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STOCHASTIC INTEGRAL INEQUALITIES WITH APPLICATIONS

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Abstract. In this paper, we study some new stochastic inequalities involving the Itô integral and give some estimates for the solutions of controlled stochastic differential equations. As applications, we utilize the stochastic integral inequalities presented in this paper to show an existence theorem of the solution for a class of stochastic differential equations and to give necessary conditions that make the solution for a class of stochastic differential equations be a diffusion process.

1. Introduction

It is well known that the stochastic integral inequalities play an important role in the development of stochastic differential and integral equations with applications ([3, 4, 5, 6, 8, 9, 10]). Recently, different types of stochastic inequalities and stochastic integral inequalities have been intensively studied by many authors (see, for example, [1, 2, 5, 7, 9, 10, 11, 12] and the references therein).

Very recently, Amano [1] studied a Gronwall type inequality for Itô integrals and showed some applications for solving the stochastic differential equations. Kuo [5] gave some estimates for Itô integrals and proved the existence of the solutions for stochastic differential equations by using the estimates under some suitable conditions. Throughout the works of Amano [1] and Kuo [5], they assumed that the stochastic processes take values in $\mathbb{R}$.

The aim of the present paper is to give some new inequalities for Itô integral under the assumption that the stochastic processes take values in $\mathbb{R}^n$. Meanwhile, we give some estimates for the solutions of controlled stochastic differential equations. Finally, as examples, we utilize the stochastic integral inequalities presented in this paper to show an existence theorem of the solution for a class of stochastic differential equations and to give necessary conditions that make the solution for a class of stochastic differential equations be a diffusion process.

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2. Stochastic Integral Inequalities

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, P)$ be a complete filtered probability space on which is defined a m-dimensional standard Brownian motion $W(t)$, such that $\{\mathscr{F}_t\}_{t\geqslant 0}$ is the natural filtration of $W(t)$. For a positive number T, $L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$ denotes the set of all $\{\mathscr{F}_t\}_{t\geqslant 0}$-progressively measurable processes $f(t)$ valued in $\mathbb{R}^n$ such that

$$E\left[\int_0^T |f(t)|^2 dt\right] < \infty.$$

THEOREM 2.1. *Let $\xi(t)$, $\eta(t)$ and $\phi(t)$ belong to $L^2_{\mathscr{F}}(0,T;\mathbb{R})$, $W(t)$ be a 1-dimensional standard Brownian motion and $f_1(t)$, $f_2(t)$, $g_1(t)$, $g_2(t) \in L^2_{\mathscr{F}}(0,T;\mathbb{R})$ such that, for all $t \in [0,T]$,*

$$|\xi(t)| \leqslant \left|\int_0^t f_1(s)ds + \int_0^t g_1(s)dW(s)\right|, \tag{2.1}$$

$$|\eta(t)| \leqslant \left|\int_0^t f_2(s)ds + \int_0^t g_2(s)dW(s)\right| \tag{2.2}$$

and

$$\begin{cases} |f_i(t)| \leqslant \alpha_{i1}|\xi(t)| + \alpha_{i2}|\eta(t)| + \alpha_{i3}|\phi(t)|, \\ |g_i(t)| \leqslant \beta_{i1}|\xi(t)| + \beta_{i2}|\eta(t)| + \beta_{i3}|\phi(t)|, \end{cases} \tag{2.3}$$

where α_{ij} and β_{ij} are nonnegative constants for $i = 1,2$ and $j = 1,2,3$. Then

$$E(\xi^2(t) + \eta^2(t)) \leqslant \exp\{9tD(t)\}9A_3^2(t)\int_0^t E\phi^2(s)ds,$$

for all $t \in [0,T]$, where $A_j(t) = \sum_{i=1}^2 \left(\sqrt{t}\alpha_{ij} + \beta_{ij}\right)$ with $j = 1,2,3$ and $D(t) = \max\{A_1^2(t), A_2^2(t)\}$.

