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THE FRIEDRICHS–WIRTINGER TYPE INEQUALITY AND ITS APPLICATION TO THE TRANSMISSION PROBLEM IN A CONICAL DOMAIN

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Abstract. We formulate the new Friedrichs-Wirtinger type inequality with the sharp constant and apply it to the investigation of the behavior of weak solutions to the transmission problem for linear elliptic divergence second order equations in a neighborhood of the boundary conical point. We establish the precise rate of decreasing of the solution.

1. Main inequalities

Integro-differential inequalities such as the Poincaré, Friedrichs, Wirtinger, Sobolev inequalities play the important role in the theory of boundary value problems for partial differential equations. Such inequalities with exact constants allow to establish **the best possible estimates** of solutions to boundary value problems for elliptic equations near a conical boundary point (see e.g. [2]). In this article we formulate the new Friedrichs-Wirtinger type inequality and apply it to the investigation of the behavior of weak solutions to the transmission problem for linear elliptic divergence second order equations in a neighborhood of the boundary conical point. We establish the precise rate of decreasing of the solution.

Let $G \subset \mathbb{R}^n$, $n \geqslant 2$ be a bounded domain with the boundary ∂G that is a smooth surface everywhere except at the origin $\mathscr{O} \in \partial G$ and near the point $\mathscr{O}$ it is a conical surface with the vertex at $\mathscr{O}$. We assume that $G = G_+ \cup G_- \cup \Sigma_0$ is divided into two subdomains G_+ and G_- by a $\Sigma_0 = G \cap \{x_n = 0\}$, where $\mathscr{O} \in \overline{\Sigma_0}$.

Let use the following notations:

- S^{n-1} : a unit sphere in $\mathbb{R}^n$ centered at $\mathscr{O}$;

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- $(r,\omega), \omega = (\omega_1, \omega_2, \dots, \omega_{n-1})$: the spherical coordinates of $x \in \mathbb{R}^n$ with the pole $\mathscr{O}$:

$$\begin{aligned} x_1 &= r\cos\omega_1, \\ x_2 &= r\cos\omega_2 \sin\omega_1, \\ &\vdots \\ x_{n-1} &= r\cos\omega_{n-1}\sin\omega_{n-2}\dots\sin\omega_1, \\ x_n &= r\sin\omega_{n-1}\sin\omega_{n-2}\dots\sin\omega_1. \end{aligned}$$

- $\mathscr{C}$: the convex rotational cone $\{x_1 > r\cos\frac{\omega_0}{2}\}$ with the vertex at $\mathscr{O}$;
- $\partial\mathscr{C}$: the lateral surface of $\mathscr{C}$: $\{x_1 = r\cos\frac{\omega_0}{2}\}$;
- Ω : a domain on the unit sphere S^{n-1} with the smooth boundary $\partial\Omega$ obtained by the intersection of the cone $\mathscr{C}$ with the sphere S^{n-1};
- $\Omega_+ = \Omega \cap \{x_n > 0\}$, $\Omega_- = \Omega \cap \{x_n < 0\} \Longrightarrow \Omega = \Omega_+ \cup \Omega_- \cup \sigma_0$;
- $\sigma_0 = \Omega \cap \{x_n = 0\}$;
- $\partial\Omega = \partial\mathscr{C} \cap S^{n-1}$, $\partial_\pm\Omega = \overline{\Omega_\pm} \cap \partial\mathscr{C}$, $\partial\Omega_\pm = \partial_\pm\Omega \cup \sigma_0$;

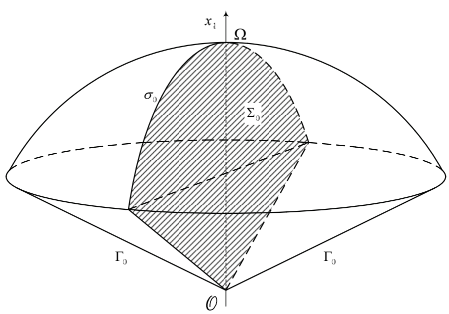

Figure 1.

We use also the standard function spaces: $C^k(\overline{G_\pm})$ with the norm $|u_\pm|_{k,G_\pm}$, the Lebesgue space $L_p(G_\pm), p \geqslant 1$ with the norm $\|u_\pm\|_{p,G_\pm}$, the Sobolev space $W^{k,p}(G_\pm)$ with the norm $\|u_\pm\|_{p,k;G_\pm}$, and introduce their direct sums $\mathbf{C}^k(\overline{G}) = C^k(\overline{G_+}) \dotplus C^k(\overline{G_-})$ with the norm $|u|_{k,G} = |u_+|_{k,G_+} + |u_-|_{k,G_-}$; $\mathbf{L}_p(G) = L_p(G_+) \dotplus L_p(G_-)$ with the norm $\|u\|_{\mathbf{L}_p(G)} = \left(\int_{G_+} |u_+(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{G_-} |u_-(x)|^p dx\right)^{\frac{1}{p}}$; $\mathbf{W}^{k,p}(G) = W^{k,p}(G_+) \dotplus W^{k,p}(G_-)$ with the norm

$$\|u\|_{p,k;G} = \left(\int_{G_+} \sum_{|\beta|=0}^{k} |D^\beta u_+(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{G_-} \sum_{|\beta|=0}^{k} |D^\beta u_-(x)|^p dx\right)^{\frac{1}{p}}.$$

We define the weighted Sobolev spaces:

$\mathbf{V}^k_{p,\alpha}(G)$ for integer $k \geqslant 0$ and real α as the space of distributions $u \in \mathscr{D}'(G)$ with the finite norm

$$\|u\|_{\mathbf{V}^k_{p,\alpha}(G)} = \left(\int\limits_{G_+} \sum_{|\beta|=0}^{k} r^{\alpha+p(|\beta|-k)} |D^\beta u_+(x)|^p \, dx\right)^{\frac{1}{p}} + \left(\int\limits_{G_-} \sum_{|\beta|=0}^{k} r^{\alpha+p(|\beta|-k)} |D^\beta u_-(x)|^p \, dx\right)^{\frac{1}{p}}.$$

We denote $\mathbf{W}^k(G) \equiv \mathbf{W}^{k,2}(G), \quad \overset{\circ}{\mathbf{W}}{}^1_\alpha(G) \equiv \mathbf{V}^1_{2,\alpha}(G)$.

Let us recall some well known formulae related to the unique sphere:

- $d\Omega = J(\omega)d\omega$ denotes the $(n-1)$-dimensional area element of the unique sphere;
- $J(\omega) = \sin^{n-2}\omega_1 \sin^{n-3}\omega_2 \dots \sin\omega_{n-2}, \quad d\omega = d\omega_1 \dots d\omega_{n-1}$;
- $d\sigma$ denotes the $(n-2)$-dimensional area element on $\partial\Omega$;
- $|\nabla_\omega u|$ is the projection of the vector ∇u onto the tangent plane to the unit sphere at the point ω, $\quad |\nabla_\omega u|^2 = \sum_{i=1}^{n-1} \frac{1}{q_i}\left(\frac{\partial u}{\partial \omega_i}\right)^2$, where $q_1 = 1$, $q_i = (\sin\omega_1 \cdots \sin\omega_{i-1})^2$, $i \geqslant 2$;
- $\Delta_\omega u = \frac{1}{J(\omega)} \sum_{i=1}^{n-1} \frac{\partial}{\partial \omega_i}\left(\frac{J(\omega)}{q_i}\frac{\partial u}{\partial \omega_i}\right)$, the Beltrami-Laplace operator.

1.1. The eigenvalue problem

Let $\Omega \subset S^{n-1}$ with smooth boundary $\partial\Omega$ be the intersection of the cone $\mathscr{C}$ with the unit sphere S^{n-1}. Let $\overrightarrow{\nu}$ be the exterior normal to $\partial\mathscr{C}$ at points of $\partial\Omega$ and $\overrightarrow{\tau}$ be the exterior with respect to Ω_+ normal to σ_0 (lying in the tangent to Ω plane). Let $\gamma(\omega)$, $\omega \in \partial\Omega$ be a positive bounded piecewise smooth function, $\sigma(\omega)$ be a positive continuous function on σ_0. We consider the eigenvalue problem for the Laplace-Beltrami operator $\triangle_\omega$ on the unit sphere

$$\begin{cases} \triangle_\omega \psi_\pm + \vartheta \psi_\pm = 0, & \omega \in \Omega_\pm, \\ [\psi]_{\sigma_0} = 0, \quad \left[a\frac{\partial \psi}{\partial \overrightarrow{\tau}}\right]_{\sigma_0} + \sigma(\omega)\psi\Big|_{\sigma_0} = 0; & \\ a_\pm \frac{\partial \psi_\pm}{\partial \overrightarrow{\nu}} + \gamma_\pm(\omega)\psi_\pm\Big|_{\partial_\pm\Omega} = 0, & \end{cases} \qquad (EVP)$$

which consists of the determination of all values ϑ (eigenvalues) for which (EVP) has a non-zero weak solutions (eigenfunctions); here:

- $a = \begin{cases} a_+, & \text{in } \overline{\Omega_+}; \\ a_-, & \text{in } \overline{\Omega_-}, \end{cases}$ $a_\pm$ are positive constants;

- $[\psi]_{\sigma_0}$ denotes the saltus of the function $\psi(x)$ on crossing σ_0, i.e. $[\psi]_{\sigma_0} = \psi_+\Big|_{\sigma_0} - \psi_-\Big|_{\sigma_0}$;
- $\left[a\frac{\partial\psi}{\partial\vec{\tau}}\right]_{\sigma_0}$ denotes the saltus of the co-normal derivative of the function $\psi(x)$ on crossing σ_0, i.e.

$$\left[a\frac{\partial\psi}{\partial\vec{\tau}}\right]_{\sigma_0} = a_+\frac{\partial\psi_+}{\partial\vec{\tau}}\Big|_{\sigma_0} - a_-\frac{\partial\psi_-}{\partial\vec{\tau}}\Big|_{\sigma_0}.$$

DEFINITION 1. Function ψ is called a **weak** solution of the problem (EVP) provided that $\psi \in \mathbf{C}^0(\overline{\Omega}) \cap \mathbf{W}^1(\Omega)$ and satisfies the integral identity

$$\int_\Omega \left\{a\frac{1}{q_i}\frac{\partial\psi}{\partial\omega_i}\frac{\partial\eta}{\partial\omega_i} - a\vartheta\psi(\omega)\eta(\omega)\right\}d\Omega + \int_{\sigma_0}\sigma(\omega)\psi(\omega)\eta(\omega)d\sigma + \int_{\partial\Omega}\gamma(\omega)\psi(\omega)\eta(\omega)d\sigma = 0$$

for all $\eta \in \mathbf{C}^0(\overline{\Omega}) \cap \mathbf{W}^1(\Omega)$.

REMARK 1. We observe that $\vartheta = 0$ is **not** an eigenvalue of (EVP). In fact, setting $\eta = \psi$ and $\vartheta = 0$ we have

$$\int_\Omega a|\nabla_\omega\psi|^2 d\Omega + \int_{\sigma_0}\sigma(\omega)|\psi(\omega)|^2 d\sigma + \int_{\partial\Omega}\gamma(\omega)|\psi(\omega)|^2 d\sigma = 0 \quad \Longrightarrow \psi \equiv 0,$$

since $a_\pm \neq 0$, $\sigma(\omega) > 0$, $\gamma(\omega) > 0$.

Now, let us introduce the following functionals on $\mathbf{C}^0(\overline{\Omega}) \cap \mathbf{W}^1(\Omega)$

$$F[\psi] = \int_\Omega a|\nabla_\omega\psi(\omega)|^2 d\Omega + \int_{\sigma_0}\sigma(\omega)|\psi(\omega)|^2 d\sigma + \int_{\partial\Omega}\gamma(\omega)|\psi(\omega)|^2 d\sigma,$$

$$G[\psi] = \int_\Omega a\psi^2(\omega)d\Omega,$$

$$H[\psi] = \int_\Omega a\Big\langle|\nabla_\omega\psi(\omega)|^2 - \vartheta\psi^2(\omega)\Big\rangle d\Omega + \int_{\sigma_0}\sigma(\omega)|\psi(\omega)|^2 d\sigma + \int_{\partial\Omega}\gamma(\omega)|\psi(\omega)|^2 d\sigma$$

and the corresponding bilinear forms

$$F(\psi,\eta) = \int_\Omega a\frac{1}{q_i}\frac{\partial\psi}{\partial\omega_i}\frac{\partial\eta}{\partial\omega_i}d\Omega + \int_{\sigma_0}\sigma(\omega)\psi(\omega)\eta(\omega)d\sigma + \int_{\partial\Omega}\gamma(\omega)\psi(\omega)\eta(\omega)d\sigma,$$

$$G(\psi,\eta)=\int\limits_{\Omega} a\psi(\omega)\eta(\omega)d\Omega.$$

We define yet the set $K=\left\{\psi\in\mathbf{W}^1(\Omega)\Big|\ G[\psi]=1\right\}$. Since $K\subset\mathbf{W}^1(\Omega)$, $F[\psi]$ is bounded from below for $\psi\in K$. The greatest lower bound of $F[\psi]$ for this family we denote by ϑ : $\inf\limits_{\psi\in K}F[\psi]=\vartheta$. We formulate the following statement:

THEOREM 1.1. *Let $\Omega\subset S^{n-1}$ be a bounded domain with the smooth boundary $\partial\Omega$. Let $\gamma(\omega)$, $\omega\in\partial\Omega$ be a positive bounded piecewise smooth function, $\sigma(\omega)$ be a positive continuous function on σ_0. There exist $\vartheta>0$ and a function $\psi\in K$ such that*

$$F(\psi,\eta)-\vartheta G(\psi,\eta)=0 \textit{ for arbitrary } \eta\in\mathbf{W}^1(\Omega).$$

In particular $F[\psi]=\vartheta$. In addition, on Ω, ψ has continuous derivatives of second order, satisfies the equation $\triangle_\omega\psi+\vartheta\psi=0$, $\omega\in\Omega$ as well as the boundary and conjunction conditions of (EVP) in the weak sense (for details see the Remark 2.19 [2]*).*

Proof. The proof is analogous to the proof of Theorem 2.18 [2] (pp. 56–59). The smoothness of ψ follows from the theory of the transmission elliptic problem in smooth domains (see e.g. §16, chapt. III [9] as well as [1]).

