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REFINED GENERALIZATIONS OF THE TRIANGLE INEQUALITY ON BANACH SPACES

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Abstract. Let X be a complex Banach space and p a real number with $p \geqslant 1$. We give a necessary and sufficient condition for complex numbers a, b and real numbers λ, μ and ν in order that the inequality

$$\frac{\|ax+by\|^p}{\lambda} \leqslant \frac{\|x\|^p}{\mu} + \frac{\|y\|^p}{\nu}$$

holds for every $x, y \in X$.

1. Introduction and main result

Let H be a Hilbert space and let $\lambda, \mu, \nu, a, b \in \mathbf{R} \setminus \{0\}$ with $\lambda = \mu a^2 + \nu b^2$. Then we recall the Euler-Lagrange type identity

$$\frac{\|x\|^2}{\mu} + \frac{\|y\|^2}{\nu} - \frac{\|ax+by\|^2}{\lambda} = \frac{\|\nu bx - \mu ay\|^2}{\lambda\mu\nu} \tag{1.1}$$

for all $x, y \in H$ ([2-7]). Therefore, if $\lambda\mu\nu > 0$, then we have the inequality

$$\frac{\|ax+by\|^2}{\lambda} \leqslant \frac{\|x\|^2}{\mu} + \frac{\|y\|^2}{\nu}, \tag{1.2}$$

for all $x, y \in H$. Applying the mapping $T\colon\ (\lambda, \mu, \nu) \longrightarrow -(\lambda, \mu, \nu)$, if $\lambda\mu\nu < 0$, then the inequality sign in (1.2) is reversed. In particular, we have the well-known inequality

$$\|x+y\|^2 \leqslant 2(\|x\|^2 + \|y\|^2), \tag{1.3}$$

called the triangle inequality of the second kind for various reasons in [9]. In particular, for any two Hilbert spaces, some natural sum and a natural triangle inequality which, in particular, implies (1.3) for the sum of same Hilbert spaces were introduced. See also [8]. Of course, we know also

$$\|x+y\|^p \leqslant 2^{p-1}(\|x\|^p + \|y\|^p) \quad (p \geqslant 1) \tag{1.4}$$

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on any Banach space. We may regard (1.2) as Hilbert space version of the classical Bohr's inequality [1]. Cheung and Pečarić [2] studied Bohr type inequalities for bounded linear operators on a complex separable Hilbert space (cf. [4]).

In this paper, we shall give a unified generalization of the inequalities (1.2) and (1.4) on Banach spaces.

Let X be a Banach space, $a,b \in \mathbf{C}, \lambda, \mu, \nu \in \mathbf{R}$ and $p \geqslant 1$. Put

$$D_p^+ = \left\{ (a,b,\lambda,\mu,\nu) : \frac{\|ax+by\|^p}{\lambda} \leqslant \frac{\|x\|^p}{\mu} + \frac{\|y\|^p}{\nu} \quad (x,y \in X) \right\}$$

and

$$D_p^- = \left\{ (a,b,\lambda,\mu,\nu) : \frac{\|ax+by\|^p}{\lambda} \geqslant \frac{\|x\|^p}{\mu} + \frac{\|y\|^p}{\nu} \quad (x,y \in X) \right\}.$$

We wish to identify the sets D_p^+ and D_p^-. Note that

$$T(D_p^+) = D_p^-$$

and so, we shall identify the set D_p^+.

For simplicity, we shall write, for example,

$$\{\lambda > 0, \mu > 0, \nu > 0\}$$

for

$$\{(\lambda,\mu,\nu) \in \mathbf{R}^3 : \lambda > 0, \mu > 0, \nu > 0\}.$$

Our main result is:

THEOREM 1.1. *Let X be a Banach space and $p > 1$. Put $p' = (1-1/p)^{-1}$. Then,*

(i) $D_p^+ \cap \{\lambda > 0, \mu > 0, \nu > 0\}$
$= \{\lambda > 0, \mu > 0, \nu > 0, |\lambda|^{1/(p-1)} \geqslant |\mu|^{1/(p-1)}|a|^{p'} + |\nu|^{1/(p-1)}|b|^{p'}\}.$

