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SHARP MEAN TRIANGLE INEQUALITY

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Abstract. By using a mean operator we shall present some sharp mean triangle inequalities in a Banach space which generalize the sharp triangle inequality with n elements and its reverse one shown recently by the last three authors in [7]. In the course of doing this we shall present a new two element triangle inequality with parameter and its reverse. Several applications will be given.

1. Introduction

In the theory of Banach spaces the triangle inequality is fundamental and important. Many authors considered this inequality (cf. Diaz and Metcalf [1], Dragomir [2], Dunkl and Williams [4], Hudzik and Landes [6], Massera and Schäffer [10], Saitoh [12], Maligranda [8], Kato, Saito and Tamura [7]). In particular Kato, Saito and Tamura [7] proved the following sharp triangle inequality and its reverse: For all nonzero elements $x_1,\dots,x_n$ in a Banach space X

$$\begin{aligned}\Big\|\sum_{i=1}^{n}x_i\Big\| &+ \left(n-\left\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\right\|\right)\min_{1\leqslant i\leqslant n}\|x_i\| \\ &\leqslant \sum_{i=1}^{n}\|x_i\| \\ &\leqslant \Big\|\sum_{i=1}^{n}x_i\Big\| + \left(n-\left\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\right\|\right)\max_{1\leqslant i\leqslant n}\|x_i\|.\end{aligned}$$

These inequalities are very useful. Indeed, they were used in [7] to give a simple proof of a characterization of uniform non-ℓ_1^n-ness for a Banach space. After that, several authors improved and generalized these inequalities (cf. Mitani, Saito, Kato and Tamura [9], Dragomir [3], Pečarić and Rajić [11], Hsu, Shaw and Wong [5], etc.).

The aim of this paper is to study sharp mean triangle inequalities in a Banach space, which generalize the above-mentioned inequalities given in [7]. In Section 2

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we shall give a new two element sharp triangle inequality with parameter, which in particular provides a simple proof of the two element sharp triangle inequality in [7, 8]. In Section 3 we shall define a notion of a mean operator and using the idea of Section 2, we shall present some sharp mean triangle inequalities. In Section 4 we shall discuss some applications.

2. Two element case

Let X be a Banach space and $x,y \in X$. We define a function $f_{x,y}$ by

$$f_{x,y}(t) = \frac{\|x+ty\| - \|x\|}{t} \quad (t > 0).$$

Then the following lemma is essential in our discussion.

LEMMA 2.1. *For every $x,y \in X$, $f_{x,y}$ is a nondecreasing function on $(0,\infty)$ and $f_{x,y}(t) \leqslant \|y\|$ for all $t > 0$.*

Proof. Let $0 < s \leqslant t$. Let $g(t) = \|x+ty\| - \|x\|$ for $t \geqslant 0$. Then as g is convex and $g(0) = 0$, we have

$$g(s) = g\left(\frac{s}{t}t\right) = g\left(\left(1-\frac{s}{t}\right)0 + \frac{s}{t}t\right) \leqslant \frac{s}{t}g(t),$$

whence $\frac{g(s)}{s} \leqslant \frac{g(t)}{t}$, or $f_{x,y}(s) \leqslant f_{x,y}(t)$. Further, we have

$$f_{x,y}(t) = \frac{\|x+ty\| - \|x\|}{t} \leqslant \lim_{s\to\infty} \frac{\|x+sy\| - \|x\|}{s} = \|y\|$$

for all $t > 0$. This completes the proof. □

From Lemma 2.1, we immediately obtain the following inequalities which are regarded as a new sharp triangle inequality and its reverse inequality.

