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SOME INEQUALITIES FOR OPERATOR MEANS AND HADAMARD PRODUCT

JAGJIT SINGH MATHARU AND JASPAL SINGH AUJLA

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Abstract. We prove some general theorems which unify results on arithmetic-geometric mean and some other related matrix inequalities. As an application we obtain some results involving Hadamard product of matrices.

1. Introduction

With a view to studying electrical network connections, Anderson and Duffin [1] introduced the concept of parallel sum of two positive semidefinite matrices. If A and B are impedance matrices of two resistive n-port networks, then their parallel sum $A:B$ is the impedance matrix of the parallel connection. The notion of geometric mean of two positive operators was first introduced by Pusz and Wornowicz [11] and was further developed by Ando [2]. Thereafter Nishio and Ando [10] and Kubo and Ando [8] considered the axiomatic concept of connections and means for pairs of positive operators. Mond and Pečarić [9] proved the mixed arithmetic-geometric mean and harmonic-geometric mean inequalities:

$$A\sharp(A\nabla B) \geqslant A\nabla(A\sharp B)$$

and

$$A!(A\sharp B) \geqslant A\sharp(A!B)$$

for positive definite matrices A,B. Here $!, \sharp$ and ∇ stand for harmonic mean, geometric mean and arithmetic mean respectively and $X \geqslant Y$ means that $X-Y$ is positive semidefinite.

In Section 2, we shall consider mixed arithmetic-geometric and related inequalities in more general context. In Section 3, we shall apply results proved in Section 2, to obtain some inequalities involving Hadamard product of matrices. In Section 4, we shall note some comments on two conjectures made by Bhatia and Kittaneh in [5].

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2. Mixed Arithmetic-Geometric Mean

In what follows, the capital letters $A,B,C,\ldots$ denote $n\times n$ positive definite matrices over the algebra of complex numbers, unless mentioned otherwise. Let $f:(0,\infty)\to(0,\infty)$ be a function. Consider a binary operation σ_f among positive matrices defined as follows:

$$A\sigma_f B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}.$$

Here $f(A)$ is defined by familiar functional calculus. The function f is called operator monotone if $A\geqslant B$ implies $f(A)\geqslant f(B)$. σ_f is the operator connection considered by Kubo and Ando [8] in case f is an operator monotone function. If the function f is operator monotone, we say σ_f is an operator connection and if in addition $f(1)=1$, σ_f is an operator mean. The binary operation σ corresponding to the function $\frac{x}{f(x)}$ is denoted by $\sigma_f^{\perp}$ and is called the dual of σ_f.

The operator connection corresponding to operator monotone function $f(x)=s+tx$, $s,t>0$, is denoted by $\nabla_{s,t}$. $\nabla_{1/2,1/2}$ is called the arithmetic mean. The operator connection corresponding to the operator monotone function $x\to\frac{x}{s+tx}, s,t>0$, is denoted by $!_{s,t}$, $!_{1/2,1/2}$ is called the harmonic mean. If $f(x)=x^s, s\geqslant 0$, we denote σ_f by $\sharp_s$. The operator mean corresponding to the operator monotone function $x\to x^{1/2}$ is called the geometric mean. Note that the parallel sum is $A:B=\frac{1}{2}(A!B)$.

THEOREM 2.1. *Let f be a positive function on $(0,\infty)$ with $f(1)=1$ and $s,t>0$. Then*

(i) $A\sigma_f(A\nabla_{s,t}B)\geqslant A\nabla_{s,t}(A\sigma_f B)$ *if and only if* $f(s+tx)\geqslant s+tf(x)$.
(ii) $A\sigma_f(A\nabla_{s,t}B)\leqslant A\nabla_{s,t}(A\sigma_f B)$ *if and only if* $f(s+tx)\leqslant s+tf(x)$.

Proof. (i) Suppose $f(s+tx)\geqslant s+tf(x)$. Then it follows that

$$f(sI+tA^{-1/2}BA^{-1/2})\geqslant sI+tf(A^{-1/2}BA^{-1/2})$$

which implies

$$A^{1/2}f(A^{-1/2}(sA+tB)A^{-1/2})A^{1/2}\geqslant sA+tA^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}$$

i.e.

$$A\sigma_f(sA+tB)\geqslant sA+t(A\sigma_f B).$$

Thus

$$A\sigma_f(A\nabla_{s,t}B)\geqslant A\nabla_{s,t}(A\sigma_f B).$$

The converse follows by replacing A by 1 and B by x.

