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A PRIORI ESTIMATES FOR ELLIPTIC EQUATIONS IN WEIGHTED SOBOLEV SPACES

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Abstract. In this paper we prove some a priori bounds for the solutions of the Dirichlet problem for elliptic equations with singular coefficients in weighted Sobolev spaces.

1. Introduction

Consider the Dirichlet problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ Lu = f, \quad f \in L^p(\Omega), \end{cases} \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^n$, $n \geqslant 3$, $p \in]1,+\infty[$ and L is the uniformly elliptic differential operator defined by

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a. \tag{1.2}$$

If Ω is bounded and sufficiently regular, problem (1.1) has been widely investigated by several authors under various hypotheses on the leading coefficients a_{ij}. Here we just refer to [10] and [11] where some $W^{2,p}$-bounds for the solutions of problem (1.1) as well as related existence and uniqueness results have been obtained, assuming the a_{ij}'s to be in VMO and $a_i = a = 0$. This latter condition has been removed in [15] and [16]. Recently, these results have been extended to the case of unbounded open sets in [7] and [8].

In this paper we extend the above quoted results to the case of open sets with singular boundary and coefficients are singular near a subset of $\partial\Omega$. More precisely, given a suitable weight function ρ and denoted by S_ρ the subset of $\partial\Omega$ where ρ goes to zero, we assume that there exist extensions a_{ij}° of a_{ij} in $VMO(\Omega_\circ) \cap L^\infty(\Omega_\circ)$ - where $\Omega_\circ$ is a regular open set containing Ω -, and the functions $(a_{ij})_{x_h}$, a_i and a are singular near S_ρ. We prove that the following a priori bound holds:

$$||u||_{W^{2,p}_s(\Omega)} \leqslant c \left(||Lu||_{L^p_s(\Omega)} + ||u||_{L^p(\Omega_1)} \right), \tag{1.3}$$

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$$\forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_{s-1}^{1,p}(\Omega), \quad \forall p \in [2, +\infty[,$$

where $c \in \mathbb{R}_+$ is independent of u, Ω_1 is a bounded open subset of Ω, $s \in \mathbb{R}$, $W_s^{2,p}(\Omega)$, $\overset{\circ}{W}{}_{s-1}^{1,p}(\Omega)$ and $L_s^p(\Omega)$ are some weighted Sobolev spaces and the weight functions are suitable powers of ρ.

2. Notation and function spaces

Let G be a Lebesgue measurable subset of $\mathbb{R}^n$ and $\Sigma(G)$ be the collection of its Lebesgue measurable subsets. If $F \in \Sigma(G)$, we denote by $|F|$ the Lebesgue measure of F and by $\mathfrak{D}(F)$ (resp. $\mathfrak{D}^0(F)$) the class of restrictions to F of functions $\zeta \in C_\circ^\infty(\mathbb{R}^n)$ (resp. $\zeta \in C_\circ^0(\mathbb{R}^n)$) such that $\bar{F} \cap \operatorname{supp} \zeta \subseteq F$. Moreover, for any function space $X(F)$, $X_{\text{loc}}(F)$ stands for the class of all functions $g : F \to \mathbb{R}$ such that $\zeta g \in X(F)$ for each $\zeta \in \mathfrak{D}(F)$.

Let Ω be an open subset of $\mathbb{R}^n$. Put

$$\Omega(x,r) = \Omega \cap B(x,r) \qquad \forall x \in \mathbb{R}^n, \quad \forall r \in \mathbb{R}_+,$$

where $B(x,r)$ is the open ball of radius r centered at x.

Denote by $\mathscr{A}(\Omega)$ the class of all measurable functions $\rho : \Omega \to \mathbb{R}_+$ such that

$$\gamma^{-1} \rho(y) \leqslant \rho(x) \leqslant \gamma \rho(y) \qquad \forall y \in \Omega, \quad \forall x \in \Omega(y, \rho(y)), \tag{2.1}$$

where $\gamma \in \mathbb{R}_+$ is independent of x and y. Given $\rho \in \mathscr{A}(\Omega)$, we define

$$S_\rho = \{z \in \partial\Omega : \lim_{x \to z} \rho(x) = 0\}.$$

It is known that

$$\rho \in L_{\text{loc}}^\infty(\bar{\Omega}), \qquad \rho^{-1} \in L_{\text{loc}}^\infty(\bar{\Omega} \setminus S_\rho), \tag{2.2}$$

and, if $S_\rho \neq \emptyset$,

$$\rho(x) \leqslant \operatorname{dist}(x, S_\rho) \qquad \forall x \in \Omega \tag{2.3}$$

(see [9], [13]).

If $r \in \mathbb{N}$, $1 \leqslant p \leqslant +\infty$, $s \in \mathbb{R}$ and $\rho \in \mathscr{A}(\Omega)$, consider the space $W_s^{r,p}(\Omega)$ of distributions u on Ω such that $\rho^{s+|\alpha|-r} \partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leqslant r$. Equipped with the norm

$$\|u\|_{W_s^{r,p}(\Omega)} = \sum_{|\alpha| \leqslant r} \|\rho^{s+|\alpha|-r} \partial^\alpha u\|_{L^p(\Omega)},$$

$W_s^{k,p}(\Omega)$ is a Banach space, separable if $1 \leqslant p < +\infty$, Hilbert space for $p = 2$ (see, for instance, [12], [2] and [14]).