THEOREM 2.2. *Let X be a Banach space and $x,y \in X$.*
(i) *Let $0 < s \leqslant 1 \leqslant t$. Then*

$$\|x+y\| + \|y\| - f_{x,y}(t) \leqslant \|x\| + \|y\| \leqslant \|x+y\| + \|x\| - f_{y,x}(s),$$

or equivalently,
(ii) *Let $0 < \alpha \leqslant 1 \leqslant \beta$. Then*

$$\begin{aligned} \|x+y\| + \alpha\|x\| + \|y\| - \|\alpha x + y\| &\leqslant \|x\| + \|y\| \\ &\leqslant \|x+y\| + \|x\| + \beta\|y\| - \|x+\beta y\|. \end{aligned}$$

Proof. (i) Since $1 \leqslant t$, by Lemma 2.1, we have $f_{x,y}(1) \leqslant f_{x,y}(t)$, that is,

$$\|x+y\| - \|x\| \leqslant f_{x,y}(t).$$

Thus we have

$$\|x+y\| + \|y\| - f_{x,y}(t) \leqslant \|x\| + \|y\|.$$

Since $s \leqslant 1$, we have $f_{y,x}(s) \leqslant f_{y,x}(1)$ and so we have the second inequality.

(ii) Putting $t = 1/\alpha$ and $s = 1/\beta$, we have (ii) from (i). This completes the proof. □

By putting $\alpha = \|y\|/\|x\|$ and $\beta = \|x\|/\|y\|$ in Theorem 2.2 (ii), we obtain the sharp triangle inequality and its reverse with two elements ([7, 8]).

COROLLARY 2.3. ([7, 8]). *Assume that* $\|x\| \geqslant \|y\| > 0$. *Then*

$$\|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|y\| \leqslant \|x\| + \|y\| \leqslant \|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|x\|.$$

3. Mean operator and some mean inequalities

Let X be a Banach space and let S be a set. We denote the Banach space of all bounded mappings of S into X by $\ell^\infty(S,X)$ with supremum norm. In particular, if $X = \mathbb{R}$, then we denote it by $\ell^\infty(S)$. An element $\mu \in \ell^\infty(S)^*$, the dual space of $\ell^\infty(S)$, is called a *mean* on $\ell^\infty(S)$ if $\mu(\mathbf{1}) = \|\mu\| = 1$.

We shall state a fundamental fact of a *mean* on $\ell^\infty(S)$.

PROPOSITION 3.1. (cf. [13, Theorem 1.4.1]). *Let* μ *be a mean on* $\ell^\infty(S)$. *Then*

(i) *If* f *is a positive function in* $\ell^\infty(S)$ *in the sense that* $f(s) \geqslant 0$ *for any* $s \in S$, *then* $\mu(f) \geqslant 0$.

(ii) *For any* $f \in \ell^\infty(S)$, $\inf_{s\in S} f(s) \leqslant \mu(f) \leqslant \sup_{s\in S} f(s)$.

Let μ be a mean on $\ell^\infty(S)$. For any $F \in \ell^\infty(S,X)$, we define a mapping $M_\mu(F)$ from X^* into $\mathbb{R}$ by

$$M_\mu(F)(x^*) = \mu(\langle F(\cdot),\, x^*\rangle)$$

for every $x^* \in X^*$. Then we have

PROPOSITION 3.2. *Let* μ *be a mean on* $\ell^\infty(S)$. *Then* M_μ *is a bounded linear operator from* $\ell^\infty(S,X)$ *into* X^{**} *with the following properties:*

(i) $\|M_\mu\| = 1$.

(ii) *If* $\mathbf{x}$ *is the constant mapping in the sense that* $\mathbf{x}(s) = x$ $(s \in S)$ *for some* $x \in X$, *then* $M_\mu(\mathbf{x}) = x$.

Proof. Take any $x^* \in X^*$. Then

$$|M_\mu(F)(x^*)| = |\mu(\langle F(\cdot), x^*\rangle)| \leqslant \|\mu\| \|(\langle F(\cdot), x^*\rangle)\| \\ = \|(\langle F(\cdot), x^*\rangle)\| \leqslant \|F\| \|x^*\|.$$

Thus we have $\|M_\mu(F)\| \leqslant \|F\|$, that is, $M_\mu(F) \in X^{**}$ and $\|M_\mu\| \leqslant 1$.