Now from the variational principle we obtain **the Friedrichs-Wirtinger type inequality**:

THEOREM 1.2. *Let $\Omega\subset S^{n-1}$. Let $\psi\in\mathbf{W}^1(\Omega)$ satisfy in the weak sense the boundary and conjunction conditions from (EVP). Let ϑ be the smallest positive eigenvalue of the problem (EVP). (It exists according to Theorem* 1.1.*) Let $\gamma(\omega)$, $\omega\in\partial\Omega$ be a positive bounded piecewise smooth function, $\sigma(\omega)$ be a positive continuous function on σ_0. Then*

$$\vartheta\int\limits_{\Omega}a\psi^2(\omega)d\Omega\leqslant\int\limits_{\Omega}a|\nabla_\omega\psi(\omega)|^2d\Omega+\int\limits_{\sigma_0}\sigma(\omega)\psi^2(\omega)d\sigma+\int\limits_{\partial\Omega}\gamma(\omega)\psi^2(\omega)d\sigma. \tag{1.1}$$

Proof. Consider the functionals $F[\psi],G[\psi],H[\psi]$ described above on $\mathbf{C}^0(\overline{\Omega})\cap\mathbf{W}^1(\Omega)$. We will find the minimum of the functional $F[\psi]$ on the set K. For this we investigate the minimization of the functional $H[\psi]$ on all functions $\psi(\omega)$, for which the integrals exist and which satisfy in the weak sense the boundary and conjunction conditions from (EVP). We use formally the Lagrange multipliers and get the Euler equation from the condition $\delta H[\psi]=0$. By the calculation of the first variation δH, we have

$$
\begin{aligned}
\delta H[\psi] &= \delta\left(\int\limits_{\Omega} a\left\{\sum_{i=1}^{N-1}\frac{1}{q_i}\left(\frac{\partial\psi}{\partial\omega_i}\right)^2 - \vartheta\psi^2(\omega)\right\}d\Omega\right.\\
&\qquad \left. + \int\limits_{\sigma_0}\sigma(\omega)\psi^2(\omega)d\sigma + \int\limits_{\partial\Omega}\gamma(\omega)\psi^2(\omega)d\sigma\right)\\
&= -2\int\limits_{\Omega} a\sum_{i=1}^{N-1}\frac{\partial}{\partial\omega_i}\left(\frac{J(\omega)}{q_i}\frac{\partial\psi}{\partial\omega_i}\right)\cdot\delta\psi(\omega)d\omega - 2\vartheta\int\limits_{\Omega} a\psi(\omega)\cdot\delta\psi(\omega)d\Omega\\
&\qquad +2\int\limits_{\partial\Omega} a\frac{\partial\psi}{\partial\overrightarrow{\nu}}\cdot\delta\psi(\omega)d\sigma + 2\int\limits_{\sigma_0}\left[a\frac{\partial\psi}{\partial\overrightarrow{\tau}}\right]\cdot\delta\psi(\omega)d\sigma\\
&\qquad +2\int\limits_{\sigma_0}\sigma(\omega)\psi(\omega)\cdot\delta\psi(\omega)d\sigma + 2\int\limits_{\partial\Omega}\gamma(\omega)\psi(\omega)\cdot\delta\psi(\omega)d\sigma\\
&= -2\int\limits_{\Omega} a\left(\triangle_\omega\psi(\omega) + \vartheta\psi(\omega)\right)\cdot\delta\psi(\omega)d\Omega\\
&\qquad +2\int\limits_{\sigma_0}\left\{\left[a\frac{\partial\psi}{\partial\overrightarrow{\tau}}\right] + \sigma(\omega)\psi(\omega)\right\}\cdot\delta\psi(\omega)d\sigma\\
&\qquad +2\int\limits_{\partial\Omega}\left\{a\frac{\partial\psi}{\partial\overrightarrow{\nu}} + \gamma(\omega)\psi(\omega)\right\}\cdot\delta\psi(\omega)d\sigma.
\end{aligned}
$$

Hence, because of $\delta H[\psi] = 0\ \forall\delta\psi \in \mathbf{C}^0(\overline{\Omega})\cap\mathbf{W}^1(\Omega)$, it follows the eigenvalue problem (EVP). Conversely, let $\vartheta, \psi(\omega)$ be a weak solution of the eigenvalue problem (EVP). From the definition of the weak eigenfunction under $\eta = \psi(\omega)$ it follows

$$0 = F[\psi] - \vartheta G[\psi] \underset{(\text{on } K)}{=} F[\psi] - \vartheta \Rightarrow \vartheta = F[u],$$

consequently, the required minimum is the least eigenvalue of the eigenvalue problem (EVP). The existence of a function $\psi \in K$ such that $F[\psi] \leqslant F[v]$ for all $v \in K$ has been proved above.

Let us define the value

$$\lambda = \frac{2 - n + \sqrt{(n-2)^2 + 4\vartheta}}{2}, \tag{1.2}$$

where ϑ is the smallest positive eigenvalue of the problem (EVP). Then the Friedrichs-Wirtinger inequality will be written in the following form

$$\lambda(\lambda + n - 2)\int\limits_{\Omega} a\psi^2(\omega)d\Omega \leqslant \int\limits_{\Omega} a|\nabla_\omega\psi|^2 d\Omega + \int\limits_{\sigma_0}\sigma(\omega)\psi^2(\omega)d\sigma + \int\limits_{\partial\Omega}\gamma(\omega)\psi^2(\omega)d\sigma, \tag{1.3}$$

$\forall\psi(\omega)\in\mathbf{W}^1(\Omega)$ satisfying the boundary and conjunction conditions from (EVP), $\sigma(\omega)\geqslant 0, \gamma(\omega)\geqslant 0$.

REMARK 2. The constant in (1.3) is the best possible.

Now we will use the following notations:

- $G_a^b=\{(r,\omega)\mid 0\leqslant a<r<b;\omega\in\Omega\}\cap G$: a layer in $\mathbb{R}^n$;
- $\Gamma_a^b=\{(r,\omega)\mid 0\leqslant a<r<b;\omega\in\partial\Omega\}\cap\partial G$: the lateral surface of the layer G_a^b;
- $G_d=G\setminus G_0^d$; $\quad \Gamma_d=\partial G\setminus\Gamma_0^d$, $d>0$;
- $\Sigma_a^b=G_a^b\cap\{x_n=0\}\subset\Sigma_0$; $\quad \Sigma_d=\Sigma_0\setminus\Sigma_0^d$, $d>0$;
- $\Omega_\rho=G_0^d\cap\{|x|=\rho\}$; $\quad 0<\rho<d$.

COROLLARY 1.3. *Let $u\in\mathbf{C}^0(\overline{G})\cap\overset{\circ}{\mathbf{W}}{}^1_{\alpha-2}(G)$, $u(\cdot,\omega)$ satisfy the boundary and conjunction conditions from (EVP) in the weak sense and λ be as above in (1.2). Let $\sigma(\omega)$, $\omega\in\sigma_0$; $\gamma(\omega)$, $\omega\in\partial\Omega$ be nonnegative bounded piecewise smooth functions. Then*

$$\int_{G_0^d} ar^{\alpha-4}u^2(x)dx\leqslant\frac{1}{\lambda(\lambda+n-2)}\Big\{\int_{G_0^d} ar^{\alpha-2}|\nabla u(x)|^2dx+\int_{\Sigma_0^d} r^{\alpha-3}\sigma(\omega)u^2(x)ds + \int_{\Gamma_0^d} r^{\alpha-3}\gamma(\omega)u^2(x)ds\Big\},\quad\forall\alpha. \tag{1.4}$$

Proof. Multiplying (1.3) by $r^{n-5+\alpha}$ and integrating over $r\in(0,d)$ we obtain the required (1.4).

LEMMA 1.4. *Let G_0^d be the conical domain and $\nabla u(\rho,\cdot)\in\mathbf{L}_2(\Omega)$ for a.e. $\rho\in(0,d)$. Assume that for a.e. $\rho\in(0,d)$*

$$U(\rho)=\int_{G_o^\rho} ar^{2-n}|\nabla u(x)|^2dx+\int_{\Sigma_0^\rho} r^{1-n}\sigma(\omega)u^2(x)ds+\int_{\Gamma_0^\rho} r^{1-n}\gamma(\omega)u^2(x)ds<\infty. \tag{1.5}$$

Then

$$\int_\Omega a\left(\rho u(\rho,\omega)\frac{\partial u}{\partial r}(\rho,\omega)+\frac{n-2}{2}u^2(\rho,\omega)\right)d\Omega\leqslant\frac{\rho}{2\lambda}U'(\rho), \tag{1.6}$$

where λ is defined by (1.2).

Proof. Writing $U(\rho)$ in spherical coordinates

$$
\begin{aligned}
U(\rho) &= \int_0^\rho r^{2-n}\left(\int_\Omega a|\nabla u(r,\omega)|^2 d\Omega\right) r^{n-1}dr \\
&\quad + \int_0^\rho r^{1-n}\Big(\int_{\sigma_0}\sigma(\omega)|u(r,\omega)|^2 d\sigma + \int_{\partial\Omega}\gamma(\omega)|u(r,\omega)|^2 d\sigma\Big) r^{n-2}dr \\
&= \int_0^\rho r\int_\Omega a\left(u_r^2+\frac{1}{r^2}|\nabla_\omega u(r,\omega)|^2\right) d\Omega dr \\
&\quad + \int_0^\rho \frac{1}{r}\left(\int_{\sigma_0}\sigma(\omega)|u(r,\omega)|^2 d\sigma + \int_{\partial\Omega}\gamma(\omega)|u(r,\omega)|^2 d\sigma\right) dr
\end{aligned}
$$

and differentiating with respect to ρ we obtain

$$
\begin{aligned}
U'(\rho) &= \int_\Omega a\left(\rho\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{\rho}|\nabla_\omega u|^2\right)\Bigg|_{r=\rho} d\Omega \\
&\quad + \frac{1}{\rho}\left(\int_{\sigma_0}\sigma(\omega)u^2(\rho,\omega)d\sigma + \int_{\partial\Omega}\gamma(\omega)u^2(\rho,\omega)d\sigma\right).
\end{aligned}
\tag{1.7}
$$

Moreover, by the Cauchy inequality, we have $\rho u\frac{\partial u}{\partial r} \leqslant \frac{\varepsilon}{2}u^2 + \frac{1}{2\varepsilon}\rho^2\left(\frac{\partial u}{\partial r}\right)^2$, $\forall\varepsilon>0$. Then

$$
\int_\Omega a\left(\rho u\frac{\partial u}{\partial r} + \frac{n-2}{2}u^2\right)\Bigg|_{r=\rho} d\Omega \leqslant \frac{\varepsilon+n-2}{2}\int_\Omega a u^2 d\Omega + \frac{\rho^2}{2\varepsilon}\int_\Omega a\left(\frac{\partial u}{\partial r}\right)^2 d\Omega.
$$

Thus choosing $\varepsilon=\lambda$ we obtain, by the Friedrichs-Wirtinger inequality (1.3),

$$
\begin{aligned}
&\int_\Omega a\left(\rho u\frac{\partial u}{\partial r} + \frac{n-2}{2}u^2\right)\Bigg|_{r=\rho} d\Omega \\
&\qquad \leqslant \frac{\varepsilon+n-2}{2\lambda(\lambda+n-2)}\int_\Omega a|\nabla_\omega u|^2 d\Omega + \frac{\rho^2}{2\varepsilon}\int_\Omega a\left(\frac{\partial u}{\partial r}\right)^2 d\Omega \\
&\qquad\quad + \frac{\varepsilon+n-2}{2\lambda(\lambda+n-2)}\left(\int_{\sigma_0}\sigma(\omega)u^2(\rho,\omega)d\sigma + \int_{\partial\Omega}\gamma(\omega)u^2(\rho,\omega)d\sigma\right) \\
&\qquad = \frac{\rho}{2\lambda}U'(\rho).
\end{aligned}
$$

We need also the well known inequalities:

$$\int_{\Gamma} v(x)ds \leqslant C\int_{G}(|v(x)|+|\nabla v(x)|)dx,\ \forall v\in \mathbf{W}^{1,1}(G),\ \forall \Gamma\subseteq \partial G, \tag{1.8}$$

$$\int_{\partial G} v^2(x)ds \leqslant \int_{G}(\delta|\nabla v(x)|^2+\frac{1}{\delta}c_0v^2(x))dx,\ \forall v\in \mathbf{W}^{1,2}(G), \forall\delta>0. \tag{1.9}$$

1.2. Quasi-distance $\mathbf{r}_\varepsilon(\mathbf{x})$

Further, we define **the function** $\mathbf{r}_\varepsilon(\mathbf{x})$ as follows. We fix the point $Q=(-1,0,...,0)\in S^{n-1}\setminus\bar{\Omega}$ and consider the unit radius-vector $\vec{l}=\mathscr{O}Q=\{-1,0,...,0\}$. We denote by $\vec{r}$ the radius-vector of the point $x\in\bar{G}$ and introduce the vector $\vec{r}_\varepsilon=\vec{r}-\varepsilon\vec{l},\ \forall\varepsilon>0$. Since $\varepsilon\vec{l}\notin G_0^d$ for all $\varepsilon\in]0,d[$, it follows that $r_\varepsilon(x)=|\vec{r}-\varepsilon\vec{l}|\neq 0$ for all $x\in\bar{G}$. It is easy to see that $r_\varepsilon(x)$ has the following properties (see for details §1.4 [2]):

1. $\exists h>0$ *such that:* $r_\varepsilon(x)\geqslant hr$ *and* $r_\varepsilon(x)\geqslant h\varepsilon,\ \forall x\in\overline{G}$, *where* $h=\begin{cases}1, & \text{if } x_1\geqslant 0\ ,\\ \sin\frac{\omega_0}{2}, & \text{if } x_1<0.\end{cases}$
2. *If* $x\in G_d$, *then* $r_\varepsilon(x)\geqslant\frac{d}{2}$ *for all* $\varepsilon\in]0,\frac{d}{2}[$,
3. $\lim\limits_{\varepsilon\to 0^+} r_\varepsilon(x)=r$, *for all* $x\in\bar{G}$.
4. $|\nabla r_\varepsilon|^2=1$, *and* $\triangle r_\varepsilon=\frac{n-1}{r_\varepsilon}$.

LEMMA 1.5. *Let* $v\in\mathbf{C}^0(\overline{G})\cap\mathbf{W}^1(G)$ *and* $v(\cdot,\omega)$ *satisfy the boundary and conjunction conditions from* (EVP). *Let* $\sigma(\omega)\geqslant 0,\ \gamma(\omega)\geqslant 0$. *Then for any* $\varepsilon>0$ *and* $d>0$

$$\int_{G_0^d} ar_\varepsilon^{\alpha-2}r^{-2}v^2(x)dx \leqslant \frac{c}{\lambda(\lambda+n-2)}\left\{\int_{G_0^d} ar_\varepsilon^{\alpha-2}|\nabla v(x)|^2dx+\int_{\Sigma_0^d} r^{-1}r_\varepsilon^{\alpha-2}\sigma(\omega)v^2(x)ds\right.$$
$$\left.+\int_{\Gamma_0^d} r^{-1}r_\varepsilon^{\alpha-2}\gamma(\omega)v^2(x)ds\right\}. \tag{1.10}$$

where $c=const(\omega_0;\alpha)>0$.

Proof. Multiplying both sides of the Friedrichs-Wirtinger inequality (1.3) by $(\rho+\varepsilon)^{\alpha-2}r^{n-3}$ and integrating over $r\in(\frac{\rho}{2},\rho)$ we obtain

$$\int_{G_{\rho/2}^\rho} a(\rho+\varepsilon)^{\alpha-2}r^{-2}v^2(x)dx$$
$$\leqslant\frac{1}{\lambda(\lambda+n-2)}\left\{\int_{G_{\rho/2}^\rho} a(\rho+\varepsilon)^{\alpha-2}|\nabla v(x)|^2dx+\int_{\Gamma_{\rho/2}^\rho} r^{-1}(\rho+\varepsilon)^{\alpha-2}\gamma(\omega)v^2(x)ds\right.$$

$$+\int\limits_{\Sigma^{\rho}_{\rho/2}} r^{-1}(\rho+\varepsilon)^{\alpha-2}\sigma(\omega)v^2(x)ds\Bigg\}, \ \forall\varepsilon>0$$

or since $\rho+\varepsilon\sim r_\varepsilon$

$$\int\limits_{G^{\rho}_{\rho/2}} ar_\varepsilon^{\alpha-2}r^{-2}v^2(x)dx \leqslant \frac{1}{\lambda(\lambda+n-2)}\Bigg\{\int\limits_{G^{\rho}_{\rho/2}} ar_\varepsilon^{\alpha-2}|\nabla v(x)|^2+\int\limits_{\Sigma^{\rho}_{\rho/2}} r^{-1}r_\varepsilon^{\alpha-2}\sigma(\omega)v^2(x)ds$$
$$+\int\limits_{\Gamma^{\rho}_{\rho/2}} r^{-1}r_\varepsilon^{\alpha-2}\gamma(\omega)v^2(x)ds\Bigg\}, \ \forall\varepsilon>0.$$

Letting $\rho=2^{-k}d$, $(k=0,1,2,...)$ and summing the obtained inequalities over all k we get the desired inequality (1.10).