(ii) The proof of this is similar to that of (i), and is therefore, not included.

COROLLARY 2.2. *Let $0<s,t<1$. Then*

$$A\sharp_s(A\nabla_{t,1-t}B)\geqslant A\nabla_{t,1-t}(A\sharp_s B).$$

Proof. The function $f(x) = x^s, 0 < s < 1$ being the concave function satisfies $f(t+(1-t)x) \geqslant t+(1-t)f(x)$, and the corollary follows.

COROLLARY 2.3. *Let $s > 1$ and $0 < t < 1$. Then*

$$A\sharp_s(A\nabla_{t,1-t}B) \leqslant A\nabla_{t,1-t}(A\sharp_s B).$$

Proof. The proof follows as in Corollary 2.2, since the function $f(x) = x^s$, $s > 1$, is a convex function.

REMARK 2.4. The special case $s = t = \frac{1}{2}$ in Corollary 2.2 is Theorem 2 in [9].

THEOREM 2.5. *Let f be a positive function on $(0,\infty)$ with $f(x^{-1}) = (f(x))^{-1}$ and $s,t > 0$. Then*
(i) $A\sigma_f(A!_{s,t}B) \leqslant A!_{s,t}(A\sigma_f B)$ if and only if $f(s+tx) \geqslant s+tf(x)$.
(ii) $A\sigma_f(A!_{s,t}B) \geqslant A!_{s,t}(A\sigma_f B)$ if and only if $f(s+tx) \leqslant s+tf(x)$.

Proof. (i) First suppose $f(s+tx) \geqslant s+tf(x)$. Then

$$f(sI+tA^{1/2}B^{-1}A^{1/2}) \geqslant sI+tf(A^{1/2}B^{-1}A^{1/2}).$$

Hence

$$A^{-1/2}f(A^{1/2}(sA^{-1}+tB^{-1})A^{1/2})A^{-1/2} \geqslant sA^{-1}+tA^{-1/2}f(A^{1/2}B^{-1}A^{1/2})A^{-1/2}.$$

Consequently

$$A^{1/2}f(A^{-1/2}(sA^{-1}+tB^{-1})^{-1}A^{-1/2})A^{1/2} \leqslant [sA^{-1}+t(A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2})^{-1}]^{-1}$$

by using that the function $x \to -x^{-1}$ is operator monotone on $(0,\infty)$ and $f(x^{-1}) = (f(x))^{-1}$. This gives the desired inequality

$$A\sigma_f(A!_{s,t}B) \leqslant A!_{s,t}(A\sigma_f B).$$

The converse follows by replacing A by 1 and B by x. The proof for (ii) is similar.

COROLLARY 2.6. *(i) $A\sharp_s(A!_{t,1-t}B) \leqslant A!_{t,1-t}(A\sharp_s B)$, if $0 < s,t < 1$.*
(ii) $A\sharp_s(A!_{t,1-t}B) \geqslant A!_{t,1-t}(A\sharp_s B)$, if $s > 1$ and $0 < t < 1$.

Proof. The proof follows from the facts that the function $x \to x^s$ is concave for $0 < s < 1$ and is convex for $s > 1$.

REMARK 2.7. The special case $s = t = \frac{1}{2}$ of Corollary 2.6 (i) is Theorem 3 in [9].

The following Lemma is well known.

LEMMA 2.8. *Let X be selfadjoint. Then the matrix $\begin{pmatrix} A & X \\ X & B \end{pmatrix}$ is positive semidefinite if and only if $XA^{-1}X \leqslant B$.*

THEOREM 2.9. *Let f, g be positive functions on $(0,\infty)$ and let $h(x) = (f(x)g(x))^{1/2}$. Then $A\sigma_h B$ is the maximum of all selfadjoint X for which $\begin{pmatrix} A\sigma_f B & X \\ X & A\sigma_g B \end{pmatrix}$ is positive semidefinite.*