Moreover, denote by $\overset{\circ}{W}{}_s^{r,p}(\Omega)$ the closure of $C_\circ^\infty(\Omega)$ in $W_s^{r,p}(\Omega)$ and put $W_s^{0,p}(\Omega) = L_s^p(\Omega)$.

We now recall the definitions of some function spaces in which the coefficients of the operator will be chosen. For $p \in [1,+\infty[$, let $K^p_s(\Omega)$ be the set of all functions $g \in L^p_{\mathrm{loc}}(\bar\Omega \setminus S_\rho)$ such that

$$\|g\|_{K^p_s(\Omega)} = \sup_{x\in\Omega} \Big(\rho^{s-n/p}(x)\, \|g\|_{L^p(\Omega(x,\rho(x)))}\Big) < +\infty, \tag{2.4}$$

endowed with the norm defined by (2.4). Obviously the space $K^p_s(\Omega)$ is a Banach space containing $L^\infty_s(\Omega)$ and $C^\infty_o(\Omega)$ as well (see [3]). Therefore we denote by $\tilde K^p_s(\Omega)$ (resp. $\overset{\circ}{K}{}^p_s(\Omega)$) the closure of $L^\infty_s(\Omega)$ (resp. $C^\infty_\circ(\Omega)$) in $K^p_s(\Omega)$.

For any function g belonging to $\tilde K^p_s(\Omega)$ or to $\overset{\circ}{K}{}^p_s(\Omega)$, it can be defined a modulus of continuity. To this aim, we need to introduce a suitable sequence of functions. It is well known that, if $S_\rho \neq \emptyset$, then there exists a function $\alpha \in C^\infty(\Omega)\cap C^{0,1}(\bar\Omega)$ which is equivalent to dist $(\cdot, S_\rho)$ (see for instance [17]). For any fixed $f \in \mathfrak{D}(\bar{\mathbb{R}}_+)$ satisfying the condition

$$0 \leqslant f \leqslant 1, \qquad f(t) = 1 \quad \text{if} \quad t \leqslant \frac{1}{2}, \qquad f(t) = 0 \quad \text{if} \quad t \geqslant 1,$$

we can consider the sequence

$$\psi_h : x \in \bar\Omega \to \Big(1 - f\big(h\,\alpha(x)\big)\Big) f\big(|x|/2h\big), \qquad h \in \mathbb{N}.$$

It is easy to prove that each ψ_h belongs to $\mathfrak{D}(\bar\Omega \setminus S_\rho)$ and

$$0 \leqslant \psi_h \leqslant 1, \qquad \psi_{h_{|\bar G_h}} = 1, \qquad \operatorname{supp}\psi_h \subset \bar G_{2h},$$

where $G_h = \{x\in\Omega \,:\, |x| < h,\ \alpha(x) > 1/h\}$. Now, it is useful to recall that for a function $g \in K^p_s(\Omega)$ the following characterizations hold (see [3]):

$$g \in \tilde K^p_s(\Omega) \ \Leftrightarrow\ \lim_{h\to+\infty} \sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega} \frac{|\Omega(x)\cap E|}{\rho^n(x)} \leqslant 1/h}} \|\chi_E\, g\|_{K^p_s(\Omega)} = 0,$$

$$g \in \overset{\circ}{K}{}^p_s(\Omega) \ \Leftrightarrow\ \lim_{h\to+\infty} \|(1-\psi_h)\, g\|_{K^p_s(\Omega)} = 0,$$

where $\Omega(x) = \Omega(x,\rho(x))$ and χ_E is the characteristic function of E. Thus a *modulus of continuity* of $g \in \tilde K^p_s(\Omega)$ is defined as a function $\tilde\omega^p_s[g] : \mathbb{N} \to \mathbb{R}_+$ such that

$$\sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega} \frac{|\Omega(x)\cap E|}{\rho^n(x)} \leqslant 1/h}} \|\chi_E\, g\|_{K^p_s(\Omega)} \leqslant \tilde\omega^p_s[g](h) \quad \forall h\in\mathbb{N}, \quad \lim_{h\to+\infty} \tilde\omega^p_s[g](h) = 0$$

and a *modulus of continuity* of $g \in \overset{\circ}{K}{}^p_s(\Omega)$ is a function $\overset{\circ}{\omega}{}^p_s[g] : \mathbb{N} \to \mathbb{R}_+$ such that

$$\|(1-\psi_h)\, g\|_{K^p_s(\Omega)} + \sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega} |\Omega(x)\cap E| \leqslant 1/h}} \|\chi_E\, g\|_{K^p_s(\Omega)} \leqslant \overset{\circ}{\omega}{}^p_s[g](h) \quad \forall h\in\mathbb{N},$$

$$\lim_{h\to+\infty} \overset{\circ}{\omega}{}^p_s[g](h) = 0 .$$

Further properties of the spaces $K^p_s(\Omega)$, $\tilde{K}^p_s(\Omega)$ and $\overset{\circ}{K}{}^p_s(\Omega)$ can be found in [3] and [9].