(ii) $D_p^+ \cap \{\lambda < 0, \mu < 0, \nu > 0\}$
$= \{\lambda < 0, \mu < 0, \nu > 0, |\lambda|^{1/(p-1)} \leqslant |\mu|^{1/(p-1)}|a|^{p'} - |\nu|^{1/(p-1)}|b|^{p'}\}.$

(iii) $D_p^+ \cap \{\lambda < 0, \mu > 0, \nu < 0\}$
$= \{\lambda < 0, \mu > 0, \nu < 0, |\lambda|^{1/(p-1)} \leqslant -|\mu|^{1/(p-1)}|a|^{p'} + |\nu|^{1/(p-1)}|b|^{p'}\}.$

(iv) $D_p^+ \cap \{\lambda < 0, \mu < 0, \nu < 0\} = \phi.$

(v) $D_p^+ \cap \{\lambda > 0, \mu > 0, \nu < 0\} = \phi.$

(vi) $D_p^+ \cap \{\lambda > 0, \mu < 0, \nu > 0\} = \phi.$

(vii) $D_p^+ \cap \{\lambda > 0, \mu < 0, \nu < 0\} = \phi.$

(viii) $D_p^+ \cap \{\lambda < 0, \mu > 0, \nu > 0\} = \mathbf{C} \times \mathbf{C} \times \mathbf{R}^3 \setminus \{\lambda\mu\nu = 0\}.$

2. Proof of Theorem 1.1

For the proof of Theorem 1.1 we need the following basic

LEMMA 2.1. *([cf. [10, Theorem 1, (iii)]). Let $p > 1$ and $\mathbf{R}^+ = \{t \in \mathbf{R} : t \geqslant 0\}$. Set*

$$S_1 = \{(A,B) \in \mathbf{R}^2 : A+B = 1, 0 \leqslant A \leqslant 1\}$$

and

$$h_p(\alpha) = \alpha(\alpha^{1/(p-1)} - 1)^{1-p} \quad for \quad \alpha \in \mathbf{R}^+, \alpha > 1.$$

If $D \subseteq \{(A,B) \in \mathbf{R}^2 : A,B \geqslant 0, A+B \geqslant 1\}$ with $S_1 \subseteq cl(D)$, the topological closure of D in $\mathbf{R}^2$, then

$$\bigcap_{(A,B)\in D} \{(\alpha,\beta) \in \mathbf{R}^+ \times \mathbf{R}^+ : \alpha A^p + \beta B^p \geqslant 1\}$$

$$= \{(\alpha,\beta) \in \mathbf{R}^+ \times \mathbf{R}^+ : \alpha > 1, \beta \geqslant h_p(\alpha)\}$$

holds.

Proof of Theorem 1.1.
If $ab = 0$, then, the results are simple; indeed, for example, if

$$a \neq 0, b = 0, \lambda > 0, \mu > 0, \nu > 0,$$

then,

$$\begin{aligned}
&\frac{\|ax+by\|^p}{\lambda} \leqslant \frac{\|x\|^p}{\mu} + \frac{\|y\|^p}{\nu} \quad (x,y \in X)\\
&\Leftrightarrow \frac{\|ax\|^p}{\lambda} \leqslant \frac{\|x\|^p}{\mu} + \frac{\|y\|^p}{\nu} \quad (x,y \in X)\\
&\Leftrightarrow \frac{\|ax\|^p}{\lambda} \leqslant \frac{\|x\|^p}{\mu} \quad (x \in X)\\
&\Leftrightarrow \frac{|a|^p}{\lambda} \leqslant \frac{1}{\mu}\\
&\Leftrightarrow |\lambda|^{1/(p-1)} \geqslant |\mu|^{1/(p-1)}|a|^{p'}\\
&\Leftrightarrow |\lambda|^{1/(p-1)} \geqslant |\mu|^{1/(p-1)}|a|^{p'} + |\nu|^{1/(p-1)}|b|^{p'},
\end{aligned}$$

that is, we obtain (i).

In the sequel we shall assume that $ab \neq 0$.