Proof. By Minkowski inequality, it follows from (2.1) and (2.2) that

$$\begin{aligned}(E(\xi^2(t) + \eta^2(t)))^{\frac{1}{2}} \leqslant{}& \left(E\left|\int_0^t f_1(s)ds\right|^2\right)^{\frac{1}{2}} + \left(E\left|\int_0^t g_1(s)dW(s)\right|^2\right)^{\frac{1}{2}} \\ &+ \left(E\left|\int_0^t f_2(s)ds\right|^2\right)^{\frac{1}{2}} + \left(E\left|\int_0^t g_2(s)dW(s)\right|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{2.4}$$

Using Cauchy-Schwarz inequality and Itô isometry, (2.4) implies that

$$\begin{aligned}(E(\xi^2(t) + \eta^2(t)))^{\frac{1}{2}} \leqslant{}& \left(t\int_0^t Ef_1^2(s)ds\right)^{\frac{1}{2}} + \left(\int_0^t Eg_1^2(s)ds\right)^{\frac{1}{2}} \\ &+ \left(t\int_0^t Ef_2^2(s)ds\right)^{\frac{1}{2}} + \left(\int_0^t Eg_2^2(s)ds\right)^{\frac{1}{2}}. \end{aligned}$$

It follows from (2.3) that

$$\left(t\int_0^t Ef_i^2(s)ds\right)^{\frac{1}{2}} \leqslant \sqrt{3t}\alpha_{i1}\left(\int_0^t E\xi^2(s)ds\right)^{\frac{1}{2}} + \sqrt{3t}\alpha_{i2}\left(\int_0^t E\eta^2(s)ds\right)^{\frac{1}{2}} + \sqrt{3t}\alpha_{i3}\left(\int_0^t E\phi^2(s)ds\right)^{\frac{1}{2}}$$

and

$$\left(\int_0^t Eg_i^2(s)ds\right)^{\frac{1}{2}} \leqslant \sqrt{3}\beta_{i1}\left(\int_0^t E\xi^2(s)ds\right)^{\frac{1}{2}} + \sqrt{3}\beta_{i2}\left(\int_0^t E\eta^2(s)ds\right)^{\frac{1}{2}} + \sqrt{3}\beta_{i3}\left(\int_0^t E\phi^2(s)ds\right)^{\frac{1}{2}}.$$

Thus,

$$E(\xi^2(t)+\eta^2(t)) \leqslant 9A_1^2(t)\int_0^t E\xi^2(s)ds + 9A_2^2(t)\int_0^t E\eta^2(s)ds + 9A_3^2(t)\int_0^t E\phi^2(s)ds \tag{2.5}$$

for all $t \in [0,T]$, where $A_j(t) = \sum_{i=1}^2(\sqrt{t}\alpha_{ij}+\beta_{ij})$ with $j = 1,2,3$.

Now we fix $t_0 \in [0,T]$ arbitrarily. Then, for any $\epsilon > 0$, (2.5) implies that

$$\frac{d}{dt}\log\left(9A_1^2(t_0)\int_0^t E\xi^2(s)ds+9A_2^2(t_0)\int_0^t E\eta^2(s)ds+9A_3^2(t_0)\int_0^{t_0} E\phi^2(s)ds+\epsilon\right) \leqslant 9\max\{A_1^2(t_0),A_2^2(t_0)\}, \quad \text{a.e. } t\in[0,t_0]. \tag{2.6}$$

Integrating (2.6) from 0 to t_0 with respect to t,

$$\log\left(\frac{9A_1^2(t_0)\int_0^{t_0} E\xi^2(s)ds + 9A_2^2(t_0)\int_0^{t_0} E\eta^2(s)ds + 9A_3^2(t_0)\int_0^{t_0} E\phi^2(s)ds + \epsilon}{9A_3^2(t_0)\int_0^{t_0} E\phi^2(s)ds + \epsilon}\right) \leqslant 9t_0\max\{A_1^2(t_0),A_2^2(t_0)\}.$$

Letting $\epsilon \to 0$, it follows from (2.5) that

$$E(\xi^2(t)+\eta^2(t)) \leqslant \exp\{9t\max\{A_1^2(t),A_2^2(t)\}\}9A_3^2(t)\int_0^t E\phi^2(s)ds.$$