If Ω has the property

$$|\Omega(x,r)| \geqslant A\, r^n \qquad \forall x \in \Omega, \quad \forall r \in]0,1], \tag{2.5}$$

where A is a positive constant independent of x and r, it is possible to consider the space $BMO(\Omega,t)$ $(t \in \mathbb{R}_+)$ of functions $g \in L^1_{\text{loc}}(\bar{\Omega})$ such that

$$[g]_{BMO(\Omega,t)} = \sup_{\substack{x\in\Omega \\ r\in]0,t]}} \fint_{\Omega(x,r)} |g - \fint_{\Omega(x,r)} g| < +\infty,$$

where

$$\fint_{\Omega(x,r)} g = |\Omega(x,r)|^{-1} \int_{\Omega(x,r)} g.$$

If $g \in BMO(\Omega) = BMO(\Omega, t_A)$, where

$$t_A = \sup_{t\in\mathbb{R}_+} \left(\sup_{\substack{x\in\Omega \\ r\in]0,t]}} \frac{r^n}{|\Omega(x,r)|} \leqslant \frac{1}{A} \right),$$

we say that $g \in VMO(\Omega)$ if $[g]_{BMO(\Omega,t)} \to 0$ for $t \to 0^+$. A function $\eta[g] : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *modulus of continuity* of g in $VMO(\Omega)$ if

$$[g]_{BMO(\Omega,t)} \leqslant \eta[g](t) \qquad \forall t \in \mathbb{R}_+, \qquad \lim_{t\to 0^+} \eta[g](t) = 0 .$$

3. Preliminary results

In our results certain regularity properties of open subsets of $\mathbb{R}^n$ will often occur; for the corresponding definitions we will refer to [1].

Let Ω be an open subset of $\mathbb{R}^n$ with the segment property and fix a function $\rho \in \mathscr{A}(\Omega) \cap L^\infty(\Omega)$ such that $S_\rho \neq \emptyset$. We suppose that there exists an open subset $\Omega_\circ$ of $\mathbb{R}^n$ with the uniform $C^{1,1}$-regularity property such that:

$$\Omega \subset \Omega_\circ \ , \quad \partial\Omega \setminus S_\rho \subset \partial\Omega_\circ . \tag{h_1}$$

REMARK 3.1. We observe that if condition (h_1) holds, then there exists $\theta \in]0, \frac{\pi}{2}[$ such that

$$\forall x \in \Omega \qquad \exists C_\theta(x) \ : \qquad \overline{C_\theta(x,\rho(x))} \subset \Omega ,$$

where $C_\theta(x)$ is an open indefinite cone with vertex in x and opening θ and $C_\theta(x,\rho(x)) = C_\theta(x) \cap B(x,\rho(x))$ (see Remark 3.1 of [9]).

The following density result can be proved.

LEMMA 3.2. *If* Ω *verifies* (h_1) *and* $p\in]1,+\infty[$, *then for every* $u\in W_s^{2,p}(\Omega)\cap \overset{\circ}{W}{}_{s-1}^{1,p}(\Omega)$ *(for some* $s\in\mathbb{R}$*) there exists a sequence* $(u_k)_{k\in\mathbb{N}}$ *such that*

$$u_k\in W_s^{2,p}(\Omega)\cap \overset{\circ}{W}{}_{s-1}^{1,p}(\Omega)\cap\mathfrak{D}^0(\bar\Omega\setminus S_\rho)\;,\;\; u_k\to u \;\text{ in }\; W_s^{2,p}(\Omega)\,. \tag{3.1}$$

Proof. Fix $u\in W_s^{2,p}(\Omega)\cap \overset{\circ}{W}{}_{s-1}^{1,p}(\Omega)$. From (h_1) and Remark 3.1 we get the existence of a sequence of functions $(\delta_k)_{k\in\mathbb{N}}\subset\mathfrak{D}(\bar\Omega\setminus S_\rho)$ such that

$$\lim_{k\to+\infty}\partial^\alpha(1-\delta_k(x))=0\quad\forall x\in\Omega\;,\quad\forall\alpha\in\mathbb{N}_0^n\,, \tag{3.2}$$

$$\sup_{k\in\mathbb{N}}|\partial^\alpha\delta_k(x)|\leqslant c_\alpha\rho^{-|\alpha|}(x)\quad\forall x\in\Omega\;,\quad\forall\alpha\in\mathbb{N}_0^n \tag{3.3}$$