1.3. The Cauchy problem for differential inequality

THEOREM 1.6. *Let $U(\rho)$ be a monotonically increasing, nonnegative differentiable function defined on $[0,2d]$ and it satisfies the problem*

$$\begin{cases} U'(\rho)-\mathscr{P}(\rho)U(\rho)+\mathscr{N}(\rho)U(2\rho)+\mathscr{Q}(\rho)\geqslant 0, & 0<\rho<d,\\ U(d)\leqslant U_0, \end{cases} \tag{CP}$$

where $\mathscr{P}(\rho),\mathscr{N}(\rho),\mathscr{Q}(\rho)$ are nonnegative continuous functions defined on $[0,2d]$ and U_0 is a constant. Then

$$U(\rho)\leqslant \exp\Bigg(\int_\rho^d \mathscr{B}(\tau)d\tau\Bigg)\Bigg\{U_0\exp\Bigg(-\int_\rho^d \mathscr{P}(\tau)d\tau\Bigg)+\int_\rho^d \mathscr{Q}(\tau)\exp\Bigg(-\int_\rho^\tau \mathscr{P}(\sigma)d\sigma\Bigg)d\tau\Bigg\} \tag{1.11}$$

with $\mathscr{B}(\rho)=\mathscr{N}(\rho)\exp\Bigg(\int_\rho^{2\rho}\mathscr{P}(\sigma)d\sigma\Bigg)$.

Proof. For the proof see §1.10 (Theorem 1.57) [2]

2. The transmission problem for linear elliptic divergence second order equations in a conical domain

The transmission problems often arise in different fields of physics and technics. For instance, one of the important problem of the electrodynamics of solid media is the study of electromagnetic processes in ferromagnetic media with different dielectric constants. These problems also arise in solid mechanics if a body consists of composite materials.

In this article we obtain **the best possible estimates** of the weak solutions of problem (L) near a conical boundary point. Analogous results were established in [2] for the Dirichlet and Robin problems in a conical domain without interfaces. Many mathematicians have considered the transmission problems. First, V.A. Il'in [6], O.A. Ladyzhenskaya and N.N. Ural'tseva [9], Z.G. Sheftel [12], M.V. Borsuk [1] studied general interface problems for second order elliptic operators in smooth domains. Later other mathematicians studied transmission problems in non-smooth domains in some particular linear cases (see the references cited in [10, 11], [4]). General *linear* interface problems in polygonal and polyhedral domains was considered in [10, 11]. Regularity results in terms of weighted Sobolev-Kondratiev spaces were obtained in [4] for two and three dimensional transmission problems for the Laplace operator. D. Kapanadze and B.-W. Schulze studied boundary-contact problems with conical [7] singularities and edge [8] singularities at the interfaces for general linear any order elliptic equations (as well as systems). They constructed the parametrix and showed the regularity with the asymptotics of solutions in weighted Sobolev-Kondratiev spaces. A principal new feature of this article is the consideration of the estimates for equations with minimally smooth coefficients in n-dimensional conic domains.

Our assumptions concerning the smoothness of the coefficients are the least restrictive possible: leading coefficients of the the equation must be Dini-continuous at the conical point $\mathscr{O}$, whereas lower coefficients can grow and we indicate the exact admissible order of power growth. In §7 we construct the examples which show that the Dini condition for leading coefficients of the equation at the conical point as well as the assumption concerning the lower order coefficients of the equation, are essential for the validity of the estimates established in the Theorem 2.3. The fact that the exponent λ in these estimates cannot be increased is shown by constructing particular solutions of the Laplace equation in the domain with the angular or conical point (see §6, Appendix as well as §7, Examples). In this sense the derived estimates are the best possible.

We consider the elliptic transmission problem

$$\begin{cases} \mathscr{L}[u] \equiv \frac{\partial}{\partial x_i}\left(a^{ij}(x)u_{x_j}\right) + a^i(x)u_{x_i} + b(x)u = f(x), & x \in G\setminus\Sigma_0; \\ [u]_{\Sigma_0} = 0, \quad \mathscr{S}[u] \equiv \left[\frac{\partial u}{\partial \nu}\right]_{\Sigma_0} + \frac{1}{|x|}\sigma\left(\frac{x}{|x|}\right)u(x) = h(x), & x \in \Sigma_0; \\ \mathscr{B}[u] \equiv \frac{\partial u}{\partial \nu} + \frac{1}{|x|}\gamma\left(\frac{x}{|x|}\right)u = g(x), & x \in \partial G\setminus\{\Sigma_0 \cup \mathscr{O}\} \end{cases} \qquad (L)$$

(summation over repeated indices from 1 to n is understood); here:

- $$u(x) = \begin{cases} u_+(x), & x \in G_+, \\ u_-(x), & x \in G_-; \end{cases} \qquad f(x) = \begin{cases} f_+(x), & x \in G_+, \\ f_-(x), & x \in G_- \end{cases} \qquad \text{etc.;}$$
- $[u]_{\Sigma_0}$ denotes the saltus of the function $u(x)$ on crossing Σ_0, , i.e. $[u]_{\Sigma_0} = u_+(x)\Big|_{\Sigma_0} - u_-(x)\Big|_{\Sigma_0}$, where $u_\pm(x)\Big|_{\Sigma_0} = \lim_{G_\pm \ni y \to x \in \Sigma_0} u_\pm(y)$;
- $\frac{\partial}{\partial \nu} = a^{ij}(x)\cos(\overrightarrow{n}, x_i)\frac{\partial}{\partial x_j}$, where $\overrightarrow{n}$ denotes the unite outward with respect to G_+ (or G) normal to Σ_0 (respectively to $\partial G\setminus\mathscr{O}$);

- $\left[\frac{\partial u}{\partial \nu}\right]_{\Sigma_0}$ denotes the saltus of the co-normal derivative of the function $u(x)$ on crossing Σ_0, i.e.

$$\left[\frac{\partial u}{\partial \nu}\right]_{\Sigma_0} = a_+^{ij}(x)\cos(\overrightarrow{n},x_i)\frac{\partial u}{\partial x_j}\Big|_{\Sigma_0} - a_-^{ij}(x)\cos(\overrightarrow{n},x_i)\frac{\partial u}{\partial x_j}\Big|_{\Sigma_0},$$

Let us recall some well known formulae related to spherical coordinates $(r,\omega_1,\ldots,\omega_{n-1})$ centered at the conical point $\mathscr{O}$

- $dx = r^{n-1}drd\Omega, \quad d\Omega_\rho = \rho^{n-1}d\Omega,$
- ds denotes the $(n-1)$-dimensional area element on ∂G; $\quad ds = r^{n-2}drd\sigma;$
- $|\nabla u|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}|\nabla_\omega u|^2,$
- $\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_\omega u,$

$C = C(\ldots)$, $c = c(\ldots)$ denote the constants depending only on the quantities appearing in parentheses. In the sequel, the same letters C, c will (generally) be used to denote different constants depending on the same set of arguments.

DEFINITION 2. The function $u(x)$ is called a *weak* solution of the problem (L) provided that $u \in \mathbf{C}^0(\overline{G}) \cap \overset{\circ}{\mathbf{W}}{}_0^1(G)$ and satisfies the integral identity

$$\begin{aligned}
&\int_G \left\{a^{ij}(x)u_{x_j}\eta_{x_i} - a^i(x)u_{x_i}\eta(x) - b(x)u(x)\eta(x)\right\}dx \\
&\quad + \int_{\Sigma_0} \frac{1}{r}\sigma(\omega)u(x)\eta(x)ds + \int_{\partial G} \frac{1}{r}\gamma(\omega)u(x)\eta(x)ds \\
&= \int_{\partial G} g(x)\eta(x)ds + \int_{\Sigma_0} h(x)\eta(x)ds - \int_G f(x)\eta(x)dx
\end{aligned} \tag{II}$$

for all functions $\eta(x) \in \mathbf{C}^0(\overline{G}) \cap \overset{\circ}{\mathbf{W}}{}_0^1(G)$.

We assume, without loss of generality, that there exists $d > 0$ such that G_0^d is *a rotational cone* with the vertex at $\mathscr{O}$ and the aperture $\omega_0 \in (0,2\pi)$, thus

$$\Gamma_0^d = \left\{(r,\omega)\Big| x_1^2 = \cot^2\frac{\omega_0}{2}\sum_{i=2}^n x_i^2;\ r \in (0,d),\ \omega_1 = \frac{\omega_0}{2}\right\}. \tag{2.1}$$

By means of the direct calculation we obtain

LEMMA 2.1.

$$x_i\cos(\vec{n},x_i)|_{\Gamma_0^d} = 0, \textit{ and } \cos(\vec{n},x_1)|_{\Gamma_0^d} = -\sin\frac{\omega_0}{2}, \tag{2.2}$$

LEMMA 2.2. *Let $u(x)$ be a weak solution of (L). For any function $\eta \in \mathbf{C}^0(\overline{G}) \cap \overset{\circ}{\mathbf{W}}{}^1_0(G)$ the equality*

$$\int\limits_{G_0^\rho} \Big\{ a^{ij}(x) u_{x_j} \eta_{x_i} + \big(f(x) - a^i(x) u_{x_i} - b(x) u(x)\big) \eta(x) \Big\} dx$$

$$= \int\limits_{\Omega_\rho} a^{ij}(x) u_{x_j} \eta(x) \cos(r, x_i) d\Omega_\rho + \int\limits_{\Gamma_0^\rho} \left(g(x) - \frac{1}{r} \gamma(\omega) u(x) \right) \eta(x) ds$$

$$+ \int\limits_{\Sigma_0^\rho} \left(h(x) - \frac{1}{r} \sigma(\omega) u(x) \right) \eta(x) ds \qquad (II)_{loc}$$

holds for a.e. $\rho \in (0, d)$.

Proof. The proof is analogous to the proof of Lemma 5.2 [2] (pp. 167–170). Regarding to the equation we assume that the following **conditions** are satisfied:

(a) the condition of the uniform ellipticity:

$$\nu_\pm \xi^2 \leqslant a^{ij}_\pm(x) \xi_i \xi_j \leqslant \mu_\pm \xi^2, \quad \forall x \in \overline{G_\pm}, \quad \forall \xi \in \mathbb{R}^n;$$
$$\nu_\pm, \mu_\pm = const > 0, \text{ and } a^{ij}(0) = a\delta_i^j,$$

where δ_i^j is the Kronecker symbol and $a = \begin{cases} a_+, & x \text{ in } \overline{G_+}, \\ a_-, & x \text{ in } \overline{G_-}, \end{cases}$ $a_\pm$ are positive constants; we denote $a_* = \min\{a_+, a_-\} > 0$, $\nu_* = \min\{\nu_-, \nu_+\}$, $\mu^* = \max\{\mu_-, \mu_+\}$;

(b) $a^{ij} \in \mathbf{C}^0(\overline{G})$, $a^i \in \mathbf{L}_p(G)$, $b, f \in \mathbf{L}_{p/2}(G) \cap \mathbf{L}_2(G)$; $n < p \leqslant 2n$, for them the inequalities

$$\left(\sum_{i,j=1}^n |a^{ij}(x) - a^{ij}(y)|^2 \right)^{\frac{1}{2}} \leqslant a\mathscr{A}(|x - y|);$$

$$|x| \left(\sum_{i=1}^n |a^i(x)|^2 \right)^{\frac{1}{2}} + |x|^2 |b(x)| \leqslant a\mathscr{A}(|x|)$$

hold for $x, y \in \overline{G_\pm}$, where $\mathscr{A}(r)$ is a monotonically increasing, nonnegative function, **continuous at 0**, *$\mathscr{A}(0) = 0$;*

(c) $b(x) \leqslant 0$ in G; $\sigma(\omega) \geqslant \nu_0 > 0$ on σ_0; $\gamma(\omega) \geqslant \nu_0 > 0$ on ∂G.

(d) there exist numbers $f_1 \geqslant 0$, $g_1 \geqslant 0$, $h_1 \geqslant 0$, $s > 1$ such that

$$|f(x)| \leqslant f_1 |x|^{s-2}, \ |g(x)| \leqslant g_1 |x|^{s-1}, \ |h(x)| \leqslant h_1 |x|^{s-1}, \ x \in G_0^d$$

$\gamma(\omega)$ is a positive bounded piecewise smooth function on $\partial\Omega$, $\sigma(\omega)$ is a positive continuous function on σ_0;

Our main result is the following theorem:

THEOREM 2.3. *Let u be a weak solution of the problem (L) and the assumptions (a)–(d) be satisfied with the function $\mathscr{A}(r)$ which is Dini-continuous at zero. Then there exist $d \in (0,1)$ and the positive constants C_0, C_1, C_2 depending only on n, ν_*, μ^*, a_*, p, λ, $\left\| \sum_{i=1}^{n} |a^i(x)|^2 \right\|_{\mathbf{L}_{p/2}(G)}$, ω_0, s, meas G, diam G and on the quantity $\int_0^1 \frac{\mathscr{A}(r)}{r} dr$ such that $\forall x \in G_0^d$*

$$|u(x)| \leqslant C_0 \Big(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\nu_0}} g_1 + \frac{1}{\sqrt{\nu_0}} h_1 \Big) \cdot \begin{cases} |x|^{\lambda}, & \text{if } s > \lambda, \\ |x|^{\lambda} \ln\Big(\frac{1}{|x|}\Big), & \text{if } s = \lambda, \\ |x|^{s}, & \text{if } s < \lambda, \end{cases} \tag{2.3}$$

where λ is defined by (1.2). Suppose, in addition, that

$$a^{ij} \in \mathbf{C}^1(G), \quad \sigma \in C^1(\sigma_0), \quad \gamma \in \mathbf{C}^1(\partial G), \quad f \in \mathbf{V}^0_{p,2p-n}(G),$$
$$h \in V^{1-1/p}_{p,2p-n}(\sigma_0), \quad g \in \mathbf{V}^{1-1/p}_{p,2p-n}(\partial G); \quad p > n$$

and

$$\tau_s =: \sup_{\rho > 0} \rho^{-s} \Big(\|h\|_{V^{1-\frac{1}{p}}_{p,2p-n}(\Sigma^{\rho}_{\rho/2})} + \|g\|_{\mathbf{V}^{1-\frac{1}{p}}_{p,2p-n}(\Gamma^{\rho}_{\rho/2})} \Big) < \infty. \tag{2.4}$$

Then for $\forall x \in G_0^d$

$$|\nabla u(x)| \leqslant C_1 \Big(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\nu_0}} g_1 + \frac{1}{\sqrt{\nu_0}} h_1 + \tau_s \Big) \cdot \begin{cases} |x|^{\lambda-1}, & \text{if } s > \lambda, \\ |x|^{\lambda-1} \ln\Big(\frac{1}{|x|}\Big), & \text{if } s = \lambda, \\ |x|^{s-1}, & \text{if } s < \lambda. \end{cases} \tag{2.5}$$

Furthermore, if $u \in \mathbf{V}^2_{p,2p-n}(G)$, then

$$\|u\|_{\mathbf{V}^2_{p,2p-n}(G_0^{\rho})} \leqslant C_2 \Big(\|u\|_{2,G} + f_1 + \frac{1}{\sqrt{\nu_0}} g_1 + \frac{1}{\sqrt{\nu_0}} h_1 + \tau_s \Big) \cdot \begin{cases} \rho^{\lambda}, & \text{if } s > \lambda, \\ \rho^{\lambda} \ln\Big(\frac{1}{\rho}\Big), & \text{if } s = \lambda, \\ \rho^{s}, & \text{if } s < \lambda. \end{cases} \tag{2.6}$$

3. Global integral estimates

At first we will obtain a global estimate for the Dirichlet integral.