Proof. Since

$$\begin{aligned} A\sigma_g B &= A^{1/2}g(A^{-1/2}BA^{-1/2})A^{1/2} \\ &= (A^{1/2}(f(A^{-1/2}BA^{-1/2})g(A^{-1/2}BA^{-1/2}))^{1/2}A^{1/2})(A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2})^{-1} \\ &\quad \times (A^{1/2}(f(A^{-1/2}BA^{-1/2})g(A^{-1/2}BA^{-1/2}))^{1/2}A^{1/2}) \\ &= (A\sigma_h B)(A\sigma_f B)^{-1}(A\sigma_h B), \end{aligned}$$

it follows from Lemma 2.8 that $\begin{pmatrix} A\sigma_f B & A\sigma_h B \\ A\sigma_h B & A\sigma_g B \end{pmatrix}$ is positive semidefinite. Now for selfadjoint Y, suppose that $\begin{pmatrix} A\sigma_f B & Y \\ Y & A\sigma_g B \end{pmatrix}$ is positive semidefinite. This implies

$$\begin{aligned} &\begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix} \begin{pmatrix} A\sigma_f B & Y \\ Y & A\sigma_g B \end{pmatrix} \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix} \\ &\qquad = \begin{pmatrix} f(A^{-1/2}BA^{-1/2}) & A^{-1/2}YA^{-1/2} \\ A^{-1/2}YA^{-1/2} & g(A^{-1/2}BA^{-1/2}) \end{pmatrix} \end{aligned}$$

is positive semidefinite. Hence again by Lemma 2.8,

$$A^{-1/2}YA^{-1/2}(f(A^{-1/2}BA^{-1/2}))^{-1}A^{-1/2}YA^{-1/2} \leqslant g(A^{-1/2}BA^{-1/2}),$$

which further implies

$$\begin{aligned} &[(f(A^{-1/2}BA^{-1/2}))^{-1/2}A^{-1/2}YA^{-1/2}(f(A^{-1/2}BA^{-1/2}))^{-1/2}]^2 \\ &\qquad \leqslant (f(A^{-1/2}BA^{-1/2}))^{-1}g(A^{-1/2}BA^{-1/2}). \end{aligned}$$

Using that the function $x \to x^{1/2}$ is operator monotone, we get

$$\begin{aligned} &(f(A^{-1/2}BA^{-1/2}))^{-1/2}A^{-1/2}YA^{-1/2}(f(A^{-1/2}BA^{-1/2}))^{-1/2} \\ &\qquad \leqslant (f(A^{-1/2}BA^{-1/2}))^{-1/2}(g(A^{-1/2}BA^{-1/2}))^{1/2}. \end{aligned}$$

Consequently,

$$Y \leqslant A^{1/2}\left(f(A^{-1/2}BA^{-1/2})g(A^{-1/2}BA^{-1/2})\right)^{1/2}A^{1/2} = A\sigma_h B.$$

This completes a proof.

COROLLARY 2.10. [2] *$A\sharp_{1/2}B$ is the maximum of all selfadjoint X for which $\begin{pmatrix} A & X \\ X & B \end{pmatrix}$ is positive semidefinite.*

Proof. The proof follows on taking $f(x) = 1$ and $g(x) = x$ in the Theorem 2.9.
For a proof of the following lemma reader is referred to [2].

LEMMA 2.11. *The harmonic mean $A!B$ is the maximum of all selfadjoint X for which*

$$\begin{pmatrix} 2A & 0 \\ 0 & 2B \end{pmatrix} \geqslant \begin{pmatrix} X & X \\ X & X \end{pmatrix}.$$

THEOREM 2.12. *Let f,g be positive functions on $(0,\infty)$ and let $h(x) = [(f(x))^{-1} + (g(x))^{-1}]^{-1}$. Then $A\sigma_h B$ is the maximum of all selfadjoint X for which*

$$\begin{pmatrix} A\sigma_f B & 0 \\ 0 & A\sigma_g B \end{pmatrix} \geqslant \begin{pmatrix} X & X \\ X & X \end{pmatrix}.$$