This completes the proof. □

THEOREM 2.2. *Let $x(t)$ and $y(t)$ belong to $L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$. Assume that $f : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $g : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous functions, and there exists constants $L_1, L_2 > 0$ such that, for all $t\in[0,T]$,*

$$|f(t,x)-f(t,\bar{x})| + |g(t,x)-g(t,\bar{x})| \leqslant L_1|x-\bar{x}| + L_2|y-\bar{y}|, \tag{2.7}$$

for all $x,\bar{x},y,\bar{y}\in\mathbb{R}^n$, *and*

$$|x(t)|\leqslant\left|\int_0^t f(s,x(s))ds+\int_0^t g(s,x(s))dW(s)\right|, \tag{2.8}$$

where $|g|=\sqrt{\mathrm{tr}(g\cdot g^T)}$, $dW_idW_j=\delta_{ij}dt$, $\delta_{ij}=1$, *if* $i=j$ *and* $\delta_{ij}=0$, *if* $i\neq j$. *Then*

$$E|x(t)|^2\leqslant 9(\sqrt{t}+1)^2\exp\{9L_1^2t(\sqrt{t}+1)^2\}\left(C^2+L_2^2E\int_0^t|y(s)|^2ds\right)$$

for all $t\in[0,T]$, *where* $C=L_1|f(t,0)|+L_2|g(t,0)|$.

Proof. It follows from (2.7) that

$$|f(t,x)|+|g(t,x)|\leqslant C+L_1|x|+L_2|y| \tag{2.9}$$

where $C=L_1|f(t,0)|+L_2|g(t,0)|$. By Minkowski and Cauchy-Schwarz inequalities, (2.8) implies

$$(E|x(t)|^2)^{\frac{1}{2}}\leqslant\left(tE\int_0^t|f(s,x(s))|^2ds\right)^{\frac{1}{2}}+\left(E\int_0^t|g(s,x(s))|^2ds\right)^{\frac{1}{2}}.$$

It follows from (2.9) that

$$\left(tE\int_0^t|f(s,x(s))|^2ds\right)^{\frac{1}{2}}\leqslant\sqrt{3t}C+\sqrt{3t}L_1\left(E\int_0^t|x(s)|^2ds\right)^{\frac{1}{2}}+\sqrt{3t}L_2\left(E\int_0^t|y(s)|^2ds\right)^{\frac{1}{2}}$$

and

$$\left(E\int_0^t|g(s,x(s))|^2ds\right)^{\frac{1}{2}}\leqslant\sqrt{3}C+\sqrt{3}L_1\left(E\int_0^t|x(s)|^2ds\right)^{\frac{1}{2}}+\sqrt{3}L_2\left(E\int_0^t|y(s)|^2ds\right)^{\frac{1}{2}}.$$

Therefore,

$$E|x(t)|^2\leqslant 9(\sqrt{t}+1)^2\left(C^2+L_1^2E\int_0^t|x(s)|^2ds+L_2^2E\int_0^t|y(s)|^2ds\right)$$

for all $t\in[0,T]$.

By using the same method of Theorem 2.1, we have

$$E|x(t)|^2\leqslant 9(\sqrt{t}+1)^2\exp\{9L_1^2t(\sqrt{t}+1)^2\}\left(C^2+L_2^2E\int_0^t|y(s)|^2ds\right).$$

This completes the proof. □

COROLLARY 2.1. *Let* $x(t)$ *and* $y(t)$ *belong to* $L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$ *and* $x(t)$ *be a step function which satisfies*

$$x(t)=x(t_n),\quad t\in[t_n,t_{n+1}),$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$. *Assume that* $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ *are continuous functions, such that*

$$|x(t_n)| \leqslant \left| \int_0^{t_n} f(s,x(s))ds + \int_0^{t_n} g(s,x(s))dW(s) \right|,$$

for $n = 0,1,2,...,N$. *If there exist constants* $L_1, L_2 > 0$ *satisfying (2.7) for all* $t \in [0,T]$, *then we have*

$$E|x(t)|^2 \leqslant 9(\sqrt{t}+1)^2 \exp\{9L_1^2 t(\sqrt{t}+1)^2\} \left(C^2 + L_2^2 E \int_0^t |y(s)|^2 ds \right)$$

for all $t \in [0,T]$, *where* $C = L_1|f(t,0)| + L_2|g(t,0)|$.