THEOREM 3.1. *Let $u(x)$ be a weak solution of the problem (L). Let assumptions (a)–(c) be satisfied. Suppose, in addition, that $h \in L_2(\Sigma_0)$, $g \in \mathbf{L}_2(\partial G)$ Then the inequality*

$$\int_G \nu|\nabla u|^2 dx + \int_{\Sigma_0} \frac{\sigma(\omega)}{r} u^2(x) ds + \int_{\partial G} \frac{\gamma(\omega)}{r} u^2(x) ds$$
$$\leqslant C\left\{ \int_G u^2(x)dx + \int_G f^2(x)dx + \frac{1}{\nu_0}\int_{\Sigma_0} h^2(x)ds + \frac{1}{\nu_0}\int_{\partial G} g^2(x)ds \right\} \quad (3.1)$$

holds, where the constant $C > 0$ depends only on $p, n, \nu_*, \left\| \sum_{i=1}^n |a^i(x)|^2 \right\|_{\mathbf{L}_{p/2}(G)}$ and meas G, diam G.

Proof. Setting in (II) $\eta(x) = u(x)$ and using the classical Hölder inequality, by assumptions $(a), (c)$, we get

$$\int_G \nu|\nabla u|^2 dx + \int_{\Sigma_0} \frac{\sigma(\omega)}{r} u^2(x) ds + \int_{\partial G} \frac{\gamma(\omega)}{r} u^2(x) ds$$
$$\leqslant \int_G \sqrt{\sum_{i=1}^n |a^i(x)|^2} |u(x)||\nabla u(x)| dx + \int_{\Sigma_0} |u(x)||h(x)| ds$$
$$+ \int_{\partial G} |u(x)||g(x)| ds + \int_{\partial G} |u(x)||f(x)| dx. \quad (3.2)$$

Further, by assumptions $(b), (c)$, as well as by the Cauchy inequality and the integral Hölder inequality, we have:

$$\int_G \sqrt{\sum_{i=1}^n |a^i(x)|^2} |u(x)||\nabla u(x)| dx$$
$$\leqslant \frac{\varepsilon}{2}\int_G \nu|\nabla u(x)|^2 dx + \frac{1}{2\varepsilon\nu_*}\int_G \sum_{i=1}^n |a^i(x)|^2 |u(x)|^2 dx$$
$$\leqslant \frac{\varepsilon}{2}\int_G \nu|\nabla u(x)|^2 dx + \frac{1}{2\varepsilon\nu_*}\left(\int_G \left(\sum_{i=1}^n |a^i(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} \cdot \left(\int_G |u(x)|^{\frac{2p}{p-2}} dx \right)^{\frac{p-2}{p}},$$
$$p > 2, \ \forall \varepsilon > 0.$$

Now we apply the inequality

$$\|u\|^2_{\mathbf{L}_{\frac{2p}{p-2}}(G)} \leqslant \delta\|\nabla u\|^2_{\mathbf{L}_2(G)} + c(\delta, p, n, \text{meas } G)\|u\|^2_{\mathbf{L}_2(G)}, \quad p > n, \ \forall\delta > 0$$

(see, for example, (2.19) §2, chapter II [9]); hence it follows that

$$\int_G \sqrt{\sum_{i=1}^n |a^i(x)|^2}|u(x)||\nabla u(x)|dx \leqslant \frac{\varepsilon}{2}\int_G \nu|\nabla u(x)|^2 dx + \frac{1}{2\varepsilon\nu_*^2}\left\|\sum_{i=1}^n |a^i(x)|^2\right\|_{\mathbf{L}_{p/2}(G)} \times \int_G \left(\delta\nu|\nabla u(x)|^2 + c(\delta, p, n, \text{meas } G)u^2\right)dx,$$

$$\forall\varepsilon > 0, \ \forall\delta > 0. \tag{3.3}$$

We choose $\delta = \frac{\varepsilon^2\nu_*^2}{\left\|\sum\limits_{i=1}^n |a^i(x)|^2\right\|_{\mathbf{L}_{p/2}(G)}}$. As a result we obtain from (3.2)–(3.3)

$$(1-\varepsilon)\int_G \nu|\nabla u(x)|^2 dx + \int_{\Sigma_0}\frac{\sigma(\omega)}{r}u^2(x)ds + \int_{\partial G}\frac{\gamma(\omega)}{r}u^2(x)ds \tag{3.4}$$

$$\leqslant c\int_G |u(x)|^2 dx + \int_{\Sigma_0}|u(x)||h(x)|ds + \int_{\partial G}|u(x)||g(x)|ds + \int_{\partial G}|u(x)||f(x)|dx, \tag{3.5}$$

where $c = const\left(\varepsilon, p, n, \nu_*, \left\|\sum\limits_{i=1}^n |a^i(x)|^2\right\|_{\mathbf{L}_{p/2}(G)}, \text{meas } G\right)$. Further, by the Cauchy inequality, in virtue of the assumption (c), we obtain

$$\int_{\Sigma_0}|u(x)||h(x)|ds = \int_{\Sigma_0}\left(\sqrt{\frac{\sigma(\omega)}{r}}|u(x)|\right)\left(\sqrt{\frac{r}{\sigma(\omega)}}|h(x)|\right)ds \leqslant \frac{1}{2}\int_{\Sigma_0}\frac{\sigma(\omega)}{r}u^2(x)ds + \frac{\text{diam}\, G}{2\nu_0}\int_{\Sigma_0}h^2(x)ds;$$

$$\int_{\partial G}|u(x)||g(x)|ds = \int_{\partial G}\left(\sqrt{\frac{\gamma(\omega)}{r}}|u(x)|\right)\left(\sqrt{\frac{r}{\gamma(\omega)}}|g(x)|\right)ds \leqslant \frac{1}{2}\int_{\partial G}\frac{\gamma(\omega)}{r}u^2(x)ds + \frac{\text{diam}\, G}{2\nu_0}\int_{\partial G}g^2(x)ds;$$

$$\int_{\partial G}|u(x)||f(x)|dx \leqslant \frac{1}{2}\int_{\partial G}|u(x)|^2 dx + \frac{1}{2}\int_{\partial G}|f(x)|^2 dx.$$

Hence and from (3.4) with $\varepsilon = \frac{1}{2}$ we get the desired inequality (3.1)
Now we will obtain a global estimate for the weighted Dirichlet integral.

THEOREM 3.2. *Let $u(x)$ be a weak solution of the problem (L) and λ be as in (1.2). Let assumptions (a)–(c) be satisfied. Suppose, in addition, that*

$$f \in \overset{\circ}{\mathbf{W}}{}^{0}_{\alpha}(G), \int_{\Sigma_0} r^{\alpha-1} h^2(x)ds < \infty, \int_{\partial G} r^{\alpha-1} g^2(x)ds < \infty, \quad 4-n \leqslant \alpha \leqslant 2.$$

Then $u \in \overset{\circ}{\mathbf{W}}{}^{1}_{\alpha-2}(G)$ and

$$\int_G a\left(r^{\alpha-2}|\nabla u(x)|^2 + r^{\alpha-4}u^2(x)\right)dx + \int_{\Sigma_0} r^{\alpha-3}\sigma(\omega)u^2(x)ds + \int_{\partial G} r^{\alpha-3}\gamma(\omega)u^2(x)ds$$
$$\leqslant C\Big\{\int_G \left(u^2(x) + (1+r^\alpha)f^2(x)\right)dx + \int_{\Sigma_0} r^{\alpha-1}h^2(x)ds + \int_{\partial G} r^{\alpha-1}g^2(x)ds\Big\}, \qquad (3.6)$$

where the constant $C>0$ depends only on $p, n, \nu_*, \mu^*, a_*, \nu_0, \alpha, \lambda$, $\left\|\sum_{i=1}^n |a^i(x)|^2\right\|_{\mathbf{L}_{p/2}(G)}$, meas G.

Proof. Setting in (II) $\eta(x) = r_\varepsilon^{\alpha-2}u(x)$, with regard to $\eta_{x_i} = r_\varepsilon^{\alpha-2}u_{x_i} + (\alpha-2)r_\varepsilon^{\alpha-3}\frac{x_i-\varepsilon l_i}{r_\varepsilon}u(x)$ we obtain

$$\int_G a r_\varepsilon^{\alpha-2}|\nabla u(x)|^2dx + \int_{\Sigma_0} r^{-1}r_\varepsilon^{\alpha-2}\sigma(\omega)u^2(x)ds + \int_{\partial G} r^{-1}r_\varepsilon^{\alpha-2}\gamma(\omega)u^2(x)ds$$
$$= \frac{2-\alpha}{2}\int_G a r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)(u^2)_{x_i}dx + (2-\alpha)\int_G \left(a^{ij}(x)-a^{ij}(0)\right)r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)u_{x_j}u(x)dx$$
$$-\int_G \left(a^{ij}(x)-a^{ij}(0)\right)r_\varepsilon^{\alpha-2}u_{x_i}u_{x_j}dx + \int_G \left(a^i(x)u_{x_i} + b(x)u(x) - f(x)\right)r_\varepsilon^{\alpha-2}u(x)dx$$
$$+\int_{\Sigma_0} r_\varepsilon^{\alpha-2}u(x)h(x)ds + \int_{\partial G} r_\varepsilon^{\alpha-2}u(x)g(x)ds. \qquad (3.7)$$

We transform the first integral on the right by integrating by parts

$$\int_G a r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\frac{\partial u^2}{\partial x_i}dx$$
$$= \int_{G_+} a_+ r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\frac{\partial u_+^2}{\partial x_i}dx + \int_{G_-} a_- r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\frac{\partial u_-^2}{\partial x_i}dx$$
$$= -\int_G a u^2(x)\frac{\partial}{\partial x_i}\left(r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\right)dx + \int_{\partial G_+} a_+u_+^2(x)r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\cos(\overrightarrow{n},x_i)ds$$
$$+\int_{\partial G_-} a_-u_-^2(x)r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\cos(\overrightarrow{n},x_i)ds$$

$$= -\int\limits_G au^2(x)\frac{\partial}{\partial x_i}\left(r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\right)dx + \int\limits_{\partial G} au^2(x)r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\cos(\overrightarrow{n},x_i)ds$$
$$+ [a]_{\Sigma_0}\int\limits_{\Sigma_0} u^2(x)r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\cos(\overrightarrow{n},x_i)ds, \qquad (3.8)$$

because of $[u]_{\Sigma_0} = 0$. By elementary calculation we have:

1) $\frac{\partial}{\partial x_i}\left(r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\right) = nr_\varepsilon^{\alpha-4} + (\alpha-4)(x_i-\varepsilon l_i)r_\varepsilon^{\alpha-5}\frac{x_i-\varepsilon l_i}{r_\varepsilon} = (n+\alpha-4)r_\varepsilon^{\alpha-4};$

2) because of $\cos(\overrightarrow{n},x_i)\Big|_{\Sigma_0} = \cos(x_n,x_i) = \delta_i^n$,

$$(x_i-\varepsilon l_i)\cos(\overrightarrow{n},x_i)\Big|_{\Sigma_0} = \delta_i^n(x_i-\varepsilon l_i)\Big|_{\Sigma_0} = (x_n-\varepsilon l_n)\Big|_{\Sigma_0} = x_n\Big|_{\Sigma_0} = 0,$$

since $\Sigma_0 = \{x_n = 0\}\cap G$ and $l_n = 0$;

3) from the representation $\partial G = \Gamma_0^d \cup \Gamma_d$ and by (2.2),

$$(x_i-\varepsilon l_i)\cos(\overrightarrow{n},x_i)\Big|_{\Gamma_0^d} = -\varepsilon\sin\frac{\omega_0}{2} \implies$$
$$\int\limits_{\partial G} au^2(x)r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\cos(\overrightarrow{n},x_i)ds$$
$$= -\varepsilon\sin\frac{\omega_0}{2}\int\limits_{\Gamma_0^d} au^2(x)r_\varepsilon^{\alpha-4}ds + \int\limits_{\Gamma_d} au^2(x)r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\cos(\overrightarrow{n},x_i)ds.$$

Hence and from (3.8) it follows

$$\frac{2-\alpha}{2}\int\limits_G ar_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\frac{\partial u^2}{\partial x_i}dx$$
$$= \frac{(2-\alpha)(4-n-\alpha)}{2}\int\limits_G ar_\varepsilon^{\alpha-4}u^2(x)dx - \varepsilon\frac{2-\alpha}{2}\sin\frac{\omega_0}{2}\int\limits_{\Gamma_0^d} au^2(x)r_\varepsilon^{\alpha-4}ds$$
$$+ \frac{2-\alpha}{2}\int\limits_{\Gamma_d} au^2(x)r_\varepsilon^{\alpha-4}(x_i-\varepsilon l_i)\cos(\overrightarrow{n},x_i)ds. \qquad (3.9)$$

From (3.7), (3.9) with regard to $4-n \leqslant \alpha \leqslant 2$ we obtain following equality:

$$\int_G a r_\varepsilon^{\alpha-2}|\nabla u(x)|^2 dx + \varepsilon \frac{2-\alpha}{2} \sin\frac{\omega_0}{2} \int_{\Gamma_0^d} a u^2(x) r_\varepsilon^{\alpha-4} ds + \int_{\Sigma_0} \frac{1}{r} r_\varepsilon^{\alpha-2}\sigma(\omega) u^2(x) ds$$

$$+ \int_{\partial G} \frac{1}{r} r_\varepsilon^{\alpha-2}\gamma(\omega) u^2(x) ds$$

$$+ (2-\alpha)\int_G \big(a^{ij}(x) - a^{ij}(0)\big) u_{x_j} r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i) u(x) dx$$

$$- \int_G r_\varepsilon^{\alpha-2}\big(a^{ij}(x) - a^{ij}(0)\big) u_{x_i} u_{x_j} dx + \int_G \big(a^i(x) u_{x_i} + b(x)u(x) - f(x)\big) r_\varepsilon^{\alpha-2} u(x) dx$$

$$+ \int_{\Sigma_0} r_\varepsilon^{\alpha-2} u(x) h(x) ds + \int_{\partial G} r_\varepsilon^{\alpha-2} u(x) g(x) ds. \tag{3.10}$$