(see Theorem 4.1 of [13]). Clearly it follows from (3.3) and (2.2) that $\delta_k u\in W^{2,p}(\Omega)\cap \overset{\circ}{W}{}^{1,p}(\Omega)$ and by (3.2), (3.3) we also obtain that

$$\delta_k\,u\;\longrightarrow\;u\quad\text{in }W_s^{2,p}(\Omega)\,. \tag{3.4}$$

Denote by $(\delta_k u)_\circ$ the extension of $\delta_k u$ to $\mathbb{R}^n$ with zero values out of Ω. Since $\mathfrak{D}(\bar\Omega\setminus S_\rho)$ is dense in $W^{2,p}(\Omega)$ (see Lemma 3.2 of [5]), easily we can prove that $(\delta_k u)_\circ\in W^{2,p}(\Omega_\circ)\cap \overset{\circ}{W}{}^{1,p}(\Omega_\circ)$. Therefore, from Lemma 4.4 of [6] we deduce that for any $k\in\mathbb{N}$ there exists a sequence $(\psi_n^k)_{n\in\mathbb{N}}$ of functions in $W^{2,p}(\Omega_\circ)\cap \overset{\circ}{W}{}^{1,2}(\Omega_\circ)\cap \mathfrak{D}^0(\bar\Omega_\circ)$ such that

$$\psi_n^k\;\longrightarrow\;(\delta_k u)_\circ\quad\text{in }W^{2,p}(\Omega_\circ)\,. \tag{3.5}$$

With easy computations we yield that $\delta_k\psi_n^k\in W_s^{2,p}(\Omega)\cap \overset{\circ}{W}{}_{s-1}^{1,p}(\Omega)\cap\mathfrak{D}^0(\bar\Omega\setminus S_\rho)$ for any $k,n\in\mathbb{N}$.

Now observe that by (3.3) and (2.2) we obtain

$$\begin{aligned}||\delta_k\psi_n^k-u||_{W_s^{2,p}(\Omega)}&\leqslant||\delta_k\psi_n^k-\delta_k^2u||_{W_s^{2,p}(\Omega)}+||\delta_k^2u-\delta_ku||_{W_s^{2,p}(\Omega)}+||\delta_ku-u||_{W_s^{2,p}(\Omega)}\\&\leqslant c(k)||\psi_n^k-(\delta_ku)_\circ||_{W^{2,p}(\Omega_\circ)}+c||\delta_ku-u||_{W_s^{2,p}(\Omega)}\,,\end{aligned} \tag{3.6}$$

where $c(k),c\in\mathbb{R}_+$.

On the other hand, by (3.5) we get that for any $k\in\mathbb{N}$ there exists $n_k\in\mathbb{N}$ such that

$$||\psi_{n_k}^k-(\delta_ku)_\circ||_{W^{2,p}(\Omega_\circ)}\leqslant\frac{1}{k\,c(k)}\,. \tag{3.7}$$

Therefore it follows from (3.4), (3.6) and (3.7) that

$$\delta_k\psi_{n_k}^k\;\longrightarrow\;u\quad\text{in }W_s^{2,p}(\Omega)\,, \tag{3.8}$$

and then the statement follows putting $u_k=\delta_k\psi_{n_k}^k$. $\square$

Observe that, if condition (h_1) holds, then it is possible to find a function $\sigma \in \mathscr{A}(\Omega)\cap C^\infty(\Omega)\cap C^{0,1}(\bar{\Omega})$ which is equivalent to ρ and such that

$$|\partial^\alpha \sigma(x)| \leqslant c_\alpha \sigma^{1-|\alpha|}(x) \qquad \forall x \in \Omega, \quad \forall \alpha \in \mathbb{N}_o^n, \tag{3.9}$$

where c_α is independent of x (see [13]). Now we introduce a sequence of functions $\eta_k, k \in \mathbb{N}$, related to the function σ. Consider a function $g \in C_\circ^\infty(\bar{\mathbb{R}}_+)$ satisfying the condition

$$0 \leqslant g \leqslant 1, \qquad g(t) = 1 \quad \text{if} \quad t \geqslant 1, \qquad g(t) = 0 \quad \text{if} \quad t \leqslant \frac{1}{2}.$$

For any $k \in \mathbb{N}$ put

$$\eta_k(x) = \frac{1}{k}\,\zeta_k(x) + \big(1 - \zeta_k(x)\big)\,\sigma(x), \qquad x \in \Omega,$$

where $\zeta_k(x) = g(k\,\sigma(x)), x \in \Omega$. Clearly $\eta_k \in C^\infty(\Omega)$ and

$$\eta_k(x) = \begin{cases} \frac{1}{k} & \text{if} \quad x \in \bar{\Omega}_k \\ \sigma(x) & \text{if} \quad x \in \Omega \setminus \Omega_{2k}, \end{cases}$$

where

$$\Omega_k = \Big\{ x \in \Omega \,:\, \sigma(x) > \frac{1}{k} \Big\}.$$

It is easy to show that for each $k \in \mathbb{N}$

$$\sigma(x) \leqslant \eta_k(x) \leqslant 2\,\sigma(x), \qquad x \in \Omega \setminus \bar{\Omega}_k, \tag{3.10}$$

$$c'_k\,\sigma(x) \leqslant \eta_k(x) \leqslant \sigma(x), \qquad x \in \Omega_k, \tag{3.11}$$

$$\big(\eta_k(x)\big)_x \leqslant c_1\,\big(\sigma(x)\big)_x, \qquad x \in \Omega, \tag{3.12}$$

where $c'_k \in \mathbb{R}_+$ depends on k and σ, and $c_1 \in \mathbb{R}_+$ depends only on n.