Proof. Since

$$\begin{aligned}
A\sigma_h B &= A^{1/2}h(A^{-1/2}BA^{-1/2})A^{1/2} \\
&= A^{1/2}(f(A^{-1/2}BA^{-1/2})^{-1} + g(A^{-1/2}BA^{-1/2})^{-1})^{-1}A^{1/2} \\
&= A^{1/2}(f(A^{-1/2}BA^{-1/2}) - f(A^{-1/2}BA^{-1/2})(f(A^{-1/2}BA^{-1/2}) \\
&\quad + g(A^{-1/2}BA^{-1/2}))^{-1}f(A^{-1/2}BA^{-1/2}))A^{1/2} \\
&= A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2} - A^{1/2}f(A^{-1/2}BA^{-1/2})(f(A^{-1/2}BA^{-1/2}) \\
&\quad + g(A^{-1/2}BA^{-1/2}))^{-1}f(A^{-1/2}BA^{-1/2})A^{1/2} \\
&= A\sigma_f B - (A\sigma_f B)A^{-1/2}(f(A^{-1/2}BA^{-1/2}) + g(A^{-1/2}BA^{-1/2}))^{-1}A^{-1/2}(A\sigma_f B) \\
&= A\sigma_f B - (A\sigma_f B)(A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2} + A^{1/2}g(A^{-1/2}BA^{-1/2})A^{1/2})^{-1}(A\sigma_f B) \\
&= A\sigma_f B - (A\sigma_f B)(A\sigma_f B + A\sigma_g B)^{-1}(A\sigma_f B) \\
&= A\sigma_f B : A\sigma_g B.
\end{aligned}$$

Hence the result follows from Lemma 2.11.

3. Some Applications

In this section, we apply results proved in Section 2, to obtain inequalities involving tensor and Hadamard product of matrices. For $A = (a_{ij})$ and $B = (b_{ij})$, $A \circ B = (a_{ij}b_{ij})$ denotes the Hadamard product of A and B. The tensor product $A \otimes B$ is the $n^2 \times n^2$ matrix

$$\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}. \tag{1}$$

However, if $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ are block matrices, then it is more convenient to represent tensor product $A \otimes B$ as the block matrix

$$\begin{pmatrix} A_{11} \otimes B_{11} & A_{11} \otimes B_{12} & A_{12} \otimes B_{11} & A_{12} \otimes B_{12} \\ A_{11} \otimes B_{21} & A_{11} \otimes B_{22} & A_{12} \otimes B_{21} & A_{12} \otimes B_{22} \\ A_{21} \otimes B_{11} & A_{21} \otimes B_{12} & A_{22} \otimes B_{11} & A_{22} \otimes B_{12} \\ A_{21} \otimes B_{21} & A_{21} \otimes B_{22} & A_{22} \otimes B_{21} & A_{22} \otimes B_{22} \end{pmatrix}. \tag{2}$$

The definition according to (2) is unitarily equivalent to the definition according to (1) and therefore will be used here. It is well known that the Hadamard product $A \circ B$ is the principal submatrix of the tensor product $A \otimes B$ and is monotone in the sense that $A \geqslant B$, $C \geqslant D$ implies $A \circ C \geqslant B \circ D$. The following theorem is an analogue of Theorem 4.1 in [3].

THEOREM 3.1.

(i) $\quad \left(\left(\sum_{i=1}^k A_i\right) \circ \left(\sum_{i=1}^k C_i\right)\right) : \left(\left(\sum_{i=1}^k B_i\right) \circ \left(\sum_{i=1}^k D_i\right)\right) \geqslant \left(\sum_{i=1}^k (A_i : B_i)\right) \circ \left(\sum_{i=1}^k (C_i : D_i)\right).$

(ii) $\quad \left(\sum_{i=1}^k A_i\right) \circ \left(\sum_{i=1}^k B_i\right) \geqslant \left(\sum_{i=1}^k (A_i + B_i)\right) \circ \left(\sum_{i=1}^k (A_i : B_i)\right).$

Proof. (i) By Lemma 2.11,

$$\begin{pmatrix} \sum_{i=1}^k A_i & 0 \\ 0 & \sum_{i=1}^k B_i \end{pmatrix} \geqslant \begin{pmatrix} \sum_{i=1}^k (A_i : B_i) & \sum_{i=1}^k (A_i : B_i) \\ \sum_{i=1}^k (A_i : B_i) & \sum_{i=1}^k (A_i : B_i) \end{pmatrix}$$

and

$$\begin{pmatrix} \sum_{i=1}^k C_i & 0 \\ 0 & \sum_{i=1}^k D_i \end{pmatrix} \geqslant \begin{pmatrix} \sum_{i=1}^k (C_i : D_i) & \sum_{i=1}^k (C_i : D_i) \\ \sum_{i=1}^k (C_i : D_i) & \sum_{i=1}^k (C_i : D_i) \end{pmatrix}.$$