Now we estimate the integral over Γ_d. Because on Γ_d : $r_\varepsilon \geqslant hr \geqslant hd \Rightarrow (\alpha-3)\ln r_\varepsilon \leqslant (\alpha-3)\ln(hd)$, since $\alpha \leqslant 2$, we have $r_\varepsilon^{\alpha-3}|_{\Gamma_d} \leqslant (hd)^{\alpha-3}$ and therefore:

$$\frac{2-\alpha}{2}\int_{\Gamma_d} a u^2(x) r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\cos(\overrightarrow{n}, x_i) ds$$

$$\leqslant \frac{2-\alpha}{2}\int_{\Gamma_d} a r_\varepsilon^{\alpha-3} u^2(x) ds \leqslant \frac{2-\alpha}{2}(hd)^{\alpha-3}\int_{\Gamma_d} a u^2(x) ds$$

$$\leqslant c_\delta \int_{G_d} u^2(x) dx + \delta \int_{G_d} |\nabla u(x)|^2 dx, \ \forall \delta > 0, \tag{3.11}$$

by (1.9). Further, by the Cauchy inequality and because of $\gamma(\omega) \geqslant \nu_0 > 0$,

$$u(x)g(x) = \Big(r^{\frac{1}{2}} \frac{1}{\sqrt{\gamma(\omega)}}|g(x)|\Big)\Big(r^{-\frac{1}{2}}\sqrt{\gamma(\omega)}|u(x)|\Big)$$

$$\leqslant \frac{\delta}{2} r^{-1}\gamma(\omega) u^2(x) + \frac{1}{2\delta\nu_0} r g^2(x), \ \forall \delta > 0;$$

taking into account property 1) of r_ε we obtain

$$\int_{\partial G} r_\varepsilon^{\alpha-2}|u(x)||g(x)| ds \leqslant \frac{\delta}{2}\int_{\partial G} r_\varepsilon^{\alpha-2}\frac{1}{r}\gamma(\omega) u^2(x) ds + \frac{1}{2\delta\nu_0}\int_{\partial G} r^{\alpha-1} g^2(x) ds, \ \forall \delta > 0. \tag{3.12}$$

Similarly, because of $\sigma(\omega) \geqslant \nu_0 > 0$,

$$\int_{\Sigma_0} r_\varepsilon^{\alpha-2}|u(x)||h(x)| ds \leqslant \frac{\delta}{2}\int_{\Sigma_0} r_\varepsilon^{\alpha-2}\frac{1}{r}\sigma(\omega) u^2(x) ds + \frac{1}{2\delta\nu_0}\int_{\Sigma_0} r^{\alpha-1} h^2(x) ds, \ \forall \delta > 0 \tag{3.13}$$

and

$$\int_G r_\varepsilon^{\alpha-2} u(x) f(x) dx \leqslant \frac{\delta}{2} \int_G a r^{-2} r_\varepsilon^{\alpha-2} u^2(x) dx + \frac{1}{2a_* \delta} \int_G r^\alpha f^2(x) dx, \ \forall \delta > 0. \quad (3.14)$$

Now we use the representation $G = G_0^d \cup G_d$. At first we estimate integrals over G_0^d. By assumption (b) and the Cauchy inequality, we obtain:

$$\begin{aligned}
&\int_{G_0^d} \Big\{ \left(a^{ij}(x) - a^{ij}(0)\right) \left(r_\varepsilon^{\alpha-2} u_{x_i} u_{x_j} + r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i) u(x) u_{x_j}\right) \\
&\qquad + r_\varepsilon^{\alpha-2} a^i(x) u_{x_i} u(x) + r_\varepsilon^{\alpha-2} b(x) u^2(x) \Big\} dx \\
&\leqslant \mathscr{A}(d) \int_{G_0^d} a \Big(r_\varepsilon^{\alpha-2} |\nabla u(x)|^2 + r_\varepsilon^{\alpha-3} |\nabla u(x)| \cdot |u(x)| \\
&\qquad + r^{-1} r_\varepsilon^{\alpha-2} |\nabla u(x)| \cdot |u(x)| + r^{-2} r_\varepsilon^{\alpha-2} u^2(x) \Big) dx \\
&\leqslant 2\mathscr{A}(d) \int_{G_0^d} a \left(r_\varepsilon^{\alpha-2} |\nabla u(x)|^2 + r^{-2} r_\varepsilon^{\alpha-2} u^2(x) + r_\varepsilon^{\alpha-4} u^2(x) \right) dx. \qquad (3.15)
\end{aligned}$$

Now we estimate integrals over G_d. By assumptions $(a), (c)$ and the Cauchy inequality and taking into account the inequality (3.3), we obtain:

$$\begin{aligned}
&\int_{G_d} \Big\{ \left(a^{ij}(x) - a^{ij}(0)\right) \left(r_\varepsilon^{\alpha-2} u_{x_i} u_{x_j} + r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i) u(x) u_{x_j}\right) \\
&\qquad + r_\varepsilon^{\alpha-2} a^i(x) u_{x_i} u(x) + r_\varepsilon^{\alpha-2} b(x) u^2(x) \Big\} dx \\
&\leqslant \mu^* \int_{G_d} \left(3 r_\varepsilon^{\alpha-2} |\nabla u(x)|^2 + r_\varepsilon^{\alpha-4} |u(x)|^2 \right) dx + \int_{G_d} r_\varepsilon^{\alpha-2} |u(x)| |\nabla u(x)| \sqrt{\sum_{i=1}^n |a^i(x)|^2} dx \\
&\leqslant C \int_{G_d} \left(\nu |\nabla u(x)|^2 + u^2(x) \right) dx, \qquad (3.16)
\end{aligned}$$

where $C = const\left(p, n, \nu_*, \mu^*, \alpha, d, \left\| \sum_{i=1}^n |a^i(x)|^2 \right\|_{\mathbf{L}_{p/2}(G)}\right)$. As a result from (3.10)–(3.16) we obtain:

$$\begin{aligned}
&\int_G a r_\varepsilon^{\alpha-2} |\nabla u(x)|^2 dx + \varepsilon \frac{2-\alpha}{2} \sin \frac{\omega_0}{2} \int_{\Gamma_0^d} a u^2(x) r_\varepsilon^{\alpha-4} ds \\
&\qquad + \int_{\Sigma_0} \frac{1}{r} r_\varepsilon^{\alpha-2} \sigma(\omega) u^2(x) ds + \int_{\partial G} \frac{1}{r} r_\varepsilon^{\alpha-2} \gamma(\omega) u^2(x) ds
\end{aligned}$$

$$\leqslant 2\mathscr{A}(d)\int_{G_0^d} a\left(r_\varepsilon^{\alpha-2}|\nabla u(x)|^2 + r^{-2}r_\varepsilon^{\alpha-2}u^2(x) + r_\varepsilon^{\alpha-4}u^2(x)\right)dx$$
$$+C\left(p,n,\nu_*,\mu^*,\alpha,d,\left\|\sum_{i=1}^n |a^i(x)|^2\right\|_{\mathbf{L}_{p/2}(G)}\right)\int_{G_d}\left(\nu|\nabla u(x)|^2+u^2(x)\right)dx$$
$$+\frac{1}{2\delta\nu_0}\int_{\partial G} r^{\alpha-1}g^2(x)ds$$
$$+\frac{1}{2\delta\nu_0}\int_{\Sigma_0} r^{\alpha-1}h^2(x)ds+\frac{\delta}{2}\int_G ar^{-2}r_\varepsilon^{\alpha-2}u^2(x)dx+\frac{1}{2a_*\delta}\int_G r^\alpha f^2(x)dx$$
$$+\frac{\delta}{2}\int_{\Sigma_0} r_\varepsilon^{\alpha-2}\frac{1}{r}\sigma(\omega)u^2(x)ds+\frac{\delta}{2}\int_{\partial G} r_\varepsilon^{\alpha-2}\frac{1}{r}\gamma(\omega)u^2(x)ds,\ \forall\delta>0. \tag{3.17}$$

In virtue of the inequality $r_\varepsilon \geqslant hr$ (see the property (1) from §1.2) we have $r_\varepsilon^{\alpha-4} \leqslant h^{-2}r^{-2}r_\varepsilon^{\alpha-2}$. Hence, by Lemma 1.5, from (3.17) it follows

$$\int_G ar_\varepsilon^{\alpha-2}|\nabla u(x)|^2dx+\int_{\Sigma_0}\frac{1}{r}r_\varepsilon^{\alpha-2}\sigma(\omega)u^2(x)ds+\int_{\partial G}\frac{1}{r}r_\varepsilon^{\alpha-2}\gamma(\omega)u^2(x)ds$$
$$\leqslant c(\lambda,\omega_0)\left(\delta+\mathscr{A}(d)\right)\Bigg\{\int_G ar_\varepsilon^{\alpha-2}|\nabla u(x)|^2dx+\int_{\Sigma_0} r^{-1}r_\varepsilon^{\alpha-2}\sigma(\omega)u^2(x)ds$$
$$+\int_{\partial G} r^{-1}r_\varepsilon^{\alpha-2}\gamma(\omega)u^2(x)ds\Bigg\}$$
$$+C\left(p,n,\nu_*,\mu^*,\alpha,d,\left\|\sum_{i=1}^n |a^i(x)|^2\right\|_{\mathbf{L}_{p/2}(G)}\right)\int_G\left(\nu|\nabla u(x)|^2+u^2(x)\right)dx$$
$$+\frac{1}{2\delta\nu_0}\int_{\partial G} r^{\alpha-1}g^2(x)ds+\frac{1}{2\delta\nu_0}\int_{\Sigma_0} r^{\alpha-1}h^2(x)ds+\frac{1}{2a_*\delta}\int_G r^\alpha f^2(x)dx,\ \forall\delta>0. \tag{3.18}$$

Now we choose $\delta = \frac{1}{4c(\lambda,\omega_0)}$ and next $d>0$ such that, by the continuity of $\mathscr{A}(r)$ at zero, $c(\lambda,\omega_0)\mathscr{A}(d) \leqslant \frac{1}{4}$. Thus we get

$$\int_G ar_\varepsilon^{\alpha-2}|\nabla u(x)|^2dx+\int_{\Sigma_0}\frac{1}{r}r_\varepsilon^{\alpha-2}\sigma(\omega)u^2(x)ds+\int_{\partial G}\frac{1}{r}r_\varepsilon^{\alpha-2}\gamma(\omega)u^2(x)ds$$
$$\leqslant C\left(p,n,\nu_*,\mu^*,a_*,\alpha,\lambda,\left\|\sum_{i=1}^n |a^i(x)|^2\right\|_{\mathbf{L}_{p/2}(G)}\right)$$

$$\times \left\{ \int_G \left(\nu|\nabla u(x)|^2 + u^2(x) + r^\alpha f^2(x)\right) dx + \frac{1}{\nu_0} \int_{\partial G} r^{\alpha-1} g^2(x) ds + \frac{1}{\nu_0} \int_{\Sigma_0} r^{\alpha-1} h^2(x) ds \right\},$$
$$\forall \varepsilon > 0. \tag{3.19}$$

We observe that the right hand side of (3.19) does not depend on ε. Therefore we can perform the passage to the limit as $\varepsilon \to +0$ by the Fatou Theorem. Hence it follows that

$$\int_G ar^{\alpha-2}|\nabla u(x)|^2 dx + \int_{\Sigma_0} r^{\alpha-3}\sigma(\omega) u^2(x) ds + \int_{\partial G} r^{\alpha-3}\gamma(\omega) u^2(x) ds$$
$$\leqslant C\left(p, n, \nu_*, \mu^*, a_*, \alpha, \lambda, \left\| \sum_{i=1}^n |a^i(x)|^2 \right\|_{\mathbf{L}_{p/2}(G)}\right)$$
$$\times \left\{ \int_G \left(\nu|\nabla u(x)|^2 + u^2(x) + r^\alpha f^2(x)\right) dx + \frac{1}{\nu_0} \int_{\partial G} r^{\alpha-1} g^2(x) ds + \frac{1}{\nu_0} \int_{\Sigma_0} r^{\alpha-1} h^2(x) ds \right\}. \tag{3.20}$$

Now applying Theorem 3.1 and Corollary 2.4 (see inequality (1.4)), from (3.20) we get the desired estimate (3.6).

4. Local integral estimates

4.1. Local estimate at the boundary

By the applying the Moser iteration method (see §8.6 [5] or §1 chapter 4 [3]), we derive a result asserting the local boundedness (near the conical point) of the weak solution of problem (L).

THEOREM 4.1. *Let $u(x)$ be a weak solution of the problem (L). Let assumptions (a)–(c) be satisfied. Suppose, in addition, that $h \in L_\infty(\Sigma_0)$, $g \in \mathbf{L}_\infty(\partial G)$. Then the inequality*

$$\sup_{G_0^{\varkappa\rho}} |u(x)| \leqslant \frac{C}{(1-\varkappa)^{n/t}} \left\{ \rho^{-n/t} \|u\|_{t,G_0^\rho} + \rho^{2(1-n/p)} \|f\|_{p/2,G_0^\rho} + \rho\left(\|g\|_{\infty,\Gamma_0^\rho} + \|h\|_{\infty,\Sigma_0^\rho} \right) \right\} \tag{4.1}$$

holds for any $t > 0$, $\varkappa \in (0,1)$ and $\rho \in (0,d)$, where the constant $C > 0$ depends only on $n, \nu_*, \mu^*, t, p, \left\| \sum_{i=1}^n |a^i(x)|^2 \right\|_{\mathbf{L}_{p/2}(G)}$.

4.2. Local integral weighted estimates

THEOREM 4.2. *Let $u(x)$ be a weak solution of the problem (L) and λ be as above in* (1.2)*. Let assumptions (a)–(d) be satisfied with the function $\mathscr{A}(r)$ which is Dini-continuous at zero.*

Then $u \in \overset{\circ}{\mathbf{W}}{}^{1}_{2-n}(G)$ *and there are* $d \in (0,1)$ *and a constant* $C > 0$ *depending only on* $n, s, \lambda, a_*, \omega_0$ *and on* $\int_0^1 \frac{\mathscr{A}(r)}{r} dr$ *such that* $\forall \rho \in (0,d)$

$$\int_{G_0^\rho} \left(ar^{2-n}|\nabla u(x)|^2 + r^{-n}u^2(x)\right) dx + \int_{\Sigma_0^\rho} r^{1-n}\sigma(\omega)u^2(x)ds + \int_{\Gamma_0^\rho} r^{1-n}\gamma(\omega)u^2(x)ds$$

$$\leqslant C\left(\|u\|^2_{2,G} + f_1^2 + \frac{1}{\nu_0}g_1^2 + \frac{1}{\nu_0}h_1^2\right) \cdot \begin{cases} \rho^{2\lambda}, & \text{if } s > \lambda, \\ \rho^{2\lambda}\ln^2\left(\frac{1}{\rho}\right), & \text{if } s = \lambda, \\ \rho^{2s}, & \text{if } s < \lambda. \end{cases} \quad (4.2)$$

Proof. From Theorem 3.2 it follows that $u(x)$ belongs to $\overset{\circ}{\mathbf{W}}{}^{1}_{2-n}(G)$, so it is enough to prove the estimate (4.2). Setting $\eta(x) = r^{2-n}u(x)$ in $(II)_{loc}$, with regard to the definition (1.5) we obtain

$$U(\rho) = \rho \int_\Omega au(x)\frac{\partial u}{\partial r}\bigg|_{r=\rho} d\Omega + \int_{\Omega_\rho} r^{2-n}u(x)\left(a^{ij}(x) - a^{ij}(0)\right)u_{x_j}\cos(r,x_i)d\Omega_\rho$$

$$+ \int_{\Gamma_0^\rho} r^{2-n}u(x)g(x)ds + \int_{\Sigma_0^\rho} r^{2-n}u(x)h(x)ds$$

$$+ \int_{G_0^\rho} \Big\{ -r^{2-n}\left(a^{ij}(x) - a^{ij}(0)\right)u_{x_i}u_{x_j} + (n-2)r^{-n}u(x)a^{ij}(x)x_iu_{x_j}$$

$$+ r^{2-n}u(x)a^i(x)u_{x_i} + r^{2-n}b(x)u^2(x) - r^{2-n}u(x)f(x)\Big\}dx. \quad (4.3)$$