THEOREM 2.3. *Let* $f(\cdot), g(\cdot)$ *belong to* $L^2_{\mathscr{F}}(0,T;\mathbb{R})$ *and*

$$X(t) = \int_0^t f(s)ds + \int_0^t g(s)dW(s).$$

Then

$$EX^2(t) \leqslant e^t \left(\int_0^t Ef^2(s)ds + \int_0^t Eg^2(s)ds \right)$$

and

$$EX^4(t) \leqslant \exp\{4t(1+2t)\} \left(8t \int_0^t Ef^4(s)ds + 4(1+t) \int_0^t Eg^4(s)ds \right).$$

Proof. Applying Itô formula to $X(t)$, we obtain

$$X^2(t) = \int_0^t 2X(s)f(s) + g^2(s)ds + \int_0^t 2X(s)g(s)dW(s). \tag{2.10}$$

Using Cauchy-Schwarz inequality,

$$\begin{aligned} X^4(t) &\leqslant 2\left(\int_0^t 2X(s)f(s) + g^2(s)ds \right)^2 + 2\left(\int_0^t 2X(s)g(s)dW(s) \right)^2 \\ &\leqslant 4t \int_0^t g^4(s)ds + 16t \int_0^t X^2(s)f^2(s)ds + 8\left(\int_0^t X(s)g(s)dW(s) \right)^2. \end{aligned} \tag{2.11}$$

Taking the expectation and using Cauchy-Schwarz inequality to (2.10) and (2.11), we have

$$\begin{aligned} EX^2(t) &\leqslant 2\left(\int_0^t EX^2(s)ds\right)^{\frac{1}{2}} \cdot \left(\int_0^t Ef^2(s)ds\right)^{\frac{1}{2}} + \int_0^t Eg^2(s)ds \\ &\leqslant \int_0^t EX^2(s)ds + \int_0^t Ef^2(s)ds + \int_0^t Eg^2(s)ds. \qquad (2.12) \\ EX^4(t) &\leqslant 4t\int_0^t Eg^4(s)ds + 16tE\int_0^t X^2(s)f^2(s)ds + 8E\int_0^t X^2(s)g^2(s)ds \\ &\leqslant 4t\int_0^t Eg^4(s)ds + 16t\left(\int_0^t EX^4(s)ds\right)^{\frac{1}{2}} \cdot \left(\int_0^t Ef^4(s)ds\right)^{\frac{1}{2}} \\ &\quad +8\left(\int_0^t EX^4(s)ds\right)^{\frac{1}{2}} \cdot \left(\int_0^t Eg^4(s)ds\right)^{\frac{1}{2}} \\ &\leqslant 4(1+2t)\int_0^t EX^4(s)ds + 8t\int_0^t Ef^4(s)ds + 4(1+t)\int_0^t Eg^4(s)ds. \end{aligned}$$

Let us fix a nonnegative number $t_0 \leqslant T$ arbitrarily. Then, for any $\epsilon > 0$, (2.12) implies that

$$\frac{d}{dt}\log\left(\int_0^t EX^2(s)ds + \int_0^{t_0} Ef^2(s)ds + \int_0^{t_0} Eg^2(s)ds + \epsilon\right) \leqslant 1, \text{ a.e. } t \in [0, t_0]. \quad (2.13)$$

Integrating (2.13) from 0 to t_0 with respect to t,

$$\log\left(\frac{\int_0^{t_0} EX^2(s)ds + \int_0^{t_0} Ef^2(s)ds + \int_0^{t_0} Eg^2(s)ds + \epsilon}{\int_0^{t_0} Ef^2(s)ds + \int_0^{t_0} Eg^2(s)ds + \epsilon}\right) \leqslant t_0.$$

Letting $\epsilon \to 0$, it follows from (2.12) that

$$EX^2(t) \leqslant e^t\left(\int_0^t Ef^2(s)ds + \int_0^t Eg^2(s)ds\right).$$

By using the same method for proving Theorem 2.1, we have

$$EX^4(t) \leqslant \exp\{4t(1+2t)\}\left(8t\int_0^t Ef^4(s)ds + 4(1+t)\int_0^t Eg^4(s)ds\right).$$

