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SCHUR CONVEXITY AND HADAMARD'S INEQUALITY

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Abstract. Suppose that I is an open interval and $f : I \to \mathbb{R}$ is a continuous function. If

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt - f(\frac{x+y}{2}), & x,y \in I, x \neq y, \\ 0, & x = y \in I, \end{cases}$$

and

$$G(x,y) = \begin{cases} \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(t)dt, & x,y \in I, x \neq y, \\ 0, & x = y \in I, \end{cases}$$

then $F(x,y)$ and $G(x,y)$ are Schur convex (concave) on I^2 if and only if f is convex (concave) on I.

1. Introduction

For the convenience of the readers, we recall some definitions and related results as follows.

DEFINITION 1.1. Let $D \subset \mathbb{R}^n$ be a convex set (if $n = 1$, then D is an open interval). A real-valued function $f : D \to \mathbb{R}$ is said to be convex on D if

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in D$ and $\lambda \in [0,1]$. And f is said to be concave if $-f$ is convex.

DEFINITION 1.2. Suppose that I is an interval with nonempty interior. A real-valued function $F : I^n \to \mathbb{R}$ is said to be Schur convex on I^n if

$$F(x_1, x_2, \cdots, x_n) \leqslant F(y_1, y_2, \cdots, y_n)$$

for each two $n-$tuples $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in I^n$ with $x \prec y$, that is

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \cdots, n-1$$

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and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the ith largest component in x. And F is said to be Schur concave if $-F$ is Schur convex.

The following novel and interesting result which uncover the relationship between convexity and Schur convexity was established by Elezović and Pečarić [8]:

THEOREM A. *Let f be a continuous function on an interval I, and*

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt, & x,y \in I, x \neq y, \\ f(x), & x = y \in I. \end{cases}$$

Then $F(x,y)$ is Schur convex on I^2 if and only if f is convex on I.

In the recent past, both convexities have been the subject of intensive research. In particular, many remarkable inequalities and properties for convex functions and Schur convex functions can be found in the literature [10, 12, 16].

In [10] Marshall and Olkin proved

THEOREM B. *Suppose that $I \subset \mathbb{R}$ is an interval and $\varphi : I^n \to \mathbb{R}$ is a continuous symmetric function. If φ is differentiable on I^n, then φ is Schur convex (concave) on I^n if and only if*

$$(x_i - x_j)\left(\frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_j}\right) \geqslant (\leqslant)\ 0$$

for all $x_i, x_j \in I$, $i,j = 1,2,3,\cdots,n$.

Let I be an interval and $f : I \to \mathbb{R}$ be a convex (concave) function on I, then the well-known Hadamard's inequality can be expressed as:

$$f\left(\frac{x+y}{2}\right) \leqslant (\geqslant) \frac{1}{y-x}\int_x^y f(t)dt \leqslant (\geqslant) \frac{f(x)+f(y)}{2}. \tag{1.1}$$

A large number of generalizations and improvements for the Hadamard's inequality (1.1) have been made recently. For example, it has been extended to more general classes of function [2, 6, 7, 9, 14, 15, 18]. The generalization in [9] can be interpreted as a relation between different means, which leads to further extensions [13]. A different interpretation of (1.1) as a relation between means has suggested a Hadamard's inequality for convex functions on the three-sphere [1]. In another direction, Hadamard's inequality has been interpolated by suitable defined maps [3, 17]. Some results for associated Lipschitzian maps are given in [4, 11]. In [5], Dragomir and Pearce considered a different aspect and examine the differences between two sides of the inequalities in (1.1). They uncovered a number of quasilinearlity and monotonicity properties underlying Hadamard's inequality, two different motifs for superadditive and supermultiplicative are explored.

The purpose of this paper is to prove the following two results:

THEOREM 1.1. *Suppose that I is an open interval and $f: I \to \mathbb{R}$ is a continuous function. If*

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt - f(\frac{x+y}{2}), & x,y \in I, x \neq y, \\ 0, & x = y \in I, \end{cases} \tag{1.2}$$

then $F(x,y)$ is Schur convex (concave) on I^2 if and only if f is convex (concave) on I.

