
$$\leqslant 2^{(\frac{1}{q}-1)_+}\left(\sup_{0<t\leqslant v(\tau)}t^{\alpha_1}\sup_{Q\in G_t}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff_1\|_{L_q}+\sup_{0<t\leqslant v(\tau)}t^{\alpha_1}\sup_{Q\in G_t}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff_0\|_{L_q}\right)$$
$$\leqslant 2^{(\frac{1}{q}-1)_+}\left(\sup_{t>0}t^{\alpha_1}\sup_{Q\in G_t}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff_1\|_{L_q}+\tau\sup_{t>0}\sup_{Q\in G_t}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff_1\|_{L_q}\right).$$

Since the representation $f=f_0+f_1$ is arbitrary

$$\sup_{0<t\leqslant v(\tau)}t^{\alpha_1}\sup_{Q\in G_t}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff\|_{L_q}\leqslant 2^{(\frac{1}{q}-1)_+}K(\tau,f;N^{\alpha_1,\beta,\infty},N^{0,\beta,\infty}).$$

Therefore for $0<r<\infty$ we get

$$\left(\int_0^\infty\left(\tau^{-\theta}K(\tau,f;N^{\alpha_1,\beta,\infty},N^{0,\beta,\infty})\right)^r\frac{d\tau}{\tau}\right)^{\frac{1}{r}}$$
$$\geqslant 2^{-(\frac{1}{q}-1)_+}\left(\int_0^\infty\left(\tau^{-\theta}\sup_{t\leqslant v(\tau)}t^{\alpha_1}\sup_{Q\in G_t}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff\|_{L_q}\right)^r\frac{d\tau}{\tau}\right)^{\frac{1}{r}}$$
$$=2^{-(\frac{1}{q}-1)_+}\left(\alpha_1\int_0^\infty\left(u^{-\theta\alpha_1}\sup_{t\leqslant u}t^{\alpha_1}\sup_{Q\in G_u}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff\|_{L_q}\right)^r\frac{du}{u}\right)^{\frac{1}{r}}$$
$$\geqslant 2^{-(\frac{1}{q}-1)_+}\alpha_1^{\frac{1}{r}}\left(\int_0^\infty\left(u^{-\theta\alpha_1}\sup_{t\leqslant u}t^{\alpha_1}\sup_{Q\in G_u}\frac{1}{|Q|^{\beta}}\|F^{-1}\chi_Q Ff\|_{L_q}\right)^r\frac{du}{u}\right)^{\frac{1}{r}}$$

$$\geqslant 2^{-(\frac{1}{q}-1)_+}\alpha_1^{\frac{1}{r}}\|f\|_{N^{\alpha,\beta,r}(L_q)},$$

i.e.

$$(N^{\alpha_1,\beta,\infty}(L_q),N^{0,\beta,\infty}(L_q))_{\theta,r}\hookrightarrow N^{\alpha,\beta,r}(L_q).$$

The argument for $r=\infty$ is similar. Lemma 2 is proved. □

Let f be a Lebesgue measurable function on $\mathbb{R}$. The distribution function of f is defined by

$$m(\sigma,f)=|\{x\in\mathbb{R}:|f(x)|>\sigma|.$$

The function

$$f^*(t)=\inf\{\sigma\geqslant 0:m(\sigma,f)\leqslant t\}$$

is the non-increasing rearrangement of f.

Let $1\leqslant p<\infty, 0<\tau\leqslant\infty$. A function f belongs to the Lorentz space $L_{p\tau}$, if f is measurable on $\mathbb{R}$ and for $\tau<\infty$

$$\|f\|_{L_{p\tau}}=\left(\int_0^\infty\left(t^{\frac{1}{p}}f^*(t)\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}}<\infty$$

and for $\tau=\infty$

$$\|f\|_{L_{p\infty}}=\sup_{t>0}t^{\frac{1}{p}}f^*(t)<\infty.$$

THEOREM 1. *Let $1<p<q\leqslant\infty, 0<\tau\leqslant\infty$, and $0\leqslant\alpha<1-\frac{1}{p}+\frac{1}{q}$. Then*

$$L_{p\tau}\hookrightarrow N^{\alpha,\beta,\tau}(L_q),$$

where $\beta=\alpha+\frac{1}{p}-\frac{1}{q}$

Proof. Let $1\leqslant r<q\leqslant\infty$. It is well known that the characteristic function χ_Q of a segment Q is a Fourier multiplier from L_r to L_q. Moreover, there exists $c_4(r,q)>0$, depending only on r and q, such that

$$\left\|F^{-1}\chi_QFf\right\|_{L_q}\leqslant c_4(r,q)|Q|^{\frac{1}{r}-\frac{1}{q}}\|f\|_{L_r}, \tag{2}$$

for every $f\in L_r$.