Conversely, for any $x \in X$, we have

$$(M_\mu(\mathbf{x}))(x^*) = \mu(\langle \mathbf{x}(\cdot), x^*\rangle) = \mu(\langle x, x^*\rangle \mathbf{1}) = \langle x, x^*\rangle.$$

Thus we have $M_\mu(\mathbf{x}) = x$ and so $\|M_\mu(\mathbf{x})\| = \|x\| = \|\mathbf{x}\|$. Thus we obtain $\|M_\mu\| = 1$. This completes the proof. □

In this paper, by Proposition 3.2, we shall call M_μ a *mean operator* on $\ell^\infty(S,X)$ which is a generalization of a mean on $\ell^\infty(S)$.

EXAMPLE 3.3. Let $S = \{1,\dots,n\}$ and let X be a Banach space. Then we consider the standard arithmetic mean on $\ell^\infty(S)$, that is,

$$\mu(f) = \frac{1}{n}\sum_{i=1}^{n} f(i).$$

Then it is clear that, for every $F \in \ell^\infty(S,X)$,

$$M_\mu(F) = \frac{1}{n}\sum_{i=1}^{n} F(i).$$

EXAMPLE 3.4. Let X be a Banach space and let μ be a mean on $\ell^\infty(S)$. Suppose that μ is finite in the sense that there exist $s_1,\dots,s_n \in S$ and non-negative numbers $\lambda_1,\dots,\lambda_n$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\mu(f) = \sum_{i=1}^n \lambda_i f(s_i)$ for all $f \in \ell^\infty(S)$. Since

$$\langle M_\mu(F), x^*\rangle = \mu(\langle F(\cdot), x^*\rangle) = \sum_{i=1}^{n} \lambda_i \langle F(s_i), x^*\rangle = \langle \sum_{i=1}^{n} \lambda_i F(s_i), x^*\rangle$$

for all $x^* \in X^*$, we obtain

$$M_\mu(F) = \sum_{i=1}^{n} \lambda_i F(s_i).$$

Therefore we shall obtain the following mean triangle inequality.

THEOREM 3.5. *Let X be a Banach space and μ a mean on $\ell^\infty(S)$. Then for every $F \in \ell^\infty(S,X)$,*

$$\|M_\mu(F)\| \leqslant \mu(\|F(\cdot)\|).$$

Proof. Let $F \in \ell^\infty(S,X)$. Take any $\varepsilon > 0$. Since $M_\mu(F) \in X^{**}$ by Proposition 3.2, there exists a norm one element $x^* \in X^*$ such that

$$\|M_\mu(F)\| \leqslant \langle M_\mu(F), x^* \rangle + \varepsilon.$$

Since $\langle F(s), x^* \rangle \leqslant |\langle F(s), x^* \rangle| \leqslant \|F(s)\|$ for any $s \in S$, by Proposition 3.1, we have

$$\|M_\mu(F)\| \leqslant \langle M_\mu(F), x^* \rangle + \varepsilon = \mu(\langle F(\cdot), x^* \rangle) + \varepsilon \leqslant \mu(\|F(\cdot)\|) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof. $\square$

For $F \in \ell^\infty(S,X)$ and $\alpha \in \ell^\infty(S)$, we define αF by

$$(\alpha F)(s) = \alpha(s) F(s)$$

for all $s \in S$. Further, for $\alpha, \beta \in \ell^\infty(S)$, we consider the usual order $\alpha \leqslant \beta$ of $\ell^\infty(S)$ by

$$\alpha(s) \leqslant \beta(s) \text{ for all } s \in S.$$

Then we have the following monotone property.

PROPOSITION 3.6. *Let X be a Banach space and μ a mean on $\ell^\infty(S)$. For any $F, G \in \ell^\infty(S,X)$, put*

$$U(F,G,\alpha) = \mu(\|(\alpha F)(\cdot)\|) - \|M_\mu(\alpha F + G)\|.$$

If $\alpha, \beta \in \ell^\infty(S)$ such that $\mathbf{0} \leqslant \alpha \leqslant \beta$, then $U(F,G,\alpha) \leqslant U(F,G,\beta)$.