This completes the proof. □

THEOREM 2.4. *Let $x(t)$ and $y(t)$ belong to $L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$. Assume that $f : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, $g : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous functions, and there exist a constant $L > 0$ such that, for all $t \in [0,T]$,*

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| + |g(t,x,y) - g(t,\bar{x},\bar{y})| \leqslant L(|x-\bar{x}| + |y-\bar{y}|) \quad (2.14)$$

for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}^n$ and

$$x(t) = \xi + \int_0^t f(s,x(s),y(s))ds + \int_0^t g(s,x(s),y(s))dW(s),$$

where $|g| = \sqrt{\mathrm{tr}(g\cdot g^T)}$, $dW_i dW_j = \delta_{ij}dt$, $\delta_{ij} = 1$, *if* $i = j$ *and* $\delta_{ij} = 0$, *if* $i \neq j$. *Then*

$$E|x(t)|^4 \leqslant \exp\{6t(C+1)(1+2t)\}\left(3|\xi|^4 + 6C(1+2t)\left(t + E\int_0^t |y(s)|^4)ds\right)\right)$$

and

$$E|x(t)-\xi|^4 \leqslant \exp\{4t(1+2t)\}\left(\alpha(t) + \beta(t)E\int_0^t |y(s)|^4)ds\right)$$

for all $t\in[0,T]$, *where*

$$\begin{aligned}
\alpha(t) &= Ct + 3Ct\exp\{6t(C+1)(1+2t)\}(|\xi|^4 + 2Ct(1+2t)),\\
\beta(t) &= C + 6C^2t(1+2t)\exp\{6t(C+1)(1+2t)\},\\
C &= \max\{27L^4, 27(|f(t,0,0)| + |g(t,0,0)|)^4\}.
\end{aligned}$$

Proof. Applying Itô formula to $x(t)$, we obtain

$$\begin{aligned}
|x(t)|^2 \leqslant\ & |\xi|^2 + \int_0^t 2|x(s)||f(s,x(s),y(s))| + |g(s,x(s),y(s))|^2 ds\\
&+2\sum_{i=1}^n\sum_{j=1}^m\int_0^t x_i(s)g_{ij}(s,x(s),y(s))dW_j(s).
\end{aligned}$$

Using Cauchy-Schwarz inequality,

$$\begin{aligned}
|x(t)|^4 \leqslant\ & 3|\xi|^4 + 6t\int_0^t |g(s,x(s),y(s))|^4 ds + 24t\int_0^t |x(s)|^2|f(s,x(s),y(s))|^2 ds\\
&+12\left(\sum_{i=1}^n\sum_{j=1}^m\int_0^t (x_i(s)-\xi_i)g_{ij}(s,x(s),y(s))dW_j(s)\right)^2. \qquad (2.15)
\end{aligned}$$

Taking the expectation and using Cauchy-Schwarz inequality to (2.15), we have

$$\begin{aligned}
E|x(t)|^4 \leqslant\ & 3|\xi|^4 + 6tE\int_0^t |g(s,x(s),y(s))|^4 ds + 24tE\int_0^t |x(s)|^2|f(s,x(s),y(s))|^2 ds\\
&+12E\int_0^t |x(s)|^2|g(s,x(s),y(s))|^2 ds\\
\leqslant\ & 3|\xi|^4 + 6t\int_0^t E|g(s,x(s),y(s))|^4 ds\\
&+24t\left(\int_0^t E|x(s)|^4 ds\right)^{\frac12}\cdot\left(\int_0^t E|f(s,x(s),y(s))|^4 ds\right)^{\frac12}\\
&+12\left(\int_0^t E|x(s)|^4 ds\right)^{\frac12}\cdot\left(\int_0^t E|g(s,x(s),y(s))|^4 ds\right)^{\frac12}\\
\leqslant\ & 3|\xi|^4+6(1+2t)\left(\int_0^t E|x(s)|^4 ds+E\int_0^t |f(s,x(s),y(s))|^4+|g(s,x(s),y(s))|^4 ds\right)
\end{aligned}$$