THEOREM 1.2. *Suppose that I is an open interval and $f: I \to \mathbb{R}$ is a continuous function. If*

$$G(x,y) = \begin{cases} \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(t)dt, & x,y \in I, x \neq y, \\ 0, & x = y \in I, \end{cases} \tag{1.3}$$

then $G(x,y)$ is Schur convex (concave) on I^2 if and only if f is convex (concave) on I.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

LEMMA 2.1. (see [16]). *Let $I \subset \mathbb{R}$ be an open interval and $f: I \to \mathbb{R}$ be a continuous function, then f is convex (concave) on I if and only if*

$$\frac{1}{y-x}\int_x^y f(t)dt \leqslant (\geqslant) \frac{f(x)+f(y)}{2}$$

or

$$\frac{1}{y-x}\int_x^y f(t)dt \geqslant (\leqslant) f\left(\frac{x+y}{2}\right)$$

for all $x,y \in I$ with $x \neq y$.

LEMMA 2.2. *Suppose that $F(x,y)$ and $G(x,y)$ are defined as in (1.2) and (1.3), respectively. If f has continuous second order derivatives on I, then*

$$\frac{\partial F}{\partial x}|_{(t_0,t_0)} = \frac{\partial F}{\partial y}|_{(t_0,t_0)} = \frac{\partial G}{\partial x}|_{(t_0,t_0)} = \frac{\partial G}{\partial y}|_{(t_0,t_0)} = 0$$

for all $t_0 \in I$.

Proof. For any $t_0 \in I$, from (1.2) and (1.3) together with the L'Hospital's rule we clearly see that

$$\begin{aligned}\frac{\partial F}{\partial x}|_{(t_0,t_0)} &= \lim_{t\to 0}\frac{F(t_0+t,t_0)-F(t_0,t_0)}{t} \\ &= \lim_{t\to 0}\frac{\frac{1}{t}\int_{t_0}^{t_0+t} f(x)dx - f(\frac{1}{2}t+t_0)}{t} \\ &= \lim_{t\to 0}\frac{\int_{t_0}^{t_0+t} f(x)dx - tf(\frac{1}{2}t+t_0)}{t^2} \\ &= \lim_{t\to 0}\frac{f'(t_0+t)-f'(t_0+\frac{1}{2}t)-\frac{1}{4}tf''(t_0+\frac{t}{2})}{2} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial G}{\partial x}|_{(t_0,t_0)} &= \lim_{t\to 0}\frac{G(t_0+t,t_0)-G(t_0,t_0)}{t}\\ &= \lim_{t\to 0}\frac{\frac{t}{2}(f(t_0)+f(t_0+t))-\int_{t_0}^{t_0+t}f(t)dt}{t^2}\\ &= \lim_{t\to 0}\frac{tf''(t_0+t)}{4}\\ &= 0.\end{aligned}$$

Making use of similar arguments for $\frac{\partial F}{\partial y}$ and $\frac{\partial G}{\partial y}$ we get $\frac{\partial F}{\partial y}|_{(t_0,t_0)} = \frac{\partial G}{\partial y}|_{(t_0,t_0)} = 0$. □

LEMMA 2.3. *Suppose that $I \subset \mathbb{R}$ is an open interval and $f : I \to \mathbb{R}$ is twice differentiable and convex on I. If*

$$h(x,y) = \frac{1}{2}(y-x)^2(f'(y)-f'(x))+2\int_x^y f(t)dt-(y-x)(f(y)+f(x)), \tag{2.1}$$

then $h(x,y) \geqslant 0$ for all $x,y \in I$ with $y \geqslant x$ and $h(x,y) \leqslant 0$ for all $x,y \in I$ with $x \geqslant y$.

Proof. From (2.1) we clearly see that

$$h(x,x) = h(y,y) = 0 \tag{2.2}$$

for all $x,y \in I$.