Now we transform some integrals on the right:

$$(n-2)\int_{G_0^\rho} r^{-n}u(x)a^{ij}(x)x_iu_{x_j}dx$$

$$= \frac{n-2}{2}\int_{G_0^\rho} ar^{-n}x_i\frac{\partial u^2}{\partial x_i}dx + (n-2)\int_{G_0^\rho} r^{-n}u(x)\left(a^{ij}(x) - a^{ij}(0)\right)x_iu_{x_j}dx; \quad (4.4)$$

by the Gauss-Ostrogradskiy divergence theorem

$$\int_{G_0^\rho} ar^{-n}x_i\frac{\partial u^2}{\partial x_i}dx = -\int_{G_0^\rho} au^2(x)\left(nr^{-n} - nr^{-n}\right)dx + \rho^{-n}\int_{\Omega_\rho} au^2(x)x_i\cos(r,x_i)d\Omega_\rho$$

$$+ [a]_{\Sigma_0}\int_{\Sigma_0^\rho} r^{-n}u^2(x)x_i\cos(n,x_i)ds + \int_{\Gamma_0^\rho} ar^{-n}u^2(x)x_i\cos(n,x_i)ds. \quad (4.5)$$

Hence, since by Lemma 2.1 $x_i \cos(n,x_i)\Big|_{\Gamma_0^\rho} = 0$ and $x_i \cos(r,x_i)\Big|_{\Omega_\rho} = \rho;\ x_i \cos(n,x_i)\Big|_{\Sigma_0} = x_i \cos(x_n,x_i)\Big|_{\Sigma_0} = x_n\Big|_{\Sigma_0} = 0,$ we have

$$\frac{n-2}{2}\int\limits_{G_0^\rho} a r^{-n} x_i \frac{\partial u^2}{\partial x_i} dx = \frac{n-2}{2}\int\limits_{\Omega} a u^2(x) d\Omega. \tag{4.6}$$

Because of $b(x) \leqslant 0$ and Lemma 1.4, from (4.3)–(4.6) it follows that

$$\begin{aligned} U(\rho) \leqslant{}& \frac{\rho}{2\lambda}U'(\rho) + \rho^{2-n}\int\limits_{\Omega_\rho} u(x)\left(a^{ij}(x) - a^{ij}(0)\right)u_{x_j}\cos(r,x_i)d\Omega_\rho + \int\limits_{\Gamma_0^\rho} r^{2-n}u(x)g(x)ds \\ &+ \int\limits_{\Sigma_0^\rho} r^{2-n}u(x)h(x)ds + \int\limits_{G_0^\rho}\Big\{-r^{2-n}\left(a^{ij}(x)-a^{ij}(0)\right)u_{x_i}u_{x_j} \\ &+ (n-2)r^{-n}u(x)\left(a^{ij}(x)-a^{ij}(0)\right)x_i u_{x_j} + r^{2-n}u(x)a^i(x)u_{x_i} - r^{2-n}u(x)f(x)\Big\}dx. \end{aligned} \tag{4.7}$$

Hence, in virtue of assumption (b), it follows that

$$\begin{aligned} U(\rho) \leqslant{}& \frac{\rho}{2\lambda}U'(\rho) + \rho\mathscr{A}(\rho)\int\limits_{\Omega} a|u(x)||\nabla u(x)|d\Omega + \int\limits_{\Gamma_0^\rho} r^{2-n}|u(x)||g(x)|ds \\ &+ \int\limits_{\Sigma_0^\rho} r^{2-n}|u(x)||h(x)|ds + c_1(n)\mathscr{A}(\rho)\int\limits_{G_0^\rho} a\left(r^{2-n}|\nabla u(x)|^2 + r^{1-n}|u(x)||\nabla u(x)|\right)dx \\ &+ \int\limits_{G_0^\rho} r^{2-n}|u(x)||f(x)|dx. \end{aligned} \tag{4.8}$$

We shall obtain an upper bound for each integral on the right. At first, applying the Cauchy and Friedrichs-Wirtinger inequalities (see (1.3)) with regard to (1.7), we have

$$\int\limits_{\Omega} a\rho|u(x)||\nabla u(x)|d\Omega \leqslant \frac{1}{2}\int\limits_{\Omega} a\left(\rho^2|\nabla u(x)|^2 + |u(x)|^2\right)d\Omega \leqslant c_2(\lambda)\rho U'(\rho); \tag{4.9}$$

$$\int\limits_{G_0^\rho} a r^{1-n}|u(x)||\nabla u(x)|dx \leqslant \int\limits_{G_0^\rho} a\left(r^{2-n}|\nabla u(x)|^2 + r^{-n}|u(x)|^2\right)dx \leqslant c_3(\lambda)U(\rho) \tag{4.10}$$

in virtue of inequality (1.4); and with $\forall\delta > 0$

$$\int\limits_{\Gamma_0^\rho} r^{2-n}|u(x)||g(x)|ds = \int\limits_{\Gamma_0^\rho}\left(r^{\frac{1-n}{2}}\sqrt{\gamma(\omega)}|u(x)|\right)\left(r^{\frac{3-n}{2}}\frac{1}{\sqrt{\gamma(\omega)}}|g(x)|\right)ds$$
$$\leqslant \frac{\delta}{2}\int\limits_{\Gamma_0^\rho} r^{1-n}\gamma(\omega)|u(x)|^2 ds + \frac{1}{2\delta\nu_0}\int\limits_{\Gamma_0^\rho} r^{3-n}|g(x)|^2 ds; \tag{4.11}$$

$$\int\limits_{\Sigma_0^\rho} r^{2-n}|u(x)||h(x)|ds = \int\limits_{\Sigma_0^\rho}\left(r^{\frac{1-n}{2}}\sqrt{\sigma(\omega)}|u(x)|\right)\left(r^{\frac{3-n}{2}}\frac{1}{\sqrt{\sigma(\omega)}}|g(x)|\right)ds$$
$$\leqslant \frac{\delta}{2}\int\limits_{\Sigma_0^\rho} r^{1-n}\sigma(\omega)|u(x)|^2 ds + \frac{1}{2\delta\nu_0}\int\limits_{\Sigma_0^\rho} r^{3-n}|h(x)|^2 ds; \tag{4.12}$$

$$\int\limits_{G_0^\rho} r^{2-n}|u(x)||f(x)|dx \leqslant \frac{\delta}{2a_*}\int\limits_{G_0^\rho} ar^{-n}|u(x)|^2 dx + \frac{1}{2\delta}\int\limits_{G_0^\rho} r^{4-n}|f(x)|^2 dx$$
$$\leqslant \frac{\delta}{2a_*}c_4(\lambda)U(\rho) + \frac{1}{2\delta}\int\limits_{G_0^\rho} r^{4-n}|f(x)|^2 dx \tag{4.13}$$

in virtue of inequality (1.4). Thus from (4.8)–(4.13) we get

$$[1 - c_5(n,\lambda,a_*)(\delta + \mathscr{A}(\rho))]\,U(\rho)$$
$$\leqslant \frac{\rho}{2\lambda}\left(1 + c_6(\lambda)\mathscr{A}(\rho)\right)U'(\rho)$$
$$+ \frac{1}{2\delta}\left\{\int\limits_{G_0^\rho} r^{4-n}|f(x)|^2 dx + \frac{1}{\nu_0}\int\limits_{\Gamma_0^\rho} r^{3-n}|g(x)|^2 ds + \frac{1}{\nu_0}\int\limits_{\Sigma_0^\rho} r^{3-n}|h(x)|^2 ds\right\},\ \forall\delta > 0.$$

But, by the condition (d),

$$\int\limits_{G_0^\rho} r^{4-n}|f(x)|^2 dx + \frac{1}{\nu_0}\int\limits_{\Gamma_0^\rho} r^{3-n}|g(x)|^2 ds + \frac{1}{\nu_0}\int\limits_{\Sigma_0^\rho} r^{3-n}|h(x)|^2 ds \leqslant 2c_0\left(f_1^2 + \frac{1}{\nu_0}g_1^2 + \frac{1}{\nu_0}h_1^2\right)\cdot\rho^{2s},$$

where c_0 depends only on n, s, ω_0. Hence we obtain

$$U'(\rho) - \frac{2\lambda}{\rho}\cdot\frac{1 - c_5(\delta + \mathscr{A}(\rho))}{1 + c_6\mathscr{A}(\rho)}U(\rho) + 2\lambda c_0\left(f_1^2 + \frac{1}{\nu_0}g_1^2 + \frac{1}{\nu_0}h_1^2\right)\frac{\delta^{-1}}{1 + c_6\mathscr{A}(\rho)}\rho^{2s-1} \geqslant 0,$$
$$\forall\delta > 0.$$

Thus we have the differential inequality (CP) §2.4 with

$$\mathscr{P}(\rho) = \frac{2\lambda}{\rho}\cdot[1 - c_5(n,\lambda,a_*)(\delta + \mathscr{A}(\rho))],\ \forall\delta > 0; \quad \mathscr{N}(\rho) \equiv 0;$$

$$\mathscr{Q}(\rho) = \frac{\lambda}{s} c_0 \left(f_1^2 + \frac{1}{\nu_0} g_1^2 + \frac{1}{\nu_0} h_1^2 \right) \cdot \delta^{-1} \rho^{2s-1}, \ \forall \delta > 0; \tag{4.14}$$

$$U_0 = C \Big\{ \int_G \left(u^2 + (1 + r^{4-n}) f^2(x) \right) dx + \int_{\Sigma_0} r^{3-n} h^2(x) ds + \int_{\partial G} r^{3-n} g^2(x) ds \Big\},$$

by (3.6) with $\alpha = 4 - n$.

1) $\underline{s > \lambda}$. Choosing $\delta = \rho^{\varepsilon}$, $\forall \varepsilon > 0$,

$$\mathscr{P}(\rho) = \frac{2\lambda}{\rho} \cdot [1 - c_5(n, \lambda, a_*)(\rho^{\varepsilon} + \mathscr{A}(\rho))];$$

$$\mathscr{Q}(\rho) = \frac{\lambda}{s} c_0 \left(f_1^2 + \frac{1}{\nu_0} g_1^2 + \frac{1}{\nu_0} h_1^2 \right) \cdot \rho^{2s-1-\varepsilon}.$$

Since $\mathscr{P}(\rho) = \frac{2\lambda}{\rho} - \frac{\mathscr{K}(\rho)}{\rho}$, where $\mathscr{K}(\rho)$ satisfies the Dini condition at zero we have

$$-\int_\rho^\tau \mathscr{P}(s) ds = -2\lambda \ln \left(\frac{\tau}{\rho} \right) + \int_\rho^\tau \frac{\mathscr{K}(s)}{s} ds \leqslant \ln \left(\frac{\rho}{\tau} \right)^{2\lambda} + \int_0^d \frac{\mathscr{K}(r)}{r} dr \implies$$

$$\exp \left(-\int_\rho^d \mathscr{P}(\tau) d\tau \right) \leqslant \left(\frac{\rho}{d} \right)^{2\lambda} \exp \left(\int_0^d \frac{\mathscr{K}(\tau)}{\tau} d\tau \right) = K_0 \left(\frac{\rho}{d} \right)^{2\lambda};$$

$$\exp \left(-\int_\rho^\tau \mathscr{P}(\tau) d\tau \right) \leqslant \left(\frac{\rho}{\tau} \right)^{2\lambda} \exp \left(\int_0^d \frac{\mathscr{K}(\tau)}{\tau} d\tau \right) = K_0 \left(\frac{\rho}{\tau} \right)^{2\lambda}.$$

We have as well:

$$\int_\rho^d \mathscr{Q}(\tau) \exp \Big(-\int_\rho^\tau \mathscr{P}(\sigma) d\sigma \Big) d\tau \leqslant \frac{\lambda c_0 K_0}{s} \left(f_1^2 + \frac{1}{\nu_0} g_1^2 + \frac{1}{\nu_0} h_1^2 \right) \rho^{2\lambda} \int_\rho^d \tau^{2s-2\lambda-\varepsilon-1} d\tau$$

$$\leqslant \frac{\lambda c_0 K_0}{s} \left(f_1^2 + \frac{1}{\nu_0} g_1^2 + \frac{1}{\nu_0} h_1^2 \right) \cdot \frac{d^{s-\lambda}}{s-\lambda} \rho^{2\lambda},$$

since $s > \lambda$ and we can choose $\varepsilon = s - \lambda$.

Now we apply Theorem 1.6: then from (1.11) by virtue of the deduced inequalities and with regard to (1.4) for $\alpha = 4 - n$ we obtain the statement (4.2) for $s > \lambda$.

2) $\underline{s = \lambda.}$ Taking in (4.14) any function $\delta(\rho) > 0$ instead of $\delta > 0$, we obtain the problem (CP) with

$$\mathscr{P}(\rho) = \frac{2\lambda (1 - \delta(\rho))}{\rho} - c_5 \frac{\mathscr{A}(\rho)}{\rho}; \quad \mathscr{N}(\rho) = 0;$$

$$\mathscr{Q}(\rho) = c_0 \left(f_1^2 + \frac{1}{\nu_0} g_1^2 + \frac{1}{\nu_0} h_1^2 \right) \cdot \delta^{-1}(\rho) \rho^{2\lambda - 1}.$$

We choose $\delta(\rho) = \frac{1}{2\lambda \ln\left(\frac{ed}{\rho}\right)}$, $0 < \rho < d$, where e is the Euler number. Then we obtain

$$-\int_{\rho}^{\tau} \mathscr{P}(\sigma)d\sigma \leqslant \ln\left(\frac{\rho}{\tau}\right)^{2\lambda} + \int_{\rho}^{\tau} \frac{d\sigma}{\sigma \ln\left(\frac{ed}{\sigma}\right)} + c_5 \int_0^d \frac{\mathscr{A}(\tau)}{\tau} d\tau$$

$$= \ln\left(\frac{\rho}{\tau}\right)^{2\lambda} + \ln\left(\frac{\ln\left(\frac{ed}{\rho}\right)}{\ln\left(\frac{ed}{\tau}\right)}\right) + c_5 \int_0^d \frac{\mathscr{A}(\tau)}{\tau} d\tau \Longrightarrow$$

$$\Longrightarrow \quad \exp\left(-\int_{\rho}^{d} \mathscr{P}(\tau)d\tau\right) \leqslant \left(\frac{\rho}{d}\right)^{2\lambda} \ln\left(\frac{ed}{\rho}\right) \exp\left(c_5 \int_0^d \frac{\mathscr{A}(\tau)}{\tau} d\tau\right),$$

$$\exp\left(-\int_{\rho}^{\tau} \mathscr{P}(\sigma)d\sigma\right) \leqslant \left(\frac{\rho}{\tau}\right)^{2\lambda} \cdot \frac{\ln\left(\frac{ed}{\rho}\right)}{\ln\left(\frac{ed}{\tau}\right)} \exp\left(c_5 \int_0^d \frac{\mathscr{A}(\tau)}{\tau} d\tau\right).$$

In this case we also have

$$\int_{\rho}^{d} \mathscr{Q}(\tau) \exp\left(-\int_{\rho}^{\tau} \mathscr{P}(\sigma)d\sigma\right) d\tau$$

$$\leqslant c_6 \left(f_1^2 + \frac{1}{\nu_0} g_1^2 + \frac{1}{\nu_0} h_1^2\right) \rho^{2\lambda} \ln\left(\frac{ed}{\rho}\right) \cdot \int_{\rho}^{d} \frac{d\tau}{\tau \delta(\tau) \ln\left(\frac{ed}{\tau}\right)}$$

$$\leqslant 2\lambda c_6 \left(f_1^2 + \frac{1}{\nu_0} g_1^2 + \frac{1}{\nu_0} h_1^2\right) \cdot \rho^{2\lambda} \ln^2\left(\frac{ed}{\rho}\right).$$

Now we apply Theorem 1.6, and from (1.11), by virtue of the deduced inequalities, we obtain

$$U(\rho) \leqslant \widetilde{c}_6 (U_0 + f_1^2 + \frac{1}{\gamma_0} g_1^2 + \frac{1}{\sigma_0} h_1^2) \rho^{2\lambda} \ln^2 \frac{1}{\rho}, \quad 0 < \rho < d < \frac{1}{e}.$$

Thus we proved the statement (4.2) for $s = \lambda$.