Proof. Let $\alpha, \beta \in \ell^\infty(S)$ with $\mathbf{0} \leqslant \alpha \leqslant \beta$ and let $F, G \in \ell^\infty(S,X)$. By Theorem 3.5, we have

$$\begin{aligned}
\|M_\mu(\beta F + G)\| - \|M_\mu(\alpha F + G)\| &\leqslant \|M_\mu((\beta - \alpha)F)\| \\
&\leqslant \mu(\|((\beta - \alpha)F)(\cdot)\|) \\
&= \mu((\beta(\cdot) - \alpha(\cdot))\|F(\cdot)\|) \\
&= \mu(\beta(\cdot)\|F(\cdot)\| - \alpha(\cdot)\|F(\cdot)\|) \\
&= \mu(\beta(\cdot)\|F(\cdot)\|) - \mu(\alpha(\cdot)\|F(\cdot)\|) \\
&= \mu(\|\beta(\cdot)F(\cdot)\|) - \mu(\|\alpha(\cdot)F(\cdot)\|).
\end{aligned}$$

Then we have

$$\mu(\|(\alpha F)(\cdot)\|) - \|M_\mu(\alpha F + G)\| \leqslant \mu(\|\beta(\cdot)F(\cdot)\|) - \|M_\mu(\beta F + G)\|.$$

That is, $U(F,G,\alpha) \leqslant U(F,G,\beta)$. This completes the proof. $\square$

By Proposition 3.6, we obtain the following sharp mean triangle inequality.

THEOREM 3.7. *Let μ be a mean on $\ell^\infty(S)$. If $\alpha, \beta \in \ell^\infty(S)$ satisfying $\mathbf{0} \leqslant \alpha \leqslant \mathbf{1} \leqslant \beta$, then*

$$\begin{aligned}\|M_\mu(F+G)\| \;+\; & \mu(\|(\alpha F)(\cdot)\|) + \mu(\|G(\cdot)\|) - \|M_\mu(\alpha F+G)\| \\ & \leqslant \mu(\|F(\cdot)\|) + \mu(\|G(\cdot)\|) \\ & \leqslant \|M_\mu(F+G)\| + \mu(\|F(\cdot)\|) + \mu(\|(\beta G)(\cdot)\|) - \|M_\mu(F+\beta G)\|.\end{aligned}$$

for all $F, G \in \ell^\infty(S,X)$.

Proof. Since $\mathbf{0} \leqslant \alpha \leqslant \mathbf{1}$, by Proposition 3.6, we have

$$U(F,G,\alpha) \leqslant U(F,G,\mathbf{1}).$$

That is,

$$\mu(\|(\alpha F)(\cdot)\|) - \|M_\mu(\alpha F+G)\| \leqslant \mu(\|F(\cdot)\|) - \|M_\mu(F+G)\|.$$

Thus, we have

$$\|M_\mu(F+G)\| + \mu(\|(\alpha F)(\cdot)\|) + \mu(\|G(\cdot)\|) - \|M_\mu(\alpha F+G)\| \leqslant \mu(\|F(\cdot)\|) + \mu(\|G(\cdot)\|).$$

Since $\mathbf{1} \leqslant \beta$, by Proposition 3.6, we have

$$U(G,F,\mathbf{1}) \leqslant U(G,F,\beta).$$

Thus, we similarly have the second inequality. This completes the proof. $\square$

By Theorem 3.7, we obtain a sharp mean inequality.