(i) For $x,y \in X$ with $x+y \neq 0$, set

$$A^1_{x,y} = \frac{\|x\|}{\|x+y\|}, \quad B^1_{x,y} = \frac{\|y\|}{\|x+y\|}$$

and

$$D_1 = \{(A^1_{x,y}, B^1_{x,y}) \in \mathbf{R}^2 : x,y \in X, x+y \neq 0\}.$$

Then, we see that $D_1 \subseteq \{(A,B) \in \mathbf{R}^2 : A,B \geqslant 0, A+B \geqslant 1\}$ and $S_1 \subseteq cl(D_1)$. Note that for $\lambda > 0, \mu > 0, \nu > 0$,

$$\frac{\|ax+by\|^p}{\lambda} \leqslant \frac{\|x\|^p}{\mu} + \frac{\|y\|^p}{\nu} \quad (x,y \in X)$$

if and only if

$$1 \leqslant \frac{\lambda}{\mu|a|^p}A^p + \frac{\lambda}{\nu|b|^p}B^p \quad ((A,B) \in D_1).$$

Therefore we have from Lemma 2.1 that

$$D_p^+ \cap \{\lambda > 0, \mu > 0, \nu > 0\}$$

$$= \left\{\lambda > 0, \mu > 0, \nu > 0, \frac{\lambda}{\mu|a|^p} > 1, \frac{\lambda}{\nu|b|^p} \geqslant \frac{\lambda}{\mu|a|^p}\left(\left(\frac{\lambda}{\mu|a|^p}\right)^{1/(p-1)} - 1\right)^{1-p}\right\}.$$

If $\lambda > 0, \mu > 0, \nu > 0$, then

$$\frac{\lambda}{\nu|b|^p} \geqslant \frac{\lambda}{\mu|a|^p}\left(\left(\frac{\lambda}{\mu|a|^p}\right)^{1/(p-1)} - 1\right)^{1-p}$$

can be rewritten as

$$\lambda^{1/(p-1)} \geqslant \mu^{1/(p-1)}|a|^{p'} + \nu^{1/(p-1)}|b|^{p'}.$$

Also if $\lambda > 0, \mu > 0, \nu > 0$ and $\lambda^{1/(p-1)} \geqslant \mu^{1/(p-1)}|a|^{p'} + \nu^{1/(p-1)}|b|^{p'}$, then $\lambda^{1/(p-1)} > \mu^{1/(p-1)}|a|^{p'}$ and hence $\frac{\lambda}{\mu|a|^p} > 1$. Therefore, we obtain the desired result.

(ii) For $x,y \in X$ with $x \neq 0$, set

$$A_{x,y}^2 = \frac{\|x+y\|}{\|x\|}, \quad B_{x,y}^2 = \frac{\|y\|}{\|x\|}$$

and

$$D_2 = \{(A_{x,y}^2, B_{x,y}^2) \in \mathbf{R}^2 : x,y \in X, x \neq 0\}.$$

Then, we see that $D_2 \subseteq \{(A,B) \in \mathbf{R}^2 : A,B \geqslant 0, A+B \geqslant 1\}$ and $S_1 \subseteq cl(D_2)$. Note that for $\lambda < 0, \mu < 0, \nu > 0$,

$$\frac{\|ax+by\|^p}{\lambda} \leqslant \frac{\|x\|^p}{\mu} + \frac{\|y\|^p}{\nu} \quad (x,y \in X)$$

if and only if

$$1 \leqslant \frac{\mu|a|^p}{\lambda}A^p + \frac{-\mu|a|^p}{\nu|b|^p}B^p \quad ((A,B) \in D_2).$$

Therefore we have from Lemma 2.1 that

$$D_p^+ \cap \{\lambda < 0, \mu < 0, \nu > 0\}$$

$$= \left\{ \lambda < 0, \mu < 0, \nu > 0, \frac{\mu|a|^p}{\lambda} > 1, \frac{-\mu|a|^p}{\nu|b|^p} \geqslant \frac{\mu|a|^p}{\lambda}\left(\left(\frac{\mu|a|^p}{\lambda}\right)^{1/(p-1)} - 1\right)^{1-p} \right\}.$$