3) $\underline{0 < s < \lambda.}$ Now similarly to case 1) with regard to (4.14) we have

$$\exp\left(-\int_{\rho}^{d} \mathscr{P}(\tau)d\tau\right) \leqslant \left(\frac{\rho}{d}\right)^{2\lambda(1-\delta)} \exp\left(c_5 \int_0^d \frac{\mathscr{A}(\tau)}{\tau} d\tau\right) = c_7 \left(\frac{\rho}{d}\right)^{2\lambda(1-\delta)}.$$

In this case we also have

$$\int_\rho^d \mathscr{Q}(\tau)\exp\Big(-\int_\rho^\tau \mathscr{P}(\sigma)d\sigma\Big)d\tau \leqslant c_0\left(f_1^2+\frac{1}{\nu_0}g_1^2+\frac{1}{\nu_0}h_1^2\right)\cdot\delta^{-1}\rho^{2\lambda(1-\delta)}\int_\rho^d \tau^{2s-2\lambda(1-\delta)-1}d\tau \leqslant c_8\left(f_1^2+\frac{1}{\nu_0}g_1^2+\frac{1}{\nu_0}h_1^2\right)\cdot\rho^{2s},$$

if we choose $\delta\in(0,\frac{\lambda-s}{\lambda})$.

Now we apply Theorem 1.6, and then from (1.11), by virtue of the deduced inequalities, we obtain

$$U(\rho)\leqslant c_9\Big(U_0\rho^{2\lambda(1-\delta)}+\left(f_1^2+\frac{1}{\nu_0}g_1^2+\frac{1}{\nu_0}h_1^2\right)\cdot\rho^{2s}\Big)\leqslant c_{10}\left(U_0+f_1^2+\frac{1}{\nu_0}g_1^2+\frac{1}{\nu_0}h_1^2\right)\rho^{2s},$$

because of $\delta\in(0,\frac{\lambda-s}{\lambda})$. Thus we proved the statement (4.2) for $s<\lambda$.

5. The power modulus of continuity at the conical point for weak solutions

Proof of Theorem 2.3. We define the function

$$\psi(\rho)=\begin{cases}\rho^\lambda, & \text{if } s>\lambda,\\ \rho^\lambda\ln\left(\frac{1}{\rho}\right), & \text{if } s=\lambda,\\ \rho^s, & \text{if } s<\lambda\end{cases} \tag{5.1}$$

for $0<\rho<d$

By Theorem 4.1 about the local bound of the weak solution modulus we have

$$\sup_{G_0^{\rho/2}}|u(x)|\leqslant C\Big\{\rho^{-n/2}\|u\|_{2,G_0^\rho}+\rho^{2(1-n/p)}\|f\|_{p/2,G_0^\rho}+\rho\left(\|g\|_{\infty,\Gamma_0^\rho}+\|h\|_{\infty,\Sigma_0^\rho}\right)\Big\} \tag{5.2}$$

where $C=C\left(n,\nu_*,\mu^*,p,\left\|\sum_{i=1}^n|a^i(x)|^2\right\|_{\mathbf{L}_{p/2}(G)}\right)$ and $p>n$. Now, by Theorem 4.2, we have

$$\rho^{-n/2}\|u\|_{2,G_0^\rho}\leqslant 2^{n/2}\left(\int_{G_0^\rho} r^{-n}u^2(x)dx\right)^{1/2} \leqslant C\Big(\|u\|_{2,G}+\|f\|_{2,G}+\|g\|_{2,\partial G}+\|h\|_{2,\Sigma_0}+f_1+\frac{1}{\sqrt{\nu_0}}g_1+\frac{1}{\sqrt{\nu_0}}h_1\Big)\psi(\rho). \tag{5.3}$$

Further, by the assumption (d), we obtain

$$\rho^{2(1-n/p)}\|f\|_{p/2,G_0^\rho}+\rho\left(\|g\|_{\infty,\Gamma_0^\rho}+\|h\|_{\infty,\Sigma_0^\rho}\right)\leqslant c\left(f_1+\frac{1}{\sqrt{\nu_0}}g_1+\frac{1}{\sqrt{\nu_0}}h_1\right)\psi(\rho). \tag{5.4}$$

From (5.2)–(5.4) it follows that

$$\sup_{G_{\rho/4}^{\rho/2}}|u(x)|\leqslant C\Big(\|u\|_{2,G}+\|f\|_{2,G}+\|g\|_{2,\partial G}+\|h\|_{2,\Sigma_0}+f_1+\frac{1}{\sqrt{\nu_0}}g_1+\frac{1}{\sqrt{\nu_0}}h_1\Big)\psi(\rho).$$

Putting now $|x|=\frac{1}{3}\rho$ we obtain finally the desired estimate (2.3).

Now we consider two sets $G_{\rho/4}^{2\rho}$ and $G_{\rho/2}^{\rho}\subset G_{\rho/4}^{2\rho}$, $\rho>0$. We perform the change of variables $x=\rho x'$ and $u(\rho x')=\psi(\rho)v(x')$. Then the function $v(x')$ satisfies the problem

$$\begin{cases}\frac{\partial}{\partial x_i'}\left(a^{ij}(\rho x')v_{x_j'}\right)+\rho a^i(\rho x')v_{x_i'}+\rho^2 b(\rho x')v=\frac{\rho^2}{\psi(\rho)}f(\rho x'),\ x\in G_{1/2}^1,\\ [v(x')]_{\Sigma_{1/2}^1}=0,\ \left[\frac{\partial v}{\partial \nu'}\right]_{\Sigma_{1/2}^1}+\frac{1}{|x'|}\sigma(\omega)v(x')=\frac{\rho}{\psi(\rho)}h(\rho x'),\ x\in\Sigma_{1/2}^1,\\ \frac{\partial v}{\partial\nu'}+\frac{1}{|x'|}\gamma(\omega)v(x')=\frac{\rho}{\psi(\rho)}g(\rho x'),\ x\in\Gamma_{1/2}^1,\end{cases} \tag{L”}$$

By the Sobolev Imbedding Theorems, we have

$$\sup_{x'\in G_{1/2}^1}|\nabla' v(x')|\leqslant c\|v\|_{\mathbf{W}^{2,p}(G_{1/2}^1)},\quad p>n. \tag{5.5}$$

By the local L^p-*a priori* estimate [12] for the solution of the equation of the (L”) inside the domains $\left(G_{1/4}^2\right)_\pm$ and near smooth portions of the boundaries $\Sigma_{1/4}^2\cup\Gamma_{1/4}^2$, we have

$$\|v\|_{\mathbf{W}^{2,p}(G_{1/2}^1)}\leqslant c\Big\{\frac{\rho^2}{\psi(\rho)}\|f\|_{\mathbf{L}^p(G_{1/4}^2)}+\frac{\rho}{\psi(\rho)}\|h\|_{W^{1-1/p,p}(\Sigma_{1/4}^2)}$$
$$+\frac{\rho}{\psi(\rho)}\|g\|_{\mathbf{W}^{1-1/p,p}(\Gamma_{1/4}^2)}+\|v\|_{\mathbf{L}^p(G_{1/4}^2)}\Big\}. \tag{5.6}$$

Returning back to the variables x, from (5.5) and (5.6), it follows that

$$\sup_{G_{\rho/2}^\rho}|\nabla u|\leqslant c\rho^{-1}\{\rho^{-n/p}\|u\|_{\mathbf{L}^p(G_{\rho/4}^{2\rho})}+\rho^{2-n/p}\|f\|_{p,G_{\rho/4}^{2\rho}}$$
$$+\rho^{2-n/p}\|g\|_{\mathbf{V}_{p,0}^{1-1/p}(\Gamma_{\rho/4}^{2\rho})}+\rho^{2-n/p}\|h\|_{V_{p,0}^{1-1/p}(\Sigma_{\rho/4}^{2\rho})}\}$$

and

$$\rho^{2-n/p}\|u\|_{\mathbf{V}_{p,0}^2(G_{\rho/2}^\rho)}\leqslant c\{\rho^{-n/p}\|u\|_{\mathbf{L}^p(G_{\rho/4}^{2\rho})}+\rho^{2-n/p}\|f\|_{p,G_{\rho/4}^{2\rho}}$$
$$+\rho^{2-n/p}\|g\|_{\mathbf{V}_{p,0}^{1-1/p}(\Gamma_{\rho/4}^{2\rho})}+\rho^{2-n/p}\|h\|_{V_{p,0}^{1-1/p}(\Sigma_{\rho/4}^{2\rho})}\}$$

or

$$\sup_{G^{\rho}_{\rho/2}} |\nabla u| \leqslant c\rho^{-1}\big\{|u|_{0,G^{2\rho}_{\rho/4}} + \|f\|_{\mathbf{V}^0_{p,2p-n}(G^{2\rho}_{\rho/4})} + \|g\|_{\mathbf{V}^{1-1/p}_{p,2p-n}(\Gamma^{2\rho}_{\rho/4})} + \|h\|_{V^{1-1/p}_{p,2p-n}(\Sigma^{2\rho}_{\rho/4})}\big\}$$

and

$$\|u\|_{\mathbf{V}^2_{p,2p-n}(G^{\rho}_{\rho/2})} \leqslant c\big\{|u|_{0,G^{2\rho}_{\rho/4}} + \|f\|_{\mathbf{V}^0_{p,2p-n}(G^{2\rho}_{\rho/4})} + \|g\|_{\mathbf{V}^{1-1/p}_{p,2p-n}(\Gamma^{2\rho}_{\rho/4})} + \|h\|_{V^{1-1/p}_{p,2p-n}(\Sigma^{2\rho}_{\rho/4})}\big\}$$

Hence, because of (2.3), (2.4) and the assumption (d), the required inequalities (2.5) and (2.6) follow.

6. Appendix

Here we consider the two dimensional transmission problem for the Laplace operator in an angular domain and investigate the corresponding eigenvalue problem. Suppose $n=2$, the domain G lies inside the corner $G_0 = \{(r,\omega)\,|r>0;\ -\frac{\omega_0}{2} < \omega < \frac{\omega_0}{2}\}$, $\omega_0 \in]0,2\pi[$; $\mathscr{O} \in \partial G$ and in the some neighborhood of $\mathscr{O}$ the boundary ∂G coincides with the sides of the corner $\omega = -\frac{\omega_0}{2}$ and $\omega = \frac{\omega_0}{2}$. We denote $\Gamma_{\pm} = \{(r,\omega)\mid r>0;\ \omega = \pm\frac{\omega_0}{2}\}$, $\Sigma_0 = \{(r,\omega)\mid r>0;\ \omega=0\}$ and we put $\sigma(\omega)\Big|_{\Sigma_0} = \sigma(0) = \sigma = const > 0$, $\gamma(\omega)\Big|_{\omega=\pm\frac{\omega_0}{2}} = \gamma_{\pm} = const > 0$. We consider the following problem:

$$\begin{cases} a_{\pm}\triangle u_{\pm} = f_{\pm}(x), & x\in G_{\pm};\\ [u]_{\Sigma_0} = 0, \quad \left[a\frac{\partial u}{\partial \vec{n}}\right]_{\Sigma_0} + \frac{\sigma}{r}u(x) = h(x), & x\in\Sigma_0;\\ \alpha_{\pm}a_{\pm}\frac{\partial u_{\pm}}{\partial \vec{n}} + \frac{1}{r}\gamma_{\pm}u_{\pm}(x) = g_{\pm}(x), & x\in\Gamma_{\pm}\setminus\mathscr{O}, \end{cases} \tag{6.1}$$

where $\alpha_{\pm}\in\{0;1\}$. It is well known that the homogeneous problem $(f(x)=h(x)=g(x)=0)$ has solution of the form $u(r,\omega) = r^{\lambda}\psi(\omega)$, where λ^2 is an eigenvalue and $\psi(\omega)$ is an associated regular eigenfunction of the problem

$$\begin{cases} \psi''_{+} + \lambda^2\psi_{+}(\omega) = 0, & \omega\in\left(0,\frac{\omega_0}{2}\right);\\ \psi''_{-} + \lambda^2\psi_{-}(\omega) = 0, & \omega\in\left(-\frac{\omega_0}{2},0\right);\\ \psi_{+}(0) = \psi_{-}(0); \quad a_{+}\psi'_{+}(0) - a_{-}\psi'_{-}(0) = \sigma\psi(0); & \\ \pm\alpha_{\pm}a_{\pm}\psi'(\pm\frac{\omega_0}{2}) + \gamma_{\pm}\psi(\pm\frac{\omega_0}{2}) = 0. & \end{cases} \tag{6.2}$$

1) the case $\lambda = 0$.

In this case the solution of our equations has the form $\psi_{\pm}(\omega) = A_{\pm}\cdot\omega + B_{\pm}$. From boundary conditions we obtain $B_{+} = B_{-} = B$ and for the finding A_{+},A_{-},B we have the system

$$\begin{cases} a_+A_+ - a_-A_- - \sigma B &= 0, \\ \left(\alpha_+a_+ + \frac{\omega_0}{2}\gamma_+\right)A_+ + \gamma_+B &= 0, \\ -\left(\alpha_-a_- + \frac{\omega_0}{2}\gamma_-\right)A_- + \gamma_-B &= 0. \end{cases}$$

Since $A_+^2 + A_-^2 + B^2 \neq 0,$ the system determinant must be equal zero; this means the equality

$$\sigma\left(\alpha_+a_+ + \frac{\omega_0}{2}\gamma_+\right)\left(\alpha_-a_- + \frac{\omega_0}{2}\gamma_-\right) + a_+\gamma_+\left(\alpha_-a_- + \frac{\omega_0}{2}\gamma_-\right) + a_-\gamma_-\left(\alpha_+a_+ + \frac{\omega_0}{2}\gamma_+\right) = 0. \quad (6.3)$$

Thus, if the equality (6.3) satisfy, then $\lambda = 0$ and the corresponding eigenfunction

$$\psi(\omega) = \begin{cases} a_-\gamma_-\left\{\left(\omega - \frac{\omega_0}{2}\right)\gamma_+ - \alpha_+a_+\right\}, & \omega \in \left[0, \frac{\omega_0}{2}\right], \\ a_+\gamma_+\left\{\left(\omega + \frac{\omega_0}{2}\right)\gamma_- - \alpha_-a_-\right\}, & \omega \in \left[-\frac{\omega_0}{2}, 0\right], \end{cases}$$

if $\sigma = 0$;

$$\psi(\omega) = \begin{cases} -\gamma_+\left(\alpha_-a_- + \frac{\omega_0}{2}\gamma_-\right)\left(\omega + \frac{a_+}{\sigma}\right) - \frac{a_-\gamma_-}{\sigma}\left(\alpha_+a_+ + \frac{\omega_0}{2}\gamma_+\right), & \omega \in \left[0, \frac{\omega_0}{2}\right], \\ \gamma_-\left(\alpha_+a_+ + \frac{\omega_0}{2}\gamma_+\right)\left(\omega - \frac{a_-}{\sigma}\right) - \frac{a_+\gamma_+}{\sigma}\left(\alpha_-a_- + \frac{\omega_0}{2}\gamma_-\right), & \omega \in \left[-\frac{\omega_0}{2}, 0\right], \end{cases}$$

if $\sigma \neq 0$.