COROLLARY 3.8. *Let μ be a mean on $\ell^\infty(S)$. If $F \in \ell^\infty(S,X)$ such that $\inf_{s\in S}\|F(s)\| > 0$, then*

$$\begin{aligned}\|M_\mu(F)\| \;+\; & \left(1 - \left\| M_\mu\left(\frac{F}{\|F(\cdot)\|}\right)\right\|\right) \inf_{s\in S}\|F(s)\| \\ & \leqslant \mu(\|F(\cdot)\|) \\ & \leqslant \|M_\mu(F)\| + \left(1 - \left\| M_\mu\left(\frac{F}{\|F(\cdot)\|}\right)\right\|\right) \sup_{s\in S}\|F(s)\|.\end{aligned}$$

Proof. In the first inequality of Theorem 3.7, we put $G = \mathbf{0}$ and $\alpha(s) = \frac{\inf_{t\in S}\|F(t)\|}{\|F(s)\|}$ for all $s \in S$. Then we have the first inequality of this corollary. Similarly, putting $F = \mathbf{0}$ and $\beta(s) = \frac{\sup_{t\in S}\|G(t)\|}{\|G(s)\|}$ for all $s \in S$ in the second inequality of Theorem 3.7, we have the second one and this corollary. $\square$

4. Applications

Let $S=\{1,\dots,n\}$ and let X be a Banach space. As in Example 3.3, we define a finite mean μ on $\ell^\infty(S)$ by

$$\mu(f)=\frac{1}{n}\sum_{i=1}^{n} f(i)$$

for all $f\in\ell^\infty(S)$ and obtain

$$M_\mu(F)=\frac{1}{n}\sum_{i=1}^{n} F(i)$$

for all $F\in\ell^\infty(S,X)$. For $\alpha=(\alpha_1,\dots,\alpha_n)\in\ell^\infty(S)$ and $F\in\ell^\infty(S,X)$ (resp. $F\in\ell^\infty(S)$), we define the notation $\alpha\cdot F$ by $\alpha\cdot F=\sum_{i=1}^n\alpha_i F(i)$, and $\|F\|_1=\sum_{i=1}^n\|F(i)\|$. By Theorem 3.7, we have the following theorem.

THEOREM 4.1. *Let X be a Banach space and let $\alpha=(\alpha_1,\dots,\alpha_n)$, $\beta=(\beta_1,\dots,\beta_n)$ $\in\ell^\infty(S)$ with $\mathbf{0}\leqslant\alpha\leqslant\mathbf{1}\leqslant\beta$. Then*

$$\begin{aligned}\|\mathbf{1}\cdot(F+G)\| &+ \alpha\cdot(\|F(1)\|,\dots,\|F(n)\|)+\|G\|_1-\|\alpha\cdot F+\mathbf{1}\cdot G\|\\ &\leqslant\|F\|_1+\|G\|_1\\ &\leqslant\|\mathbf{1}\cdot(F+G)\|+\|F\|_1+\beta\cdot(\|G(1)\|,\dots,\|G(n)\|)-\|\mathbf{1}\cdot F+\beta\cdot G\|\end{aligned}$$

for all $F,G\in\ell^\infty(S,X)$.

Equivalently, if we put $F=(x_1,\cdots,x_n)$, $G=(y_1,\cdots,y_n)\in\ell^\infty(S,X)$, then

$$\begin{aligned}\|\sum_{i=1}^n(x_i+y_i)\| &+ \sum_{i=1}^n\alpha_i\|x_i\|+\sum_{i=1}^n\|y_i\|-\|\sum_{i=1}^n(\alpha_i x_i+y_i)\|\\ &\leqslant\sum_{i=1}^n\|x_i\|+\sum_{i=1}^n\|y_i\|\\ &\leqslant\|\sum_{i=1}^n(x_i+y_i)\|+\sum_{i=1}^n\|x_i\|+\sum_{i=1}^n\beta\|y_i\|-\|\sum_{i=1}^n(x_i+\beta_i y_i)\|.\end{aligned}$$

Proof. For any $F,G\in\ell^\infty(S,X)$ and $\alpha\in\ell^\infty(S)$,

$$\|M_\mu(F+G)\|=\|\frac{1}{n}\sum_{i=1}^n(F(i)+G(i))\|=\frac{1}{n}\|\mathbf{1}\cdot(F+G)\|,$$

$$\begin{aligned}\mu(\|\alpha F(\cdot)\|)+\mu(\|G(\cdot)\|)&=\frac{1}{n}(\sum_{i=1}^n(\alpha_i\|F(i)\|+\|G(i)\|))\\ &=\frac{1}{n}(\alpha\cdot(\|F(1)\|,\dots,\|F(n)\|)+\|G\|_1)\end{aligned}$$

and so on. By Theorem 3.7, we clearly have this theorem. $\square$

By Theorem 4.1, we have the following corollary.