If $\lambda < 0, \mu < 0, \nu > 0$, then

$$\frac{-\mu|a|^p}{\nu|b|^p} \geqslant \frac{\mu|a|^p}{\lambda}\left(\left(\frac{\mu|a|^p}{\lambda}\right)^{1/(p-1)} - 1\right)^{1-p}$$

can be rewritten as

$$(-\lambda)^{1/(p-1)} \leqslant (-\mu)^{1/(p-1)}|a|^{p'} - \nu^{1/(p-1)}|b|^{p'}.$$

Also if $\lambda < 0, \mu < 0, \nu > 0$ and $(-\lambda)^{1/(p-1)} \leqslant (-\mu)^{1/(p-1)}|a|^{p'} - \nu^{1/(p-1)}|b|^{p'}$, then $(-\lambda)^{1/(p-1)} < (-\mu)^{1/(p-1)}|a|^{p'}$ and hence $\frac{\mu|a|^p}{\lambda} > 1$. Therefore, we obtain the desired result.

(iii) For $x, y \in X$ with $y \neq 0$, set

$$A^3_{x,y} = \frac{\|x+y\|}{\|y\|}, \quad B^3_{x,y} = \frac{\|x\|}{\|y\|}$$

and

$$D_3 = \{(A^3_{x,y}, B^3_{x,y}) \in \mathbf{R}^2 : x, y \in X, y \neq 0\}.$$

Then, we see that $D_3 \subseteq \{(A,B) \in \mathbf{R}^2 : A, B \geqslant 0, A+B \geqslant 1\}$ and $S_1 \subseteq cl(D_3)$. Note that for $\lambda < 0, \mu > 0, \nu < 0$,

$$\frac{\|ax+by\|^p}{\lambda} \leqslant \frac{\|x\|^p}{\mu} + \frac{\|y\|^p}{\nu} \quad (x, y \in X)$$

if and only if

$$1 \leqslant \frac{-\nu|b|^p}{-\lambda}A^p + \frac{-\nu|b|^p}{\mu|a|^p}B^p \quad ((A,B) \in D_3).$$

Therefore we have from Lemma 2.1 that

$$D_p^+ \cap \{\lambda < 0, \mu > 0, \nu < 0\}$$

$$= \left\{ \lambda < 0, \mu > 0, \nu < 0, \frac{-\nu|b|^p}{-\lambda} > 1, \frac{-\nu|b|^p}{\mu|a|^p} \geqslant \frac{-\nu|b|^p}{-\lambda}\left(\left(\frac{-\nu|b|^p}{-\lambda}\right)^{1/(p-1)} - 1\right)^{1-p} \right\}.$$

If $\lambda < 0, \mu > 0, \nu < 0$, then

$$\frac{-\nu|b|^p}{\mu|a|^p} \geqslant \frac{-\nu|b|^p}{-\lambda}\left(\left(\frac{-\nu|b|^p}{-\lambda}\right)^{1/(p-1)} - 1\right)^{1-p}$$

can be rewritten as

$$(-\lambda)^{1/(p-1)} \leqslant -\mu^{1/(p-1)}|a|^{p'} + (-\nu)^{1/(p-1)}|b|^{p'}.$$

Also if $\lambda < 0, \mu > 0, \nu < 0$ and $(-\lambda)^{1/(p-1)} \leqslant -\mu^{1/(p-1)}|a|^{p'} + (-\nu)^{1/(p-1)}|b|^{p'}$, then $(-\lambda)^{1/(p-1)} < (-\nu)^{1/(p-1)}|b|^{p'}$ and hence $\frac{-\nu|b|^p}{-\lambda} > 1$. Therefore, we obtain the desired result.

The results (iv)–$(viii)$ are clear. $\square$

3. Remarks

Let ε_t be the sign function of $\mathbf{R}\setminus\{0\}$; that is, $\varepsilon_t = 1$, $(t > 0)$; $\varepsilon_t = -1$, $(t < 0)$. Then, from Theorem 1.1 we have

COROLLARY 3.1. *Let X be a Banach space, $a, b \in \mathbf{C}$, $p > 1$ and $\lambda, \mu, \nu \in \mathbf{R}\setminus\{0\}$ with $\lambda\mu\nu > 0$ but not $[\lambda > 0, \mu < 0, \nu < 0]$. Then the inequality*

$$\frac{\|ax+by\|^p}{\lambda} \leqslant \frac{\|x\|^p}{\mu} + \frac{\|y\|^p}{\nu} \quad (x, y \in X) \tag{3.1}$$

holds if and only if

$$\varepsilon_\lambda |\lambda|^{1/(p-1)} \geqslant \varepsilon_\mu |\mu|^{1/(p-1)} |a|^{p'} + \varepsilon_\nu |\nu|^{1/(p-1)} |b|^{p'}. \tag{3.2}$$

If $\lambda\mu\nu < 0$ but not $[\lambda < 0, \mu > 0, \nu > 0]$, then the inequality signs are reversed.