For any $p \in [2,+\infty[$ and $t \in \mathbb{R}$, we put:

$$V_t^p(\Omega) = \{u \in W_t^{1,2}(\Omega) \,|\, |\eta_k^{t-1} u|^{p-2} u \in W_t^{1,2}(\Omega)\ \forall k \in \mathbb{N}\},$$

$$\mathring{V}_t^p(\Omega) = \{u \in \mathring{W}_t^{1,2}(\Omega) \,|\, |\eta_k^{t-1} u|^{p-2} u \in \mathring{W}_t^{1,2}(\Omega)\ \forall k \in \mathbb{N}\}.$$

We can prove the following result, which is one of the main tools in the proof of Lemma 4.2.

LEMMA 3.3. *If* (h_1) *holds and* $u \in \mathring{W}_t^{1,2}(\Omega) \cap \mathfrak{D}^0(\bar{\Omega} \setminus S_\rho)$ *for some* $t \in \mathbb{R}$*, then* u *belongs to* $\mathring{V}_t^p(\Omega)$ *for every* $p \in [2,+\infty[$.

Proof. Fix $u \in \overset{\circ}{W}{}^{1,2}_t(\Omega) \cap \mathfrak{D}^0(\bar{\Omega} \setminus S_\rho)$. From (2.2) we deduce that u also belongs to $\overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $\operatorname{supp} u \subset \bar{\Omega} \setminus S_\rho$. Let $u_\circ$ be the extension of u to $\mathbb{R}^n$ with zero values out of Ω. Again from the density of $\mathfrak{D}(\bar{\Omega} \setminus S_\rho)$ in $W^{2,p}(\Omega)$, we deduce that $u_\circ \in \overset{\circ}{W}{}^{1,2}(\Omega_\circ) \cap L^\infty(\Omega_\circ)$. Thus, from Lemma 3.2 of [6], it follows that the function $|u_\circ|^{p-2} u_\circ$ lies in $\overset{\circ}{W}{}^{1,2}(\Omega_\circ)$ for any $p \in [2, +\infty[$. Then there exists a sequence $(g_n)_{n \in \mathbb{N}}$ of functions in $C^\infty_\circ(\Omega_\circ)$ such that

$$||g_n - |u_\circ|^{p-2} u_\circ||_{W^{1,2}(\Omega_\circ)} \longrightarrow 0 \quad \text{for } n \to +\infty. \tag{3.13}$$

First we observe that, since ρ and ρ^{-1} are locally bounded in $\bar{\Omega} \setminus S_\rho$ and $\operatorname{supp} u \subset \bar{\Omega} \setminus S_\rho$, then $u \in L^\infty_t(\Omega)$ and thus $|\eta_k^{t-1} u|^{p-2} u \in W^{1,2}_t(\Omega)$.

Now we prove that $|\eta_k^{t-1} u|^{p-2} u \in \overset{\circ}{W}{}^{1,2}_{\text{loc}}(\bar{\Omega} \setminus S_\rho)$ for any $k \in \mathbb{N}$. In fact, if ζ is a fixed element of $\mathfrak{D}(\bar{\Omega} \setminus S_\rho)$, since Ω has the segment property, then the functions $\psi_n = \zeta \eta_k^{(t-1)(p-2)} g_n$, $n \in \mathbb{N}$, belong to $C^\infty_\circ(\Omega)$; moreover, by (3.10), (3.11), (2.2) and (3.13) we have

$$\begin{aligned} ||\psi_n - \zeta |\eta_k^{t-1} u|^{p-2} u||_{W^{1,2}(\Omega)} &= ||\zeta \eta_k^{(t-1)(p-2)} g_n - \zeta |\eta_k^{t-1} u|^{p-2} u||_{W^{1,2}(\Omega)} \\ &= ||\zeta \eta_k^{(t-1)(p-2)} (g_n - |u_\circ|^{p-2} u_\circ)||_{W^{1,2}(\Omega_\circ)} \\ &\leqslant c ||g_n - |u_\circ|^{p-2} u_\circ||_{W^{1,2}(\Omega_\circ)} \to 0, \end{aligned}$$

where $c \in \mathbb{R}_+$ depends only on k, ζ, ρ, t and p.