Since the Hadamard product of matrices is monotone, it follows that

$$\begin{pmatrix} \sum_{i=1}^k A_i \circ \sum_{i=1}^k C_i & 0 \\ 0 & \sum_{i=1}^k B_i \circ \sum_{i=1}^k D_i \end{pmatrix}$$
$$\geqslant \begin{pmatrix} \sum_{i=1}^k (A_i : B_i) \circ \sum_{i=1}^k (C_i : D_i) & \sum_{i=1}^k (A_i : B_i) \circ \sum_{i=1}^k (C_i : D_i) \\ \sum_{i=1}^k (A_i : B_i) \circ \sum_{i=1}^k (C_i : D_i) & \sum_{i=1}^k (A_i : B_i) \circ \sum_{i=1}^k (C_i : D_i) \end{pmatrix}.$$

Hence again by Lemma 2.11, we get

$$\left(\left(\sum_{i=1}^k A_i\right) \circ \left(\sum_{i=1}^k C_i\right)\right) : \left(\left(\sum_{i=1}^k B_i\right) \circ \left(\sum_{i=1}^k D_i\right)\right) \geqslant \left(\sum_{i=1}^k (A_i : B_i)\right) \circ \left(\sum_{i=1}^k (C_i : D_i)\right).$$

(ii) By Lemma 2.11,

$$\begin{pmatrix} \sum_{i=1}^k A_i - \sum_{i=1}^k (A_i : B_i) & -\sum_{i=1}^k (A_i : B_i) \\ -\sum_{i=1}^k (A_i : B_i) & \sum_{i=1}^k B_i - \sum_{i=1}^k (A_i : B_i) \end{pmatrix} \geqslant 0.$$

Similarly,

$$\begin{pmatrix} \sum_{i=1}^k B_i - \sum_{i=1}^k (A_i : B_i) & -\sum_{i=1}^k (A_i : B_i) \\ -\sum_{i=1}^k (A_i : B_i) & \sum_{i=1}^k A_i - \sum_{i=1}^k (A_i : B_i) \end{pmatrix} \geqslant 0.$$

Consequently,

$$\begin{pmatrix} Z & \sum_{i=1}^k (A_i : B_i) \circ \sum_{i=1}^k (A_i : B_i) \\ \sum_{i=1}^k (A_i : B_i) \circ \sum_{i=1}^k (A_i : B_i) & Z \end{pmatrix} \geqslant 0,$$

where

$$Z = \left(\sum_{i=1}^k A_i - \sum_{i=1}^k (A_i : B_i)\right) \circ \left(\sum_{i=1}^k B_i - \sum_{i=1}^k (A_i : B_i)\right).$$

Hence we have

$$\left(\sum_{i=1}^k A_i - \sum_{i=1}^k (A_i : B_i)\right) \circ \left(\sum_{i=1}^k B_i - \sum_{i=1}^k (A_i : B_i)\right) \geqslant \left(\sum_{i=1}^k (A_i : B_i)\right) \circ \left(\sum_{i=1}^k (A_i : B_i)\right),$$

which implies

$$\left(\sum_{i=1}^k A_i\right) \circ \left(\sum_{i=1}^k B_i\right) \geqslant \left(\sum_{i=1}^k (A_i + B_i)\right) \circ \left(\sum_{i=1}^k (A_i : B_i)\right).$$

COROLLARY 3.2.

$$\left(\sum_{i=1}^k A_i\right) \circ \left(\sum_{i=1}^k A_i^{-1}\right) \geqslant \left(\sum_{i=1}^k (A_i + A_i^{-1})\right) \circ \left(\sum_{i=1}^k (A_i + A_i^{-1})^{-1}\right).$$

Proof. The proof follows by taking $B_i = A_i^{-1}$ in Theorem 3.1 (ii).