2) the case $\lambda \neq 0$.

In this case the solution of our equations has the form $\psi_\pm(\omega) = A_\pm\cos(\lambda\omega) + B_\pm\sin(\lambda\omega)$. From boundary conditions we obtain $A_+ = A_- = A$ and for the finding A, B_+, B_- we have the system

$$\begin{cases} \sigma A - \lambda a_+B_+ + \lambda a_-B_- &= 0, \\ \left(\gamma_+\cos\frac{\lambda\omega_0}{2} - \lambda\alpha_+a_+\sin\frac{\lambda\omega_0}{2}\right)A + \left(\gamma_+\sin\frac{\lambda\omega_0}{2} + \lambda\alpha_+a_+\cos\frac{\lambda\omega_0}{2}\right)B_+ &= 0, \\ \left(\gamma_-\cos\frac{\lambda\omega_0}{2} - \lambda\alpha_-a_-\sin\frac{\lambda\omega_0}{2}\right)A - \left(\gamma_-\sin\frac{\lambda\omega_0}{2} + \lambda\alpha_-a_-\cos\frac{\lambda\omega_0}{2}\right)B_- &= 0. \end{cases}$$

Since $A^2 + B_+^2 + B_-^2 \neq 0,$ the system determinant must be equal zero; this means that λ is defined from the transcendence equation

$$\begin{gathered} \sigma(\lambda^2\alpha_+\alpha_-a_+a_- + \gamma_+\gamma_-) + \lambda^2(a_+ - a_-)(\alpha_-a_-\gamma_+ - \alpha_+a_+\gamma_-) \\ +\lambda[\sigma(\alpha_-a_-\gamma_+ + \alpha_+a_+\gamma_-) + (a_+ + a_-)(\gamma_+\gamma_- - \lambda^2\alpha_+\alpha_-a_+a_-)]\sin(\lambda\omega_0) \\ +[\sigma(\lambda^2\alpha_+\alpha_-a_+a_- - \gamma_+\gamma_-) + \lambda^2(a_+ + a_-)(\alpha_-a_-\gamma_+ + \alpha_+a_+\gamma_-)]\cos(\lambda\omega_0) = 0. \end{gathered} \quad (6.4)$$

Now we investigate the special cases of the boundary conditions.

The Dirichlet problem: $\alpha_{\pm} = 0$.

The equation (6.4) takes the form $\sigma(1-\cos(\lambda\omega_0)) + \lambda(a_+ + a_-)\sin(\lambda\omega_0) = 0$. Hence we get

$$\lambda = \begin{cases} \frac{\pi}{\omega_0}, & \text{if } \sigma = 0, \\ \lambda^*, & \text{if } \sigma > 0 \end{cases}$$

where λ^* is the minimal positive root of the transcendent equation

$$\tan\frac{\lambda\omega_0}{2} = -\frac{a_+ + a_-}{\sigma}\cdot\lambda,$$

and the corresponding eigenfunction $\psi(\omega) = \begin{cases} \sin\lambda\left(\frac{\omega_0}{2} - \omega\right), & \omega \in \left[0, \frac{\omega_0}{2}\right], \\ \sin\lambda\left(\frac{\omega_0}{2} + \omega\right), & \omega \in \left[-\frac{\omega_0}{2}, 0\right]. \end{cases}$

Note that $\frac{\pi}{\omega_0} < \lambda^* < \frac{2\pi}{\omega_0}$ (see Figure 2).

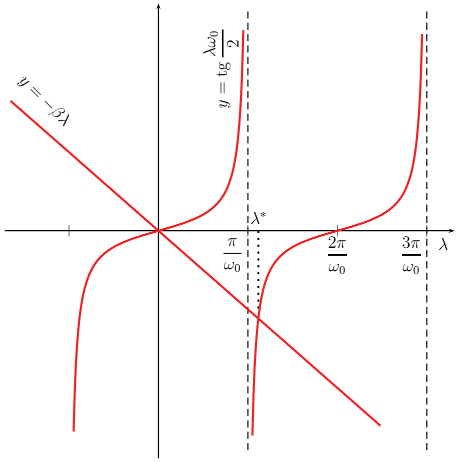

Figure 2.

The Neumann problem: $\gamma_{\pm} = 0$, $\alpha_{\pm} = 1$.

The equation (6.4) takes the form $\sigma(1+\cos(\lambda\omega_0)) - \lambda(a_+ + a_-)\sin(\lambda\omega_0) = 0$. Hence we get

$$\lambda = \begin{cases} \frac{\pi}{\omega_0}, & \text{if } \sigma = 0, \\ \lambda^*, & \text{if } \sigma > 0 \end{cases}$$

where λ^* is the minimal positive root of the transcendent equation

$$\tan\frac{\lambda\omega_0}{2} = \frac{\sigma}{a_+ + a_-}\cdot\frac{1}{\lambda}.$$

For $\lambda = \frac{\pi}{\omega_0}$ we find the corresponding eigenfunction

$$\psi(\omega) = \begin{cases} a_- \sin\frac{\pi\omega}{\omega_0}, & \omega \in \left[0, \frac{\omega_0}{2}\right], \\ a_+ \sin\frac{\pi\omega}{\omega_0}, & \omega \in \left[-\frac{\omega_0}{2}, 0\right]. \end{cases}$$

For $\lambda = \lambda^*$ we find the corresponding eigenfunction

$$\psi(\omega) = \begin{cases} \cos\lambda^*\left(\omega - \frac{\omega_0}{2}\right), & \omega \in \left[0, \frac{\omega_0}{2}\right], \\ \cos\lambda^*\left(\omega + \frac{\omega_0}{2}\right), & \omega \in \left[-\frac{\omega_0}{2}, 0\right]. \end{cases}$$

Note that $0 < \lambda^* < \frac{\pi}{\omega_0}$ (see Figure 3).

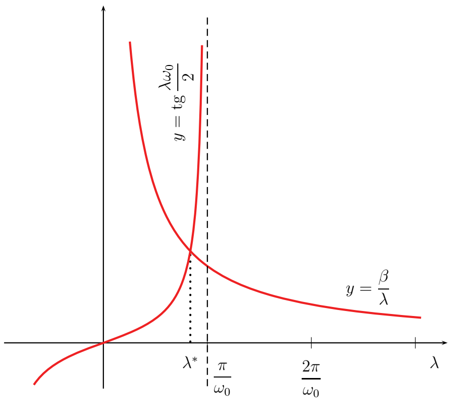

Figure 3.

Mixed problem: $\alpha_+ = 1,\ \alpha_- = 0;\ \gamma_+ = 0,\ \gamma_- = 1.$

The equation (6.4) takes the form

$$\sigma \sin(\lambda\omega_0) + \lambda(a_+ + a_-)\cos(\lambda\omega_0) = \lambda(a_+ - a_-). \tag{6.5}$$

In particular, if $\sigma = 0$, we have $\lambda = \frac{2}{\omega_0}\arctan\sqrt{\frac{a_-}{a_+}}$ and the corresponding eigenfunction

$$\psi(\omega) = \begin{cases} \cos(\lambda\omega) + \sqrt{\frac{a_-}{a_+}} \cdot \sin(\lambda\omega), & \omega \in \left[0, \frac{\omega_0}{2}\right], \\ \cos(\lambda\omega) + \sqrt{\frac{a_+}{a_-}} \cdot \sin(\lambda\omega), & \omega \in \left[-\frac{\omega_0}{2}, 0\right]. \end{cases}$$

If $\sigma > 0$, then $\lambda = \lambda^*$, where λ^* is the minimal positive root of the transcendent equation (6.5). Then we find the corresponding eigenfunction

$$\psi(\omega) = \begin{cases} \sin\frac{\lambda\omega_0}{2}\cos\lambda\left(\omega - \frac{\omega_0}{2}\right), & \omega \in \left[0, \frac{\omega_0}{2}\right], \\ \cos\frac{\lambda\omega_0}{2}\sin\lambda\left(\omega + \frac{\omega_0}{2}\right), & \omega \in \left[-\frac{\omega_0}{2}, 0\right]. \end{cases}$$

Rewriting the equation (6.5) in the form $\tan\frac{\lambda\omega_0}{2} = \frac{2\lambda a_-}{\sqrt{4\lambda^2 a_+ a_- + \sigma^2} - \sigma}$, note that $\frac{2}{\omega_0}\arctan\sqrt{\frac{\alpha_-}{\alpha_+}} < \lambda^* < \frac{\pi}{\omega_0}$ (see Figure 4).

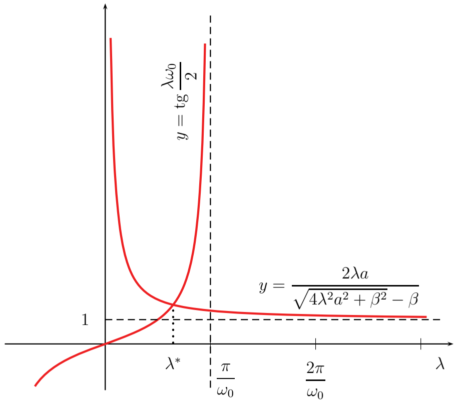

Figure 4.

The Robin problem: $\alpha_\pm = 1$; $\gamma_\pm \neq 0$.

The equation (6.4) takes the form

$$\begin{aligned}\sigma(\lambda^2 a_+ a_- + \gamma_+\gamma_-) + \lambda^2(a_+ - a_-)(a_-\gamma_+ - a_+\gamma_-)\\ + \lambda[\sigma(a_-\gamma_+ + a_+\gamma_-) + (a_+ + a_-)(\gamma_+\gamma_- - \lambda^2 a_+ a_-)]\sin(\lambda\omega_0)\\ + [\sigma(\lambda^2 a_+ a_- - \gamma_+\gamma_-) + \lambda^2(a_+ + a_-)(a_-\gamma_+ + a_+\gamma_-)]\cos(\lambda\omega_0) = 0.\end{aligned}$$

In particular, in the case of the problem without the interface ($a_+ = a_- = 1$, $\sigma = 0$) we obtain the least eigenvalue as the minimal positive root of the transcendent equation $\tan(\lambda\omega_0) = \frac{\lambda(\gamma_+ + \gamma_-)}{\lambda^2 - \gamma_+\gamma_-}$ and the corresponding eigenfunction $\psi(\omega) = \lambda\cos\left[\lambda\left(\omega - \frac{\omega_0}{2}\right)\right] - \gamma_+\sin\left[\lambda\left(\omega - \frac{\omega_0}{2}\right)\right]$ (see §10.1.7 [2]).

7. Examples

Let us present some examples which demonstrate that the assumptions on the coefficients of the operator $\mathscr{L}$ are essential for the validity of Theorem 2.3.

Let the domain $G \subset \mathbb{R}^2$ be as in §6.

EXAMPLE 1. Let us consider the function

$$u(r,\omega) = r^\lambda \left(\ln\frac{1}{r}\right)^{(\lambda-1)/(\lambda+1)} \sin(\lambda\omega)\begin{cases} a_-, & \omega \in \left[0, \frac{\omega_0}{2}\right], \\ a_+, & \omega \in \left[-\frac{\omega_0}{2}, 0\right], \end{cases}$$

where $a_\pm > 0$, $\lambda = \frac{\pi}{\omega_0}$. By direct calculations (see also the investigation of the Neumann problem in Appendix, §6) , we verify that it satisfies the transmission problem

$$\begin{cases} \frac{\partial}{\partial x_i}\left(a^{ij}(x)u_{x_j}\right) + a^i(x)u_{x_i} = 0, & x \in G \setminus \Sigma_0; \\ [u]_{\Sigma_0} = 0, \quad \left[\frac{\partial u}{\partial \nu}\right]_{\Sigma_0} = 0, & x \in \Sigma_0; \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial G \setminus \{\Sigma_0 \cup \mathscr{O}\}, \end{cases}$$

where

$$a^{11}(x) = a - \frac{2a}{\lambda+1} \cdot \frac{x_2^2}{r^2 \ln(1/r)},$$
$$a^{12}(x) = a^{21}(x) = \frac{2a}{\lambda+1} \cdot \frac{x_1 x_2}{r^2 \ln(1/r)},$$
$$a^{22}(x) = a - \frac{2a}{\lambda+1} \cdot \frac{x_1^2}{r^2 \ln(1/r)},$$
$$a^{ij}(0) = a\delta_i^j,\ i,j = 1,2;$$
$$a^1(x) = -\frac{1}{r}\mathscr{A}(r)\cos\omega, \quad a^2(x) = -\frac{1}{r}\mathscr{A}(r)\sin\omega,$$
$$\mathscr{A}(r) = \frac{2a}{(\lambda+1)\ln(1/r)}, \quad \Longrightarrow \int_0^d \frac{\mathscr{A}(r)}{r}dr = +\infty.$$

Clearly, the equation is uniformly elliptic in G_0^d for $0 < d < e^{-2}$ with the ellipticity constants

$$\nu = a - \frac{2a}{\ln(1/d)} \quad \text{and} \quad \mu = a.$$

Thus we observe that the leading coefficients of the equation are continuous but not Dini continuous at zero. From the explicit form of the solution u we have

$$|u(x)| \leqslant c|x|^{\lambda-\varepsilon}, \quad \|u\|_{\mathbf{V}^2_{p,2p-n}(G_0^\rho)} \leqslant c\rho^{\lambda-\varepsilon} \tag{7.1}$$

for all $\varepsilon > 0$. This example shows that it is not possible to replace $\lambda - \varepsilon$ in (7.1) by λ without additional assumptions concerning the modulus of continuity of the leading coefficients of the equation at zero.

EXAMPLE 2. Let $(\lambda, \psi(\omega))$ be a solution of the eigenvalue problem (6.2) (see Appendix, §6). Then the function $u(x) = r^\lambda \ln(\frac{1}{r})\psi(\omega)$ is a solution of the transmission problem

$$\begin{cases} \triangle u = -2\lambda r^{\lambda-2}\psi(\omega), & x \in G \setminus \Sigma_0; \\ [u]_{\Sigma_0} = 0, \quad \left[a\frac{\partial u}{\partial \vec{n}}\right]_{\Sigma_0} + \frac{\sigma}{r}u(x) = 0, & x \in \Sigma_0; \\ \alpha a\frac{\partial u}{\partial \vec{n}} + \frac{1}{r}\gamma u_\pm(x) = 0, & x \in \partial G \setminus \{\Sigma_0 \cup \mathscr{O}\}, \end{cases} \tag{7.2}$$

where $a > 0,\ \sigma > 0,\ \gamma > 0,\ \alpha \in \{0,1\}$.

All assumptions of Theorem 2.3 are fulfilled with $s = \lambda$. This example shows the precision of the assumption (d) and the estimate (2.3) for $s = \lambda$.

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