Hence the statement follows applying Lemma 3.3 of [5]. □

4. An a priori bound

From now on we assume that $n \geqslant 3$ and we consider the following conditions:

$$\lim_{k \to +\infty} \left(\sup_{\Omega \setminus \Omega_k} \sigma_x \right) = 0, \tag{h_2}$$

$$\begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega), \ i, j = 1, \dots, n, r \\ \exists \nu > 0 \ : \ \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geqslant \nu |\xi|^2 \ \text{ a.e. in } \Omega, \ \forall \xi \in \mathbb{R}^n, \end{cases} \tag{h_3}$$

$$b \in K_2^{n/2}(\Omega), \ \operatorname*{ess\,inf}_{\Omega} \sigma^2 b = b_\circ > 0. \tag{h_4}$$

LEMMA 4.1. *Assume that (h_1)–(h_4) hold and consider the bilinear form*

$$a_t(v, w) = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} v_{x_i} w_{x_j} + b v w \Big) \sigma^{2t} dx \ , \quad t \in \mathbb{R}, \tag{4.1}$$

with $v, w \in W_t^{1,2}(\Omega)$*; then for any* $p \in [2, +\infty[$ *there exists* $k_\circ \in \mathbb{R}_+$*, depending only on* $n, p, t, ||a_{ij}||_{L^\infty(\Omega)}, \nu, b_\circ$*, such that*

$$a_t(v, |\eta_k^{t-1} v|^{p-2} v) \geqslant (p-1)\frac{\nu}{2}\int_\Omega |\eta_k^{t-1} v|^{p-2} |\sigma^t v_x|^2 dx + \frac{b_\circ}{2}\int_\Omega |\eta_k^{t-1} v|^{p-2} |\sigma^{t-1} v|^2 dx \tag{4.2}$$

$$\forall v \in V_t^p(\Omega),\ \forall k \geqslant k_\circ .$$

Proof. Fix $p \geqslant 2$ and $v \in V_t^p(\Omega)$. Then we easily have

$$a(v, |\eta_k^{t-1} v|^{p-2} v) = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} v_{x_i} (|\eta_k^{t-1} v|^{p-2} v)_{x_j} + b v^2 |\eta_k^{t-1} v|^{p-2} \Big) \sigma^{2t} dx \tag{4.3}$$

$$\geqslant \nu(p-1)\int_\Omega |\eta_k^t v|^{p-2} |\sigma^t v_x|^2 dx + b_\circ \int_\Omega |\eta_k^{t-1} v|^{p-2} |\sigma^{t-1} v|^2 dx$$

$$+ (p-2)(t-1)\int_\Omega \sum_{i,j=1}^n a_{ij} v_{x_i} v \sigma^{2t-1} |\eta_k^{t-1} v|^{p-2} \frac{\sigma}{\eta_k} (\eta_k)_{x_j} dx.$$

On the other hand, since $(\eta_k)_{x_j} = 0$ on $\bar{\Omega}_k$ and $\dfrac{\sigma}{\eta_k} \leqslant 1$ on $\Omega \setminus \bar{\Omega}_k$, we also have

$$\Big| \int_\Omega v_{x_i} v \sigma^{2t-1} |\eta_k^{t-1} v|^{p-2} \frac{\sigma}{\eta_k} (\eta_k)_{x_j} dx \Big| \tag{4.4}$$

$$\leqslant \frac{1}{2}\int_\Omega |\eta_k^{t-1} v|^{p-2} |\sigma^t v_x|^2 (\eta_k)_x dx + \frac{1}{2}\int_\Omega |\eta_k^{t-1} v|^{p-2} |\sigma^{t-1} v|^2 (\eta_k)_x dx .$$

Therefore, from (h_2), (3.12) and (4.4), it follows that there exists $k_\circ$, depending only on $n, p, t, ||a_{ij}||_{L^\infty(\Omega)}, \nu, b_\circ$, such that

$$\Big| (p-2)(t-1)\int_\Omega \sum_{i,j=1}^n a_{ij} v_{x_i} v \sigma^{2t-1} |\eta_k^{t-1} v|^{p-2} \frac{\sigma}{\eta_k} (\eta_k)_{x_j} dx \Big| \tag{4.5}$$

$$\leqslant (p-1)\frac{\nu}{2}\int_\Omega |\eta_k^{t-1} v|^{p-2} |\sigma^t v_x|^2 dx + \frac{b_\circ}{2}\int_\Omega |\eta_k^{t-1} v|^{p-2} |\sigma^{t-1} v|^2 dx, \quad \forall k \geqslant k_\circ .$$

Hence we obtain the statement by (4.3) and (4.5). □

Now let us consider the operator

$$L_\circ = -\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \quad \text{a.e. in } \Omega$$

and suppose that the following further hypotheses are verified:

$$(a_{ij})_{x_h} \in \tilde{K}_1^{t_1}(\Omega),\ \ i, j, h = 1, \dots, n,\ \ b \in \tilde{K}_2^{t_2}(\Omega), \tag{h_5}$$

where

$$t_1 \geqslant n,\ t_1 \geqslant p,\ t_1 > p \text{ if } p = n, \quad t_2 \geqslant \frac{n}{2},\ t_2 \geqslant p,\ t_2 > p \text{ if } p = \frac{n}{2},$$

there exist extensions a°_{ij} of a_{ij} to $\Omega_\circ$ such that

$$\begin{cases} a^{\circ}_{ij} = a^{\circ}_{ji} \in L^\infty(\Omega_\circ) \cap VMO(\Omega_\circ)\,, \ \ i,j = 1,\dots,n\,, \\ \exists\, \nu_\circ > 0 \ : \ \displaystyle\sum_{i,j=1}^{n} a^{\circ}_{ij}\,\xi_i\,\xi_j \geqslant \nu_\circ|\xi|^2 \ \text{ a.e. in } \Omega_\circ\,,\ \forall \xi \in \mathbb{R}^n\,. \end{cases} \tag{h_6}$$