It follows from (2.14) that

$$|f(t,x,y)|^4 + |g(t,x,y)|^4 \leqslant C(1+|x|^4+|y|^4), \tag{2.16}$$

where $C = \max\{27L^4, 27(|f(t,0,0)| + |g(t,0,0)|)^4\}$. Thus,

$$\begin{aligned} E|x(t)|^4 &\leqslant 3|\xi|^4 + 6(1+2t)\left(E\int_0^t |x(s)|^4 ds + E\int_0^t C(1+|x(s)|^4+|y(s)|^4)ds\right) \\ &\leqslant 3|\xi|^4 + 6(1+2t)\left(Ct + (C+1)E\int_0^t |x(s)|^4 ds + CE\int_0^t |y(s)|^4 ds\right). \end{aligned} \tag{2.17}$$

Let us fix a nonnegative number $t_0 \leqslant T$ arbitrarily. Then, for any $\epsilon > 0$, (2.17) implies

$$\begin{aligned} \frac{d}{dt}\log\left(3|\xi|^4 + 6(1+2t_0)\left(Ct_0 + (C+1)E\int_0^t |x(s)|^4 ds + CE\int_0^{t_0} |y(s)|^4 ds\right) + \epsilon\right) \\ \leqslant 6(C+1)(1+2t_0), \quad \text{a.e. } t \in [0,t_0]. \end{aligned} \tag{2.18}$$

Integrating (2.18) from 0 to t_0 with respect to t, we have

$$\begin{aligned} \log\left(\frac{3|\xi|^4 + 6(1+2t_0)\left(Ct_0 + (C+1)E\int_0^{t_0} |x(s)|^4 ds + CE\int_0^{t_0} |y(s)|^4 ds\right) + \epsilon}{3|\xi|^4 + 6C(1+2t_0)\left(t_0 + E\int_0^{t_0} |y(s)|^4 ds\right) + \epsilon}\right) \\ \leqslant 6t_0(C+1)(1+2t_0). \end{aligned}$$

Letting $\epsilon \to 0$, it follows from (2.17) that

$$E|x(t)|^4 \leqslant \exp\{6t(C+1)(1+2t)\}\left(3|\xi|^4 + 6C(1+2t)\left(t + E\int_0^t |y(s)|^4 ds\right)\right). \tag{2.19}$$

Applying Itô formula to $x(t) - \xi$ and using Cauchy-Schwarz inequality,

$$\begin{aligned} |x(t)-\xi|^4 &\leqslant 4t\int_0^t |g(s,x(s),y(s))|^4 ds + 16t\int_0^t |x(s)-\xi|^2 |f(s,x(s),y(s))|^2 ds \\ &\quad + 8\left(\sum_{i=1}^n \sum_{j=1}^m \int_0^t (x_i(s)-\xi_i) g_{ij}(s,x(s),y(s)) dW_j(s)\right)^2. \end{aligned} \tag{2.20}$$

Taking the expectation and using Cauchy-Schwarz inequality, by (2.16) and (2.20), we have

$$\begin{aligned} E|x(t)-\xi|^4 &\leqslant 4(1+2t)\left(\int_0^t E|x(s)-\xi|^4 ds + E\int_0^t |f(s,x(s),y(s))|^4 + |g(s,x(s),y(s))|^4 ds\right) \\ &\leqslant 4(1+2t)\int_0^t E|x(s)-\xi|^4 ds + 4(1+2t)E\int_0^t C(1+|x(s)|^4+|y(s)|^4) ds. \end{aligned}$$

It follows from (2.19) that

$$\int_0^t E|x(s)|^4 ds \leqslant \exp\{6t(C+1)(1+2t)\}\left(3t|\xi|^4+6Ct(1+2t)\left(t+E\int_0^t |y(s)|^4 ds\right)\right).$$

Therefore,

$$E|x(t)-\xi|^4 \leqslant 4(1+2t)\left(\alpha(t)+\int_0^t E|x(s)-\xi|^4 ds+\beta(t)E\int_0^t |y(s)|^4 ds\right),$$

where

$$\alpha(t)=Ct+3Ct\exp\{6t(C+1)(1+2t)\}(|\xi|^4+2Ct(1+2t))$$

and

$$\beta(t)=C+6C^2t(1+2t)\exp\{6t(C+1)(1+2t)\}.$$

By using the same method for proving (2.19), we can show that

$$E|x(t)-\xi|^4 \leqslant \exp\{4t(1+2t)\}\left(\alpha(t)+\beta(t)E\int_0^t |y(s)|^4)ds\right).$$

This completes the proof. □

3. Applications

In this section, we utilize the stochastic integral inequalities presented in Section 2 to prove an existence theorem of the solution for a class of stochastic differential equations and to give necessary conditions that make the solution for a class of stochastic differential equations be a diffusion process.