COROLLARY 4.2. ([7]). *Let X be a Banach space and let $x_1,\dots,x_n$ be nonzero elements in X. Then*

$$\|\sum_{i=1}^n x_i\| + \left(n - \left\|\sum_{i=1}^n \frac{x_i}{\|x_i\|}\right\|\right)\min_{1\leqslant i\leqslant n}\|x_i\| \leqslant \sum_{i=1}^n \|x_i\| \leqslant \|\sum_{i=1}^n x_i\| + \left(n - \left\|\sum_{i=1}^n \frac{x_i}{\|x_i\|}\right\|\right)\max_{1\leqslant i\leqslant n}\|x_i\|.$$

Proof. Take $x_1,\dots,x_n$ in X. In Theorem 4.1, we put $F=(x_1,\dots,x_n)$, $G=\mathbf{0}$ and $\alpha_i=(\min_{1\leqslant j\leqslant n}\|x_j\|)/\|x_i\|$ for every $1\leqslant i\leqslant n$. Then we have the first inequality. For the second inequality we put $F=0$, $G=(y_1,\cdots,y_n)$ and $\beta_i=(\max_{1\leqslant i\leqslant n}\|y_i\|)/\|y_i\|$, $1\leqslant i\leqslant n$. □

An element $\mu\in\ell^\infty(\mathbb{N})^*$ is called a *Banach limit* if $\liminf_{n\to\infty}u_n\leqslant\mu(\{u_n\}_{n=1}^\infty)\leqslant\limsup_{n\to\infty}u_n$ and $\mu(\{u_n\}_{n=1}^\infty)=\mu(\{u_{n+k}\}_{n=1}^\infty)$ for all $k\geqslant 1$. If μ is a Banach limit, then μ is a mean on $\ell^\infty(\mathbb{N})$. Then we have

THEOREM 4.3. *Let X be a Banach space and let $\mu\in\ell_\infty(\mathbb{N})^*$ be a Banach limit. If $\alpha=\{\alpha_n\}_{n=1}^\infty,\beta=\{\beta_n\}_{n=1}^\infty\in\ell_\infty(\mathbb{N})$ with $\mathbf{0}\leqslant\alpha$ and $0<\limsup_{n\to\infty}\alpha_n\leqslant 1\leqslant\liminf_{n\to\infty}\beta_n$, then*

$$\begin{aligned}\|M_\mu(x+y)\| &+ \mu(\{\|\alpha_n x_n\|\}_{n=1}^\infty)+\mu(\{\|y_n\|\}_{n=1}^\infty)-\|M_\mu(\alpha x+y)\|\\ &\leqslant \mu(\{\|x_n\|\}_{n=1}^\infty)+\mu(\{\|y_n\|\}_{n=1}^\infty)\\ &\leqslant \|M_\mu(x+y)\|+\mu(\{\|x_n\|\}_{n=1}^\infty)+\mu(\{\|\beta_n y_n\|\}_{n=1}^\infty)-\|M_\mu(x+\beta y)\|\end{aligned}$$

for all $x=\{x_n\}_{n=1}^\infty,y=\{y_n\}_{n=1}^\infty\in\ell^\infty(\mathbb{N},X)$.