REMARK 3.3. When the particular case $k = 1$ in Corollary 3.2 is combined with Fiedler's Theorem $A \circ A^{-1} \geqslant I$ [7], we have

$$A \circ A^{-1} \geqslant (A + A^{-1}) \circ (A + A^{-1})^{-1} \geqslant I.$$

Thus we are getting a better lower bound for $A \circ A^{-1}$ than that in Fiedler's Theorem. For other generalizations of Fiedler's Theorem the reader is referred to [3, 4].

THEOREM 3.4. *Let f, g be positive functions on $(0, \infty)$ and $h(x) = (f(x)g(x))^{1/2}$. Then*

$$2(A\sigma_h B) \otimes (A\sigma_h B) \leqslant (A\sigma_f B) \otimes (A\sigma_g B) + (A\sigma_g B) \otimes (A\sigma_f B).$$

Proof. By Theorem 2.9 we have

$$\begin{pmatrix} A\sigma_f B & A\sigma_h B \\ A\sigma_h B & A\sigma_g B \end{pmatrix} \geqslant 0.$$

Similarly

$$\begin{pmatrix} A\sigma_g B & A\sigma_h B \\ A\sigma_h B & A\sigma_f B \end{pmatrix} \geqslant 0.$$

These implies

$$\begin{pmatrix} (A\sigma_f B) \otimes (A\sigma_g B) & (A\sigma_h B) \otimes (A\sigma_h B) \\ (A\sigma_h B) \otimes (A\sigma_h B) & (A\sigma_g B) \otimes (A\sigma_f B) \end{pmatrix} \geqslant 0 \tag{3}$$

and

$$\begin{pmatrix} (A\sigma_g B) \otimes (A\sigma_f B) & (A\sigma_h B) \otimes (A\sigma_h B) \\ (A\sigma_h B) \otimes (A\sigma_h B) & (A\sigma_f B) \otimes (A\sigma_g B) \end{pmatrix} \geqslant 0. \tag{4}$$

Adding (3) and (4), we have

$$\begin{pmatrix} (A\sigma_g B)\otimes(A\sigma_f B)+(A\sigma_f B)\otimes(A\sigma_g B) & 2(A\sigma_h B)\otimes(A\sigma_h B) \\ 2(A\sigma_h B)\otimes(A\sigma_h B) & (A\sigma_f B)\otimes(A\sigma_g B)+(A\sigma_g B)\otimes(A\sigma_f B) \end{pmatrix} \geqslant 0.$$

This implies

$$2(A\sigma_h B)\otimes(A\sigma_h B) \leqslant (A\sigma_f B)\otimes(A\sigma_g B)+(A\sigma_g B)\otimes(A\sigma_f B).$$

The following corollary is one of the main results in [12].

COROLLARY 3.5. *Let f,g be positive functions on $(0,\infty)$. Then*

$$2(A\sharp_{1/2}B)\otimes(A\sharp_{1/2}B) \leqslant (A\sigma_f B)\otimes(A\sigma_f^{\perp} B)+(A\sigma_f^{\perp} B)\otimes(A\sigma_f B).$$

Proof. Taking $g(x)=x(f(x))^{-1}$ in Theorem 3.4 we get the result.
Since Hadamard product $A\circ B$ is the principal submatrix of the tensor product $A\otimes B$, we have the following corollary.

COROLLARY 3.6. *Let f,g be positive functions on $(0,\infty)$ and $h(x)=(f(x)g(x))^{1/2}$. Then*

(i) $(A\sigma_h B)\circ(A\sigma_h B)\leqslant(A\sigma_f B)\circ(A\sigma_g B)$.
(ii) $(A\sharp_{1/2}B)\circ(A\sharp_{1/2}B)\leqslant(A\sigma_f B)\circ(A\sigma_f^{\perp} B)$.

The special case of Corollary 3.6 (ii) when $f(x)=\dfrac{1+x}{2}$ is proved in [3].