Note that if $p = 2$, then (3.2) is reduced to

$$\lambda \geqslant \mu a^2 + \nu b^2.$$

Therefore, Corollary 3.1 is a generalization of the inequality (1.2).

In Corollary 3.1, we put, for $\alpha, \beta > 0$

$$\lambda = (\alpha+\beta)^{p-1}, \mu = \alpha^{p-1}|a|^{-p}, \nu = \beta^{p-1}|b|^{-p}.$$

Then, $\lambda\mu\nu > 0$ and (λ, μ, ν) satisfies the condition (3.2). Therefore, we have from Corollary 3.1 that for $\alpha, \beta > 0$ and for $x, y \in X$,

$$\|x+y\|^p \leqslant (\alpha+\beta)^{p-1} \left(\frac{\|x\|^p}{\alpha^{p-1}} + \frac{\|y\|^p}{\beta^{p-1}} \right) \quad (p \geqslant 1).$$

In particular, we obtain the inequality (1.4).

In general, we see some deep relationship between inequalities and convexity. See, for example, the recent book [1]. For the functional $F(x) = \|x\|^p, p > 1$, (1.4) implies the mid point convexity

$$F\left(\frac{x+y}{2}\right) \leqslant \frac{F(x)+F(y)}{2}.$$

Moreover, the above generalization of the inequality can be rewritten as

$$F(ax + (1-a)y) \leqslant aF(x) + (1-a)F(y),$$

where $a = \frac{\alpha}{\alpha+\beta}$, i.e. the inequality means that the functional $F(x)$ is not only midpoint convex but also it is in fact convex (cf. [1]). At this moment, we can not, however, refer to some relationship between our inequalities and convexity.

4. The case $p = 1$

For $p = 1$, we obtain

THEOREM 4.1.

$$
\begin{aligned}
D_1^+ = &\{\lambda < 0, \mu > 0, \nu > 0\} \cup \{\lambda > 0, \mu > 0, \nu > 0, \lambda \geqslant \mu|a|, \lambda \geqslant \nu|b|\}\\
&\cup \{\lambda < 0, \mu < 0, \nu > 0, \lambda \geqslant \mu|a|, -\nu|b| \geqslant \mu|a|\}\\
&\cup \{\lambda < 0, \mu > 0, \nu < 0, \lambda \geqslant \nu|b|, -\mu|a| \geqslant \nu|b|\}.
\end{aligned}
$$

Proof of Theorem 4.1.

For the case $ab = 0$, the desired result follows from an easy observation. So, we shall assume that $ab \neq 0$.

Let $(a, b, \lambda, \mu, \nu) \in D_1^+$. Then,

$$\frac{\|ax + by\|}{\lambda} \leqslant \frac{\|x\|}{\mu} + \frac{\|y\|}{\nu}, \tag{4.1}$$

for all $x, y \in X$. If $\lambda > 0, \mu < 0$, then by (4.1), $|a|\|x\|/\lambda \leqslant \|x\|/\mu$ must hold for all $x \in X$. This is a contradiction. Similarly, we obtain a contradiction for the case of $\lambda > 0, \nu < 0$. Also, if $\lambda < 0, \mu < 0, \nu < 0$, then

$$\frac{\|x\|}{-\mu} + \frac{|a|}{|b|}\frac{\|x\|}{-\nu} \leqslant 0$$

must hold for all $x \in X$. This is a contradiction. If $\lambda > 0, \mu > 0, \nu > 0$, then we have $\lambda \geqslant \mu|a|$ and $\lambda \geqslant \nu|b|$ by putting $x = e$, the unit element, $y = 0$ and $x = 0, y = e$ in (4.1), respectively. If $\lambda < 0, \mu < 0, \nu > 0$, then we have $\lambda \geqslant \mu|a|$ and $-\nu|b| \geqslant \mu|a|$ since (4.1) can be rewritten as

$$\frac{\|x + y\|}{-\mu|a|} \leqslant \frac{\|x\|}{-\lambda} + \frac{\|y\|}{\nu|b|} \quad (x, y \in X).$$