For any fixed $x \in I$, let $y \in I$ and $y > x$. Then (2.1) leads to

$$\frac{\partial h(x,y)}{\partial y} = (f(y)-f(x))-(y-x)f'(x)+\frac{1}{2}(y-x)^2 f''(y). \tag{2.3}$$

Making use of the Lagrange mean value theorem and the convexity of f we clearly see that

$$f''(y) \geqslant 0 \tag{2.4}$$

and

$$f(y)-f(x) \geqslant (y-x)f'(x). \tag{2.5}$$

Therefore, $h(x,y) \geqslant 0$ for all $x,y \in I$ with $y \geqslant x$ follows from (2.2)-(2.5).

Next, for any fixed $y \in I$, let $x \in I$ and $x > y$. Then (2.1) leads to

$$\frac{\partial h(x,y)}{\partial x} = (f(y)-f(x))-(y-x)f'(y)-\frac{1}{2}(y-x)^2 f''(x). \tag{2.6}$$

From the convexity of f on I and the Lagrange mean value theorem we clearly see that

$$f''(x) \geqslant 0 \tag{2.7}$$

and

$$f(y)-f(x) \leqslant (y-x)f'(y). \tag{2.8}$$

Therefore, $h(x,y) \leqslant 0$ for all $x,y \in I$ with $x \geqslant y$ follows from (2.2) and (2.6)-(2.8). □

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1.

Necessity: If $F(x,y)$ is Schur convex on I^2, then from (1.2) and Definition 1.2 together with the fact that $(\frac{x+y}{2}, \frac{x+y}{2}) \prec (x,y)$ we have

$$\frac{1}{y-x}\int_x^y f(t)dt \geqslant f\left(\frac{x+y}{2}\right) \tag{3.1}$$

for all $x,y \in I$ with $y \neq x$.

Therefore, f is convex on I follows from (3.1) and Lemma 2.1.

Sufficiency: If f is convex on I, then using standard approximation argument it is enough to prove the theorem for convex polynomials. We divide the proof into two cases.

Case 1. If $x = y \in I$, then Lemma 2.2 leads to

$$(y-x)\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right) = 0. \tag{3.2}$$

Case 2. If $x \neq y \in I$, then (1.2) leads to

$$(y-x)\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right) = (f(y)+f(x)) - \frac{2}{y-x}\int_x^y f(t)dt. \tag{3.3}$$

From the Hadamard's inequality (1.1) and (3.3) we clearly see that

$$(y-x)\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right) \geqslant 0. \tag{3.4}$$

Therefore, F is Schur convex on I^2 follows from (3.2) and (3.4) together with Theorem B.

It follows from the similar arguments as above that F is Schur concave on I^2 if and only if f is concave on I. $\square$

Proof of theorem 1.2.

Necessity: If $G(x,y)$ is Schur convex on I^2, then from (1.3) and Definition 1.2 together with the fact $(\frac{x+y}{2}, \frac{x+y}{2}) \prec (x,y)$ we get

$$\frac{1}{y-x}\int_x^y f(t)dt \leqslant \frac{f(x)+f(y)}{2} \tag{3.5}$$

for all $x,y \in I$ with $y \neq x$.

Therefore, f is convex on I follows from (3.5) and Lemma 2.1.

Sufficiency: If f is convex on I, then using standard approximation argument it is enough to prove the theorem for convex polynomials. We divide the proof into two cases.

Case A. $x = y \in I$, then Lemma 2.2 leads to

$$(y-x)\left(\frac{\partial G}{\partial y}-\frac{\partial G}{\partial x}\right)=0. \tag{3.6}$$

Case B. If $x \neq y \in I$, then (1.3) leads to

$$(y-x)\left(\frac{\partial G}{\partial y}-\frac{\partial G}{\partial x}\right)=\frac{1}{2}(y-x)(f'(y)-f'(x))+\frac{2}{y-x}\int_x^y f(t)dt-(f(y)+f(x)). \tag{3.7}$$

From (3.7) and Lemma 2.3 we clearly see that

$$(y-x)\left(\frac{\partial G}{\partial y}-\frac{\partial G}{\partial x}\right)\geqslant 0. \tag{3.8}$$

Therefore, G is Schur convex on I^2 follows from (3.6) and (3.8) together with Theorem B.

It follows from the similar arguments as above that G is Schur concave on I^2 if and only if f is concave on I. □

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