4. Bhatia and Kittaneh conjectures

Bhatia and Kittaneh [5] proved the inequalities

$$tr(AB)\leqslant tr\left(\frac{A+B}{2}\right)^2$$

and

$$tr(A^2B^2)\leqslant tr\left(\frac{A+B}{2}\right)^4$$

and conjectured that

$$tr(A^mB^m)\leqslant tr\left(\frac{A+B}{2}\right)^{2m},\ m=1,2,\ldots.$$

(Here $tr(X)$ denotes trace of X.) Taking $A=\begin{pmatrix}4&-5\\-5&7\end{pmatrix}$, $B=\begin{pmatrix}9&-1\\-1&1\end{pmatrix}$ and $m=3$ shows that the inequality

$$tr(A^3B^3)\leqslant tr\left(\frac{A+B}{2}\right)^6$$

does not hold. However here we prove a related result (Corollary 4.2). Bhatia and Kittaneh also proved that

$$\left\| \sqrt{|AB|} \right\| \leqslant \left\| \frac{A+B}{2} \right\|$$

for some unitarily invariant norms (for example p-norms for $p \geqslant 2$ and trace norm) and conjectured that it is true for all unitarily invariant norms. Here we also prove a companion result when $A \geqslant 3B$. Let us denote by $s_1(X) \geqslant s_2(X) \geqslant \ldots \geqslant s_n(X)$ the singular value of X and by $\lambda_1(X) \geqslant \lambda_2(X) \geqslant \ldots \geqslant \lambda_n(X)$ the eigenvalues of X arranged in decreasing order when these are all real.

THEOREM 4.1. [6] *Let X be a $n \times n$ matrix over complex numbers. Then*

$$tr((X^*)^m X^m) \leqslant tr(X^*X)^m$$

for all positive integers m.

Proof. By the submultiplicativity of operator norm we have

$$||X^m|| \leqslant ||X||^m$$

that is,

$$s_1(X^m) \leqslant s_1^m(X).$$

Taking square on both sides, we get

$$s_1^2(X^m) \leqslant s_1^{2m}(X),$$

which further implies

$$s_1((X^*)^m X^m) \leqslant s_1^m(X^*X)$$

i.e.,

$$\lambda_1((X^*)^m X^m) \leqslant \lambda_1^m(X^*X).$$

Let $1 \leqslant k \leqslant n$. Replacing X by $\wedge^k X$, the antisymmetric tensor power of X, we get

$$\prod_{j=1}^{k} \lambda_j((X^*)^m X^m) \leqslant \prod_{j=1}^{k} \lambda_j^m(X^*X), \ 1 \leqslant k \leqslant n. \tag{5}$$

This implies

$$tr((X^*)^m X^m) \leqslant tr(X^*X)^m.$$

COROLLARY 4.2. *Let A, B be selfadjoint. Then*

$$tr(((AB)^*)^m (AB)^m) \leqslant tr(A^2B^2)^m$$

for all positive integers m.

Proof. In Theorem 4.1 by replacing X by AB we get the desired result.

COROLLARY 4.3. *Let A,B be such that $A \geqslant 3B$ (or $B \geqslant 3A$). Then*

$$\prod_{j=1}^{k} \sqrt{\lambda_j(AB)} \leqslant \prod_{j=1}^{k} \lambda_j \left(\frac{A+B}{2}\right), \ 1 \leqslant k \leqslant n.$$

Proof. Taking $X = \begin{pmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{pmatrix}$ and $m = 2$ in inequality (5) we have

$$\prod_{j=1}^{k} \lambda_j(A^2 + A^{1/2}BA^{1/2}) \leqslant \prod_{j=1}^{k} \lambda_j((A+B)^2)$$

i.e.,

$$\prod_{j=1}^{k} \lambda_j(A^{1/2}(A+B)A^{1/2}) \leqslant \prod_{j=1}^{k} \lambda_j((A+B)^2).$$

Since $A \geqslant 3B$, we have $A^{1/2}(A+B)A^{1/2} \geqslant 4A^{1/2}BA^{1/2}$. Therefore, the above inequality implies

$$\prod_{j=1}^{k} \lambda_j(4AB) \leqslant \prod_{j=1}^{k} \lambda_j(A+B)^2,$$

which gives the desired result.

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Jagjit Singh Matharu
Department of Mathematics
National Institute of Technology
Jalandhar 144011, Punjab
India
e-mail: matharujs@yahoo.com

Jaspal Singh Aujla
Department of Mathematics
National Institute of Technology
Jalandhar 144011, Punjab
India
e-mail: aujlajs@yahoo.com