Let $0\leqslant\alpha\leqslant 1-\frac{1}{p}$. Then $\frac{1}{p}-\frac{1}{q}<\beta\leqslant 1-\frac{1}{q}$ and there exists p_0 such that $1<p_0<p$ and $\beta=\frac{1}{p_0}-\frac{1}{q}$. By (2) with $r=p_0$

$$\|f\|_{N^{0,\beta,\infty}(L_q)}=\sup_{t>0}\sup_{Q\in G_t}\frac{1}{|Q|^\beta}\left\|F^{-1}\chi_QFf\right\|_{L_q}\leqslant c_5\|f\|_{L_{p_0}} \tag{3}$$

for every $f\in L_{p_0}$, where $c_5=c_4(p_0,q)$.

Next let $p<p_1<q$ and $\alpha_1=\frac{1}{p_0}-\frac{1}{p_1}$. Taking into account (2) with $r=p_1$, we get

$$\sup_{t>0}\sup_{Q\in G_t}\frac{1}{|Q|^{\frac{1}{p_1}-\frac{1}{q}}}\left\|F^{-1}\chi_QFf\right\|_{L_q}\leqslant c_6\|f\|_{L_{p_1}}$$

for every $f \in L_{p_1}$, where $c_6 = c_4(p_1, q)$.

Since

$$\sup_{t>0} \sup_{Q \in G_t} \frac{1}{|Q|^{\frac{1}{p_1} - \frac{1}{q}}} \left\|F^{-1}\chi_Q F f\right\|_{L_q} = \sup_{t>0} \sup_{Q \in G_t} \frac{|Q|^{\alpha_1}}{|Q|^{\beta}} \left\|F^{-1}\chi_Q F f\right\|_{L_q}$$
$$\geqslant \sup_{t>0} t^{\alpha_1} \sup_{Q \in G_t} \frac{1}{|Q|^{\beta}} \left\|F^{-1}\chi_Q F f\right\|_{L_q} = \|f\|_{N^{\alpha_1,\beta,\infty}(L_q)},$$

we have

$$\|f\|_{N^{\alpha_1,\beta,\infty}(L_q)} \leqslant c_6 \|f\|_{L_{p_1}} \tag{4}$$

for every $f \in L_{p_1}$. Inequalities (3) and (4) mean that

$$L_{p_0} \hookrightarrow N^{0,\beta,\infty}(L_q),$$

$$L_{p_1} \hookrightarrow N^{\alpha_1,\beta,\infty}(L_q),$$

and consequently

$$L_{p_0} + L_{p_1} \hookrightarrow N^{0,\beta,\infty}(L_q) + N^{\alpha_1,\beta,\infty}(L_q).$$

Let us denote by I the corresponding embedding operator. By (3) and (4) we have

$$I : L_{p_0} \to N^{0,\beta,\infty}(L_q),$$

and

$$I : L_{p_1} \to N^{\alpha_1,\beta,\infty}(L_q).$$

Moreover, in both cases the operator I is bounded.

Let $\theta \in (0,1)$ such, that $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_0}$. Since

$$\alpha = \beta - \frac{1}{p} + \frac{1}{q} = \frac{1}{p_0} - \frac{1}{p} = (1-\theta)\alpha_1.$$

By the interpolation properties of L_p we get

$$I : L_{p\tau} = \left(L_{p_1}, L_{p_0}\right)_{\theta\tau} \to \left(N^{\alpha_1,\beta,\infty}(L_q), N^{0,\beta,\infty}(L_q)\right)_{\theta\tau},$$

and the operator I is bounded. Thus,

$$L_{p\tau} \hookrightarrow \left(N^{\alpha_1,\beta,\infty}(L_q), N^{0,\beta,\infty}(L_q)\right)_{\theta\tau},$$

and hence the statement of theorem follows by Lemma 2.