EXAMPLE 3.1. We consider the following n-dimensional stochastic differential equation

$$dX_t = f(t,X_t)dt+g(t,X_t)dW(t), \quad 0 \leqslant t \leqslant T; \quad X_0 = X(0), \tag{3.1}$$

where $f : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ and $g : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous functions and $f(t,0)=g(t,0)=0$. If there exist $L_1, L_2 > 0$ are constants satisfying (2.7) for all $t \in [0,T]$, then the solution of stochastic differential equation (3.1) exists.

Proof. By using the Picard's method, we construct a sequence $\{Y_t^{(k)}\}$ defined by $Y_t^{(0)} = X(0)$ and

$$Y_t^{(k+1)} = Y_t^{(0)} + \int_0^t f(s,Y_s^{(k)})ds + \int_0^t g(s,Y_s^{(k)})dW(s).$$

It follows that

$$Y_t^{(k+1)} - Y_t^{(k)} = \int_0^t f(s,Y_s^{(k)}) - f(s,Y_s^{(k-1)})ds + \int_0^t g(s,Y_s^{(k)}) - g(s,Y_s^{(k-1)})dW(s).$$

By Theorem 2.2 with $L_1 = 0$ and $L_2 = L$, we have

$$\begin{aligned} E|Y_t^{(k+1)} - Y_t^{(k)}|^2 &\leqslant 9L^2(\sqrt{t}+1)^2 E\int_0^t |Y_t^{(k)} - Y_t^{(k-1)}|^2 ds \\ &\leqslant \frac{(9tL^2(\sqrt{t}+1)^2)^k}{k!} \sup_{0\leqslant s\leqslant t} E|Y_t^{(1)} - Y_t^{(0)}|^2. \end{aligned}$$

Consequently, it can be proved that $\{Y_t^{(k)}\}$ is a Cauchy sequence in $L^2(\lambda \times \mathbb{P})$, where λ denotes Lebesgue measure on $[0,T]$. Therefore, the solution of (3.1) exists. □

EXAMPLE 3.2. Consider a one-dimensional stochastic integral equation

$$X_t = \xi + \int_a^t f(s,X_s)ds + \int_a^t g(s,X_s)dW(s), \quad a \leqslant t \leqslant b \tag{3.2}$$

where $\xi \in \mathbb{R}^n$, $f(t,X_t)$ and $g(t,X_t)$ satisfy (2.14). If the solution X_t of (3.2) is a diffusion process, then $P_{s,x}(t,\cdot)$, the transition probabilities of X_t, should satisfy

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{|y-x|\geqslant c} P_{t,x}(t+\epsilon, dy) = 0 \tag{3.3}$$

for any $t \in [0,T]$ and $c > 0$, where $P_{s,x}(t,A) = P(X_t \in A | X_s = x)$ for $A \in \mathfrak{B}(\mathbb{R}^n)$.

Proof. Using Chebyshev inequality and Theorem 2.4,

$$\begin{aligned} \int_{|y-x|\geqslant c} P_{t,x}(t+\epsilon, dy) &= P(|X_{t+\epsilon} - x| \geqslant c | X_t = x) \\ &= P(|X_{t+\epsilon}^{t,x} - x|^4 \geqslant c^4) \\ &\leqslant \frac{1}{c^4} E(|X_{t+\epsilon}^{t,x} - x|^4) \\ &\leqslant \frac{\alpha\epsilon}{c^4} \exp\{4\epsilon(1+2\epsilon)\}, \end{aligned}$$

where $X_t^{s,x}$ is the solution of equation (3.2) with initial condition $X_s = x$. Letting $\epsilon \to 0$, it is easy to verify that (3.3) holds. □

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