Then we define

$$L_t = L_\circ - \sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\big((a_{ij})_{x_j} + 2t a_{ij}\sigma_{x_j}\sigma^{-1}\big)\Big)\frac{\partial}{\partial x_i} + b\ \ ,\quad t\in\mathbb{R}\,.$$

Note that, by (h_2) and Lemma 2.1 of [9], σ_x belongs to $\overset{\circ}{K}{}^{t_1}_{0}(\Omega)$. Thus, under the assumptions (h_1) - (h_4), for any $p \in]1,+\infty[$ and $t\in\mathbb{R}$ the operator $L_t\ :\ W^{2,p}_{t+1}(\Omega) \longrightarrow L^p_{t+1}(\Omega)$ is bounded (see for instance [9], Theorem 3.1).

We are now able to prove the main result of this section.

LEMMA 4.2. *If conditions $(h_1),(h_2)$, (h_4) - (h_6) hold, then for any $p\in[2,+\infty[$ and $s\in\mathbb{R}$, there exists a constant $c\in\mathbb{R}_+$ such that*

$$||u||_{W^{2,p}_s(\Omega)} \leqslant c\,||L_{s-1}u||_{L^p_s(\Omega)} \quad \forall u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_{s-1}(\Omega)\,, \tag{4.6}$$

where c depends on $\Omega, n, p, s, \rho, t_1, t_2, b_\circ, \nu_\circ$, $||(a_{ij})_{x_h}||_{K^{t_1}_1(\Omega)}$, $\tilde{\omega}^{t_1}_1[(a_{ij})_{x_h}]$, $||a^{\circ}_{ij}||_{L^\infty(\Omega_\circ)}$, $\eta[a^{\circ}_{ij}]$, $||b||_{K^{t_2}_2(\Omega)}$ and $\tilde{\omega}^{t_2}_2[b]$.

Proof. First suppose $u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_{s-1}(\Omega) \cap \mathfrak{D}^0(\bar{\Omega}\setminus S_\rho)$. From Theorem 4.1 of [4] it follows that

$$||u||_{W^{2,p}_s(\Omega)} \leqslant c_1\big(||L_{s-1}u||_{L^p_s(\Omega)} + ||u||_{L^p_{s-2}(\Omega)}\big)\,, \tag{4.7}$$

where c_1 depends on $\Omega, n, p, s, \rho, t_1, t_2, \nu_\circ$, $||(a_{ij})_{x_h}||_{K^{t_1}_1(\Omega)}$, $\tilde{\omega}^{t_1}_1[(a_{ij})_{x_h}]$, $||a^{\circ}_{ij}||_{L^\infty(\Omega_\circ)}$, $\eta[a^{\circ}_{ij}]$, $||b||_{K^{t_2}_2(\Omega)}$ and $\tilde{\omega}^{t_2}_2[b]$.

We now provide a bound for $||u||_{L^p_{s-2}(\Omega)}$. For any $k \geqslant k_\circ$, where $k_\circ$ is that defined in Lemma 4.1, by Lemmas 3.3 and 4.1, we deduce that

$$\begin{aligned} (p-1)\frac{\nu}{2}\int_\Omega |\eta_k^{s-2}u|^{p-2}|\sigma^{s-1}u_x|^2dx &+ \frac{b_\circ}{2}\int_\Omega |\eta_k^{s-2}u|^{p-2}|\sigma^{s-2}u|^2dx \\ &\leqslant a_{s-1}(u, |\eta_k^{s-2}u|^{p-2}u) \\ &= \int_\Omega\Big(\sum_{i,j=1}^{n} a_{ij}u_{x_i}(|\eta_k^{s-2}u|^{p-2}u)_{x_j} + bu|\eta_k^{s-2}u|^{p-2}u\Big)\sigma^{2(s-1)}dx \\ &= \int_\Omega L_{s-1}u\,|\eta_k^{s-2}u|^{p-2}\,u\,\sigma^{2(s-1)}dx\,. \end{aligned} \tag{4.8}$$

For $k = k_\circ$, since $\eta_{k_\circ}, \sigma$ and ρ are equivalent, by (4.8) we also have that

$$\int_\Omega |\rho^{s-2} u|^p dx \leqslant c_2 \int_\Omega L_{s-1} u \rho^s |\rho^{s-2} u|^{p-1} dx, \tag{4.9}$$

where $c_2 \in \mathbb{R}_+$ depend on n, p, s, ρ, ν, $b_\circ$, $||a_{ij}||_{L^\infty(\Omega)}$. Therefore

$$||u||_{L^p_{s-2}(\Omega)} \leqslant c_2 ||L_{s-1} u||_{L^p_s(\Omega)}. \tag{4.10}$$

Applying (4.7) and (4.10), the proof in the case $u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_{s-1}(\Omega) \cap \mathfrak{D}^0(\bar{\Omega} \setminus S_\rho)$ is complete.