If $1 - \frac{1}{p} \leqslant \alpha < 1 - \frac{1}{p} + \frac{1}{q}$, then the statement of theorem follows from (1). Indeed, let $0 < \tilde{\alpha} \leqslant 1 - \frac{1}{p}$, then by proved above and by (1) we have

$$L_{p\tau} \hookrightarrow N^{\tilde{\alpha},\tilde{\alpha}+\frac{1}{p}-\frac{1}{q},\tau}(L_q) \hookrightarrow N^{\alpha,\beta,\tau}(L_q).$$

Theorem 1 is proved. □

THEOREM 2. *Let* $1 < p < q \leqslant \infty$, $0 \leqslant \alpha < 1 - \frac{1}{p} + \frac{1}{q}$, $\beta = \alpha + \frac{1}{p} - \frac{1}{q}$. *If a function* $\varphi \in AC^{loc}(\mathbb{R} \setminus \{0\})$ *satisfies to the following conditions*

$$\sup_{y \in \mathbb{R}} |y|^{\frac{1}{p} - \frac{1}{q}} |\varphi(y)| \leqslant A$$

and

$$\sup_{t>0} t^{1-\alpha} \left(y^{\beta} \varphi'(y) \right)^{*}(t) \leqslant A,$$

then $\varphi \in m_p^q$ *and*

$$\|\varphi\|_{m_p^q} \leqslant c_7 A,$$

where $c_7 > 0$ *depends only on* p, q *and* α.

Proof. Let $f \in S$, $0 < a < b < \infty$. Since $\varphi \in AC([a,b])$, by applying integration by parts and the Minkowski inequalities for sums and integrals, we get

$$\begin{aligned}
\left\| \int_a^b \varphi(y)(Ff)(y) e^{iyz} dy \right\|_{L_q} &= \left\| \int_a^b \varphi(y) \left(\int_0^y (Ff)(\xi) e^{i\xi z} d\xi \right)_y' dy \right\|_{L_q} \\
&= \left\| \varphi(y) \int_0^y (Ff)(\xi) e^{i\xi z} d\xi \,|_a^b - \int_a^b \varphi'(y) \left(\int_0^y (Ff)(\xi) e^{i\xi z} d\xi \right) dy \right\|_{L_q} \\
&= 2\pi \left\| \varphi(y) \left(F^{-1} \chi_{[0,y]} Ff \right)(z)|_a^b - \int_a^b \varphi'(y) \left(F^{-1} \chi_{[0,y]} Ff \right)(z) dy \right\|_{L_q} \\
&\leqslant 2\pi (\left\| \varphi(a) \left(F^{-1} \chi_{[0,a]} Ff \right)(z) \right\|_{L_q} + \left\| \varphi(b) \left(F^{-1} \chi_{[0,b]} Ff \right)(z) \right\|_{L_q} \\
&\quad + \int_a^b |\varphi'(y)| \left\| F^{-1} \chi_{[0,y]} Ff \right\|_{L_q} dy) \equiv 2\pi \left(I_1 + I_2 + I_3 \right).
\end{aligned}$$

Taking into account (2) we have

$$I_1 = a^{\frac{1}{p} - \frac{1}{q}} |\varphi(a)| \frac{1}{a^{\frac{1}{p} - \frac{1}{q}}} \left\| F^{-1} \chi_{[0,a]} Ff \right\|_{L_q} \leqslant A c_8 \|f\|_{L_p},$$

where $c_8 = c_4(p,q)$.

Similarly,

$$I_2 \leqslant A c_8 \|f\|_{L_p}.$$

Furthermore,

$$\begin{aligned}
I_3 &= \int_a^b y^{\beta} |\varphi'(y)| \frac{1}{y^{\beta}} \left\| F^{-1} \chi_{[0,y]} Ff \right\|_{L_q} dy \\
&\leqslant \int_0^{\infty} y^{\beta} |\varphi'(y)| \left(\sup_{Q \in G_y} \frac{1}{|Q|^{\beta}} \left\| F^{-1} \chi_Q Ff \right\|_{L_q} \right) dy.
\end{aligned}$$