Proof. For any ε with $0<\varepsilon<\limsup_{n\to\infty}\alpha_n$ there exists k such that $0\leqslant\max\{\alpha_n-\varepsilon,\ 0\}<1<\beta_n+\varepsilon$ for all $n\geqslant k$. Since μ is translation-invariant and so $M_\mu(z)=M_\mu(z')$ for any $z=\{z_n\}_{n=1}^\infty\in\ell^\infty(\mathbb{N},X)$, where $z'=\{z_{k+n}\}_{n=1}^\infty$, we may assume that $\gamma(\varepsilon)_n:=\max\{\alpha_n-\varepsilon,0\}<1<\beta_n+\varepsilon$ for all $n\geqslant 1$. Applying Theorem 3.7 to $x=\{x_n\}_{n=1}^\infty$ and $y=\{y_n\}_{n=1}^\infty$, we have

$$\begin{aligned}&\|M_\mu(x+y)\|+\mu(\{\|\gamma(\varepsilon)_n x_n\|\}_{n=1}^\infty)+\mu(\{\|y_n\|\}_{n=1}^\infty)-\|M_\mu(\gamma(\varepsilon)x+y)\|\\ &\quad\leqslant \mu(\{\|x_n\|\}_{n=1}^\infty)+\mu(\{\|y_n\|\}_{n=1}^\infty)\\ &\quad\leqslant \|M_\mu(x+y)\|+\mu(\{\|x_n\|\}_{n=1}^\infty)+\mu(\{\|(\beta_n+\varepsilon)y_n\|\})-\|M_\mu(x+(\beta+\varepsilon)y)\|.\end{aligned}$$

Letting $\varepsilon\to 0$, we have

$$\begin{aligned}\|M_\mu(x+y)\| &+ \mu(\{\|\alpha_n x_n\|\}_{n=1}^\infty)+\mu(\{\|y_n\|\}_{n=1}^\infty)-\|M_\mu(\alpha x+y)\|\\ &\leqslant \mu(\{\|x_n\|\}_{n=1}^\infty)+\mu(\{\|y_n\|\}_{n=1}^\infty),\end{aligned}$$

because by Proposition 3.1

$$\begin{aligned}|\mu(\{\|\gamma(\varepsilon)_n x_n\|\}_{n=1}^{\infty}) - \mu(\{\|\alpha_n x_n\|\}_{n=1}^{\infty})| &= |\mu(\{\|\gamma(\varepsilon)_n x_n\| - \|\alpha_n x_n\|\}_{n=1}^{\infty})| \\ &\leqslant \mu(\{\|(\gamma(\varepsilon)_n - \alpha_n)x_n\|\}_{n=1}^{\infty}) \\ &\leqslant \|\gamma(\varepsilon) - \alpha\|_{\infty}\mu(\{\|x_n\|\}_{n=1}^{\infty}) \\ &\leqslant \varepsilon\mu(\{\|x_n\|\}_{n=1}^{\infty})\end{aligned}$$

for instance. Similarly we have the second inequality. This completes the proof. □

In Theorem 4.3, put

$$y = 0 \text{ and } \alpha = \left\{\frac{\liminf_{n\to\infty}\|x_n\|}{\|x_n\|}\right\}_{n=1}^{\infty}$$

and

$$x = 0 \text{ and } \beta = \left\{\frac{\limsup_{n\to\infty}\|x_n\|}{\|x_n\|}\right\}_{n=1}^{\infty}$$

in the first and second inequalities, respectively. Then we obtain the following

THEOREM 4.4. *Let X be a Banach space, $\mu \in \ell^{\infty}(\mathbb{N})^*$ a Banach limit and $\{x_n\}_{n=1}^{\infty} \in \ell^{\infty}(\mathbb{N}, X)$ such that $\liminf_{n\to\infty}\|x_n\| > 0$. Then*

$$\begin{aligned}&\|M_{\mu}(\{x_n\}_{n=1}^{\infty})\| + \left(1 - \left\|M_{\mu}\left(\left\{\frac{x_n}{\|x_n\|}\right\}_{n=1}^{\infty}\right)\right\|\right)\liminf_{n\to\infty}\|x_i\| \\ &\quad\leqslant \mu(\{\|x_n\|\}_{n=1}^{\infty}) \\ &\quad\leqslant \|M_{\mu}(\{x_n\}_{n=1}^{\infty})\| + \left(1 - \left\|M_{\mu}\left(\left\{\frac{x_n}{\|x_n\|}\right\}_{n=1}^{\infty}\right)\right\|\right)\limsup_{n\to\infty}\|x_n\|.\end{aligned}$$

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