Now suppose $u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_{s-1}(\Omega)$. By Lemma 3.2 there exists a sequence $(u_k)_{k\in\mathbb{N}}$ of functions satisfying (3.1), and hence it follows from the previous case that

$$||u_k||_{W^{2,p}_s(\Omega)} \leqslant c ||L_{s-1} u_k||_{L^p_s(\Omega)}, \quad k \in \mathbb{N}. \tag{4.11}$$

Relations (4.11) and (3.1) yield the statement also in this case. □

5. Main result

In this section we consider the operator

$$L = -\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} + a(x) \quad \text{a.e. in } \Omega. \tag{5.1}$$

We suppose that the coefficient a has the form $a = a' + b$, where b satisfies the hypotheses (h_4) and (h_5), as well as the functions $(a_{ij})_{x_h}$, a_i and a' verify the following condition:

$$(a_{ij})_{x_h},\ a_i \in \overset{\circ}{K}{}^{t_1}_1(\Omega),\ i,j,h = 1,\dots,n, \quad a' \in \overset{\circ}{K}{}^{t_2}_2(\Omega), \tag{h_7}$$

where t_1 and t_2 are as in (h_5).

Under the assumptions (h_1), (h_5) - (h_7), for any $p \in]1, +\infty[$ and $s \in \mathbb{R}$ the operator $L : W^{2,p}_s(\Omega) \longrightarrow L^p_s(\Omega)$ is bounded (see [9], Theorem 3.1).

The following a priori estimate for the operator L can be stated.

THEOREM 5.1. *If conditions $(h_1), (h_2)$, (h_4) - (h_7) hold, then for every $p \in [2, +\infty[$ and $s \in \mathbb{R}$, there exist a constant $c \in \mathbb{R}_+$ and a bounded open subset $\Omega_1 \subset\subset \Omega$, with the cone property, such that*

$$||u||_{W^{2,p}_s(\Omega)} \leqslant c \left(||Lu||_{L^p_s(\Omega)} + ||u||_{L^p(\Omega_1)} \right) \tag{5.2}$$

$$\forall u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_{s-1}(\Omega),$$

where c, Ω_1 depend on $\Omega, n, p, s, \rho, t_1, t_2, b_\circ, \nu_\circ, ||(a_{ij})_{x_h}||_{K^{t_1}_1(\Omega)}, \overset{\circ}{\omega}{}^{t_1}_1[(a_{ij})_{x_h}], ||a_i||_{K^{t_1}_1(\Omega)}$, $\overset{\circ}{\omega}{}^{t_1}_1[a_i]$, $||a'||_{K^{t_2}_2(\Omega)}$, $\overset{\circ}{\omega}{}^{t_2}_2[a']$, $||b||_{K^{t_2}_2(\Omega)}$, $\tilde{\omega}^{t_2}_2[b]$, $||a^\circ_{ij}||_{L^\infty(\Omega_\circ)}$ and $\eta[a^\circ_{ij}]$.

Proof. Let $u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}_{s-1}(\Omega)$. By Lemma 4.2 we get

$$||u||_{W_s^{2,p}(\Omega)} \leqslant c\Big(||Lu||_{L_s^p(\Omega)} + ||\sum_{i=1}^n d_i u_{x_i} + a'u||_{L_s^p(\Omega)}\Big), \tag{5.3}$$

where

$$d_i = \sum_{j=1}^n \big((a_{ij})_{x_j} + 2(s-1)a_{ij}\sigma_{x_j}\sigma^{-1}\big) + a_i \ , \quad i = 1, \dots, n.$$

On the other hand, it follows from Theorem 3.1 of [9] that for any $\varepsilon \in \mathbb{R}_+$ there exist a constant $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open subset $\Omega_\varepsilon \subset\subset \Omega$, with the cone property, such that

$$||\sum_{i=1}^n d_i u_{x_i} + a'u||_{L_s^p(\Omega)} \leqslant \varepsilon\, ||u||_{W_s^{2,p}(\Omega)} + c(\varepsilon)\big(||u_x||_{L^p(\Omega_\varepsilon)} + ||u||_{L^p(\Omega_\varepsilon)}\big), \tag{5.4}$$

where $c(\varepsilon)$ and Ω_ε depend on $\varepsilon, \Omega, n, p, s, \rho, t_1, t_2, ||(a_{ij})_{x_h}||_{K_1^{t_1}(\Omega)}, \overset{\circ}{\omega}{}^{t_1}_1[(a_{ij})_{x_h}]$, $||a_i||_{K_1^{t_1}(\Omega)}, \overset{\circ}{\omega}{}^{t_1}_1[a_i], ||a'||_{K_2^{t_2}(\Omega)}, \overset{\circ}{\omega}{}^{t_2}_2[a']$. Relations (5.3) and (5.4) complete the proof of the theorem. □

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