Since

$$\sup_{Q \in G_y} \frac{1}{|Q|^{\beta}} \left\| F^{-1} \chi_Q Ff \right\|_{L_q}$$

is a nonincreasing function, by using the inequality $\int_0^\infty FGdy \leqslant \int_0^\infty F^*G^*dy,$ we get

$$I_3 \leqslant \int_0^\infty \left(z^\beta \varphi'(z)\right)^*(y) \left(\sup_{Q\in G_y} \frac{1}{|Q|^\beta} \left\|F^{-1}\chi_Q Ff\right\|_{L_q}\right) dy$$
$$= \int_0^\infty y^{1-\alpha} \left(z^\beta \varphi'(z)\right)^*(y) y^\alpha \left(\sup_{Q\in G_y} \frac{1}{|Q|^\beta} \left\|F^{-1}\chi_Q Ff\right\|_{L_q}\right) \frac{dy}{y}$$
$$\leqslant A\|f\|_{N^{\alpha,\beta,1}(L_q)}.$$

By Theorem 1 there exists $c_9 > 0$, which depends only on p, q, α, such that

$$I_3 \leqslant Ac_9 \|f\|_{L_{p1}}.$$

Since $L_{p1} \hookrightarrow L_p$, there exists $c_{10} > 0$, depending only on p, such that

$$\|f\|_{L_p} \leqslant c_{10} \|f\|_{L_{p1}}.$$

Hence, there exists $c_{11} > 0$, depending only on p, q and α, such that

$$\left\|\int_a^b \varphi(y)(Ff)(y)e^{iyz}dy\right\|_{L_q} \leqslant c_{11}A\|f\|_{L_{p1}}$$

for every $f \in S$ and any $0 < a < b < \infty$.

Let couples of numbers (p_0, q_0) and (p_1, q_1) be such that $1 < p_0 < p < p_1 < \infty, 1 < q_0 < q < q_1 < \infty$ and

$$\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p_1} - \frac{1}{q_1} = \frac{1}{p} - \frac{1}{q}. \tag{5}$$

Similarly, one can prove that there exists $c_{12} > 0$ such that

$$\left\|\int_a^b \varphi(y)(Ff)(y)e^{iyz}dy\right\|_{L_{q_0}} \leqslant c_{12}A\|f\|_{L_{p_0 1}}$$

and

$$\left\|\int_a^b \varphi(y)(Ff)(y)e^{iyz}dy\right\|_{L_{q_1}} \leqslant c_{12}A\|f\|_{L_{p_1 1}}$$

for every $f \in S$. Choose $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. According to (5) we have

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{p_0} + \frac{1}{q_0} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} - \frac{1}{p_0} + \frac{1}{q_0}$$
$$= \theta\left(\frac{1}{p_1} - \frac{1}{p_0}\right) + \frac{1}{q_0} = \theta\left(\frac{1}{q_1} - \frac{1}{q_0}\right) + \frac{1}{q_0} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

By the Marcinkiewicz interpolation theorem we have that for every $0 < r < \infty$

$$\left\|\int_a^b \varphi(y)(Ff)(y)e^{iyz}dy\right\|_{L_{qr}} \leqslant c_{12}A\|f\|_{L_{pr}},$$

where $c_{12} > 0$ depends only on p, q, α and r.

If, in particular, $r = p$, then for some $c_{13} > 0$, which depends only on p, q and α,

$$\left\| \int_a^b \varphi(y)(Ff)(y)e^{iyz}dy \right\|_{L_q} \leqslant c_{13}A\, \|f\|_{L_p}$$

for all $f \in S$ and for all $0 < a < b < \infty$, because $L_{qp} \hookrightarrow L_q$ if $p < q$.

Since $|\varphi(y)| \leqslant A|y|^{-\frac{1}{p}+\frac{1}{q}}$, $y \in \mathbb{R} \setminus \{0\}$ and $Ff \in S$, the integral

$$\int_0^\infty \varphi(y)(Ff)(y)e^{iyz}dy$$

absolutely converges for every $z \in \mathbb{R}$.