If $\lambda < 0, \mu > 0, \nu < 0$, then $\lambda \geqslant \nu|b|$ and $-\mu|a| \geqslant \nu|b|$ since (4.1) can be rewritten by

$$\frac{\|x + y\|}{-\nu|b|} \leqslant \frac{\|x\|}{-\lambda} + \frac{\|y\|}{\mu|a|} \quad (x, y \in X).$$

Then we have

$$
\begin{aligned}
D_1^+ \subseteq &\{\lambda < 0, \mu > 0, \nu > 0\} \cup \{\lambda > 0, \mu > 0, \nu > 0, \lambda \geqslant \mu|a|, \lambda \geqslant \nu|b|\}\\
&\cup \{\lambda < 0, \mu < 0, \nu > 0, \lambda \geqslant \mu|a|, -\nu|b| \geqslant \mu|a|\}\\
&\cup \{\lambda < 0, \mu > 0, \nu < 0, \lambda \geqslant \nu|b|, -\mu|a| \geqslant \nu|b|\}.
\end{aligned}
$$

Conversely, if $\lambda < 0, \mu > 0, \nu > 0$, then $(a, b, \lambda, \mu, \nu) \in D_1^+$ is clear. If $\lambda > 0, \mu > 0, \nu > 0, \lambda \geqslant \mu|a|$ and $\lambda \geqslant \nu|b|$, then

$$\frac{\|ax + by\|}{\lambda} \leqslant \frac{\|ax\|}{\lambda} + \frac{\|by\|}{\lambda} \leqslant \frac{\|x\|}{\mu} + \frac{\|y\|}{\nu},$$

and hence $(a,b,\lambda,\mu,\nu) \in D_1^+$. If $\lambda < 0, \mu < 0, \nu > 0, \lambda \geqslant \mu|a|$ and $-\nu|b| \geqslant \mu|a|$, then

$$\frac{\|x\|}{\mu|a|} + \frac{\|y\|}{\nu|b|} \geqslant \frac{\|x\|}{\mu|a|} - \frac{\|y\|}{\mu|a|} = \frac{\|x\| - \|y\|}{\mu|a|} \geqslant \frac{\|x+y\|}{\mu|a|} \geqslant \frac{\|x+y\|}{\lambda}.$$

Therefore, (4.1) holds and hence $(a,b,\lambda,\mu,\nu) \in D_1^+$.

If $\lambda < 0, \mu > 0, \nu < 0, \lambda \geqslant \nu|b|$ and $-\mu|a| \geqslant \nu|b|$, then

$$\frac{\|x\|}{\mu|a|} + \frac{\|y\|}{\nu|b|} \geqslant \frac{\|x\|}{-\nu|b|} + \frac{\|y\|}{\nu|b|} = \frac{\|y\| - \|x\|}{\nu|b|} \geqslant \frac{\|x+y\|}{\nu|b|} \geqslant \frac{\|x+y\|}{\lambda}.$$

Therefore, (4.1) holds and hence $(a,b,\lambda,\mu,\nu) \in D_1^+$. Then we have

$$\begin{aligned} D_1^+ \supseteq & \{\lambda < 0, \mu > 0, \nu > 0\} \cup \{\lambda > 0, \mu > 0, \nu > 0, \lambda \geqslant \mu|a|, \lambda \geqslant \nu|b|\} \\ & \cup \{\lambda < 0, \mu < 0, \nu > 0, \lambda \geqslant \mu|a|, -\nu|b| \geqslant \mu|a|\} \\ & \cup \{\lambda < 0, \mu > 0, \nu < 0, \lambda \geqslant \nu|b|, -\mu|a| \geqslant \nu|b|\}. \end{aligned}$$

Consequently we obtain the desired result. □

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