Hence, for every $z \in \mathbb{R}$

$$\lim_{n\to\infty} \left| \int_{\frac{1}{n}}^n \varphi(y)(Ff)(y)e^{iyz}dy \right| = \left| \int_0^\infty \varphi(y)(Ff)(y)e^{iyz}dy \right|,$$

and by the Fatou theorem

$$\left\| \int_0^\infty \varphi(y)(Ff)(y)e^{iyz}dy \right\|_{L_q} \leqslant \sup_{n\in N} \left\| \int_{\frac{1}{n}}^n \varphi(y)(Ff)(y)e^{iyz}dy \right\|_{L_q} \leqslant c_{13}A\, \|f\|_{L_p}.$$

Similarly,

$$\left\| \int_{-\infty}^0 \varphi(y)(Ff)(y)e^{iyz}dy \right\|_{L_q} \leqslant c_{13}A\, \|f\|_{L_p},$$

and therefore

$$\left\|T_\varphi(f)\right\|_{L_q} = \left\|F^{-1}\varphi Ff\right\|_{L_q} = \frac{1}{2\pi} \left\| \int_{-\infty}^\infty \varphi(y)(Ff)(y)e^{iyz}dy \right\|_{L_q} \leqslant \frac{c_{13}A}{\pi} \|f\|_{L_p}$$

for every $f \in S$. This means that

$$\left\|T_\varphi(f)\right\|_{L_q} \leqslant \frac{c_{13}A}{\pi} \|f\|_{L_p}$$

for every $f \in L_p$. Hence $\varphi \in m_p^q$ and

$$\|\varphi\|_{m_p^q} \leqslant \frac{c_{13}A}{\pi}.$$

Theorem 2 is proved. $\square$

The function

$$f^\star(t) = \inf\{\sigma > 0 : \tilde{m}(\sigma, f) \geqslant t\},$$

where

$$\tilde{m}(\sigma, f) = |\{x \in \mathbb{R} : |f(x)| < \sigma|,$$

is the non-decreasing rearrangement of f.

LEMMA 3. *Let f,g be measurable functions on $\mathbb{R}$ such that $f^*(t), g^\star(t) < \infty$ for all $t > 0$. Then*

$$\sup_{t>0} f^*(t) g^\star(t) \leqslant \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)g(x)|.$$

Proof. Let $1 \leqslant p < \infty$. Since $(|g|^p)^\star(t) = (g^\star(t))^p, (|f|^p)^*(t) = (f^*(t))^p$, we have by the well-known inequality [2] that for every $a > 0$

$$\|f^* g^\star\|_{L_p(0,2a)} = \left\| \left(f\chi_{[-a,a]}\right)^* g^\star \right\|_{L_p(0,\infty)} \leqslant \left\| f\chi_{(-a,a)} g \right\|_{L_p(-\infty,\infty)} = \|fg\|_{L_p(-a,a)}.$$

Passing to the limit as $p \to +\infty$ by the Riesz lemma, we get

$$\operatorname*{ess\,sup}_{0<t<2a} f^*(t) g^\star(t) \leqslant \operatorname*{ess\,sup}_{-a<x<a} |f(x)g(x)|.$$

Passing to the limit as $a \to +\infty$, we obtain the required inequality because

$$\operatorname*{ess\,sup}_{t>0} f^*(t) g^\star(t) = \sup_{t>0} f^*(t) g^\star(t).$$

Lemma 3 is proved. □

REMARK 1. Note that the assumptions of Theorem 2 are weaker than the assumptions of the Lizorkin theorem, since

$$\sup_{t\in\mathbb{R}^+} t^{1-\alpha} \left(y^\beta \varphi'(y)\right)^* (t) \leqslant 2^{1-\alpha} \operatorname*{ess\,sup}_{y\in\mathbb{R}} |y|^{1+\frac{1}{p}-\frac{1}{q}} \left|\varphi'(y)\right|,$$

and there exists a function φ satisfying the assumptions of Theorem 2, but not satisfying the assumptions of the Lizorkin theorem, i.e.

$$\sup_{y\in\mathbb{R}} |y|^{\frac{1}{p}-\frac{1}{q}} |\varphi(y)| < \infty,$$

$$\sup_{t\in\mathbb{R}^+} t^{1-\alpha} \left(y^\beta \varphi'(y)\right)^* (t) < \infty,$$

but

$$\operatorname*{ess\,sup}_{y\in\mathbb{R}} |y|^{1+\frac{1}{p}-\frac{1}{q}} \left|\varphi'(y)\right| = \infty.$$

Since $\left(|y|^{1-\alpha}\right)^\star (t) = \left(\frac{t}{2}\right)^{1-\alpha}$, by Lemma 3 the first statement of the remark follows.

Let us prove the second statement. Let $1 > \gamma > \beta$ and

$$\mu_k = \left(k^{1-\beta} + k^{-\gamma} - (k+1)^{-\gamma}\right)^{\frac{1}{1-\beta}}, \ \ k \in N.$$

We define the function φ by the following formula

$$\varphi(x) = \begin{cases} k^{1-\beta} + k^{-\gamma} - x^{1-\beta}, & x \in [k, \mu_k], \\ (k+1)^{-\gamma}, & x \in [\mu_k, k+1], \end{cases} \ k \in N,$$

for $x \geqslant 1$. If $0 \leqslant x \leqslant 1$,

$$\varphi(x) = 1,$$

and if $x < 0$,

$$\varphi(x) = \varphi(-x).$$

The function $\varphi \in AC^{loc}(\mathbb{R} \setminus \{0\})$, because it is continious on $\mathbb{R}$, is even by definition and is absolutely continious on any interval $[0,1]$, $[k,\mu_k]$, $[\mu_k, k+1]$, where $k \in N$.

We show that the function φ satisfies the first condition. Let $x \in [k, k+1]$. Since $\gamma > \frac{1}{p} - \frac{1}{q}$ we have

$$|x|^{\frac{1}{p}-\frac{1}{q}} |\varphi(x)| \leqslant (k+1)^{\frac{1}{p}-\frac{1}{q}} \frac{1}{k^{\gamma}} \leqslant \frac{(k+1)^{\frac{1}{p}-\frac{1}{q}}}{k^{\frac{1}{p}-\frac{1}{q}}} \leqslant 2.$$

Note that

$$\left| x^{\beta} \varphi'(x) \right| = \begin{cases} 1-\beta, & x \in (k, \mu_k), \\ 0, & x \in (\mu_k, k+1), \end{cases} \quad k \in N.$$

If

$$I = \left\{ x : \left| x^{\beta} \varphi'(x) \right| = 1-\beta \right\},$$

then

$$|I| = \sum_{k=1}^{\infty} (\mu_k - k) = \sum_{k=1}^{\infty} k \left(\left(1 + \frac{1}{k^{1-\beta}} \left(k^{-\gamma} - (k+1)^{-\gamma} \right) \right)^{\frac{1}{1-\beta}} - 1 \right) < \infty.$$

Indeed, by using inequality $(1+x)^{\nu} - 1 \leqslant \nu 2^{\nu-1} x$ for $\nu \geqslant 1$ and $0 \leqslant x \leqslant 1$, we have

$$|I| \leqslant \frac{2^{\frac{\beta}{1-\beta}}}{1-\beta} \sum_{k=1}^{\infty} \frac{k}{k^{1-\beta}} \left(k^{-\gamma} - (k+1)^{-\gamma} \right).$$

Since the series $\sum_{k=1}^{\infty} k^{-(1-\beta+\gamma)}$ converges, by the limit comparison test the series

$$\sum_{k=1}^{\infty} \frac{k}{k^{1-\beta}} \left(k^{-\gamma} - (k+1)^{-\gamma} \right)$$

also converges.

Since the function $\left| x^{\beta} \varphi'(x) \right|$ takes only two values 0 and $1-\beta$, it follows that

$$\left(x^{\beta} \varphi'(x) \right)^{*}(t) = \begin{cases} 1-\beta, & x \in [0, |I|], \\ 0, & x > |I|. \end{cases}$$

Therefore,

$$\sup_{t>0} t^{1-\alpha} \cdot \left(x^{\beta} \varphi'(x) \right)^{*}(t) = \sup_{0<t\leqslant 2d} t^{1-\alpha} \cdot (1-\beta) = (1-\beta)(|I|)^{1-\alpha} < \infty,$$

i.e. the second condition holds.

On the other hand, if we choose an arbitrary sequence of points $\{x_k\}_{k=1}^{\infty}$ such that $x_k \in (k, \mu_k)$, then

$$\lim_{k\to\infty} |x_k|^{1+\frac{1}{p}-\frac{1}{q}} \cdot \left|\varphi'(x_k)\right| = \lim_{k\to\infty} |x_k|^{1+\frac{1}{p}-\frac{1}{q}-\beta} \cdot |x_k|^{\beta} \left|\varphi'(x_k)\right|$$
$$= \lim_{k\to\infty} |x_k|^{1+\frac{1}{p}-\frac{1}{q}-\beta} \cdot (1-\beta) = \infty.$$

Hence, the last condition holds.

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