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A REMARK ABOUT POSITIVE POLYNOMIALS

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Abstract. The following theorem is proved.

THEOREM. *Let* $P(x)=\sum_{k=0}^{2n}a_kx^k$ *be a polynomial with positive coefficients. If the inequalities* $\frac{a_{2k+1}^2}{a_{2k}a_{2k+2}}<\frac{1}{\cos^2(\frac{\pi}{n+2})}$ *hold for all* $k=0,1,\ldots,n-1$, *then* $P(x)>0$ *for every* $x\in\mathbb{R}$.

We show that the constant $\frac{1}{\cos^2(\frac{\pi}{n+2})}$ in this theorem cannot be increased. We also present some corollaries of this theorem.

1. Introduction and statement of results

Positive polynomials arise in many important branches of mathematics. In this note we give a simple sufficient condition for an even degree polynomial with positive coefficients to be positive on the real line. Before we formulate the main theorem we will mention two results which have in some sense similar character.

In 1926, Hutchinson [4, p. 327] extended the work of Petrovitch [8] and Hardy [2] or [3, pp. 95–100] and proved the following theorem.

THEOREM O. *Let* $P(x)=\sum_{k=0}^{n}a_kx^k$ *be a polynomial with positive coefficients. If the inequalities*

$$\frac{a_k^2}{a_{k-1}a_{k+1}}\geqslant 4,\quad k=1,2,\ldots,n-1, \tag{1}$$

hold, then all zeros of $P(x)$ *are real.*

In [7] it was proved that the constant 4 in Theorem A is sharp.

In [6] the authors of this note have found the smallest possible constant $d_n>0$ such that if coefficients of $P(x)=\sum_{k=0}^{n}a_kx^k$ are positive and satisfy the inequalities $\frac{a_k^2}{a_{k-1}a_{k+1}}>d_n$, $k=1,2,\ldots,n-1$, then $P(x)$ is Hurwitz stable. We recall that a real polynomial is called Hurwitz (stable) if all its zeros have negative real parts.

The following theorem is the main result of this work.

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THEOREM 1. *Let $P(x)=\sum_{k=0}^{2n} a_k x^k$ be a polynomial with positive coefficients. If the inequalities*

$$\frac{a_{2k+1}^2}{a_{2k}a_{2k+2}} < \frac{1}{\cos^2(\frac{\pi}{n+2})}$$

hold for all $k=0,1,\dots,n-1$, then $P(x)>0$ for every $x\in\mathbb{R}$.

The following theorem shows that the constant $\frac{1}{\cos^2(\frac{\pi}{n+2})}$ in Theorem 1 is sharp for every $n\in\mathbb{N}$.

THEOREM 2. *For every $n\in\mathbb{N}$ there exists a polynomial $Q(x)=\sum_{k=0}^{2n} a_k x^k$ with positive coefficients which satisfy conditions*

$$\frac{a_{2k+1}^2}{a_{2k}a_{2k+2}} = \frac{1}{\cos^2(\frac{\pi}{n+2})}, \quad k=0,1,\dots,n-1,$$

and such that $Q(x)$ has at least two real zeros.

The following statement is a simple corollary of Theorem 1.

COROLLARY 1. *Let $P(x)=\sum_{k=0}^{2n+1} a_k x^k$ be a polynomial with positive coefficients. If the inequalities*

$$\frac{a_{2k}^2}{a_{2k-1}a_{2k+1}} < \frac{4k^2-1}{4k^2}\cdot\frac{1}{\cos^2(\frac{\pi}{n+2})}$$

hold for all $k=1,2,\dots,n$, then $P(x)$ has only one real zero (counting multiplicities).

We show that the constants in the last statement are also sharp for every $n\in\mathbb{N}$.

THEOREM 3. *For every $n\in\mathbb{N}$ there exists a polynomial $Q(x)=\sum_{k=0}^{2n+1} a_k x^k$ with positive coefficients which satisfy conditions*

$$\frac{a_{2k}^2}{a_{2k-1}a_{2k+1}} = \frac{4k^2-1}{4k^2}\cdot\frac{1}{\cos^2(\frac{\pi}{n+2})}, \quad k=1,2,\dots,n,$$

and such that $Q(x)$ has at least three real zeros.

2. Proof of Theorem 1

Let $P(x)=\sum_{k=0}^{2n} a_k x^k$ be a polynomial with positive coefficients. Let us consider a quadratic form

$$\begin{aligned} Q_P(x_0,x_1,x_2,\dots,x_n) &= a_0x_0^2+a_1x_0x_1+a_2x_1^2+a_3x_1x_2+a_4x_2^2+\dots \\ &\quad +a_{2n-2}x_{n-1}^2+a_{2n-1}x_{n-1}x_n+a_{2n}x_n^2 \\ &= \sum_{k=0}^{n} a_{2k}x_k^2+\sum_{k=0}^{n-1} a_{2k+1}x_kx_{k+1}. \end{aligned} \tag{2}$$

For every $x\in\mathbb{R}$ we have

$$P(x)=Q_P(1,x,x^2\dots,x^n). \tag{3}$$

Thus, if the quadratic form Q_P is positive-definite, then for every $x \in \mathbb{R}$ we have $P(x) > 0$. It remains to prove that under assumptions of Theorem 1 the quadratic form Q_P is positive-definite. The following $(n+1)\times(n+1)$ matrix corresponds to the quadratic form Q_P

$$M_{Q_p} := \left\| \begin{array}{ccccccc} a_0 & \frac{a_1}{2} & 0 & 0 & \dots & 0 & 0 \\ \frac{a_1}{2} & a_2 & \frac{a_3}{2} & 0 & \dots & 0 & 0 \\ 0 & \frac{a_3}{2} & a_4 & \frac{a_5}{2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \frac{a_{2n-3}}{2} & a_{2n-2} & \frac{a_{2n-1}}{2} \\ 0 & 0 & 0 & \dots & 0 & \frac{a_{2n-1}}{2} & a_{2n} \end{array} \right\| . \tag{4}$$

By Sylvester's Criterion for positive definiteness (see, for example, [1, chapter 10, §4]) we need to show that all leading principal minors of the matrix M_{Q_p} are positive. To do this, we will use the following theorem from [5].

THEOREM A. *Let $M = (a_{ij})$ be an $m\times m$ matrix with the properties*
(a) $a_{ij} > 0$ $(1 \leqslant i,j \leqslant m)$ and
(b) $a_{ij}a_{i+1,j+1} > 4\cos^2\frac{\pi}{m+1}\, a_{i,j+1}a_{i+1,j}$ $(1 \leqslant i,j \leqslant m-1)$.
Then all minors of M are positive.

In [5] it is also shown that the constant $c_m := 4\cos^2\frac{\pi}{m+1}$ in the statement of Theorem A is the smallest possible not only in the class of $m\times m$ matrices with positive entries but also for the class of $m\times m$ Toeplitz matrices and for the class of $m\times m$ Hankel matrices.

Consider the following $(n+1)\times(n+1)$ symmetrical Toeplitz matrix

$$T(\varepsilon_1,\dots,\varepsilon_{n-1}) := \left\| \begin{array}{ccccccc} a_0 & \frac{a_1}{2} & \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_{n-2} & \varepsilon_{n-1} \\ \frac{a_1}{2} & a_2 & \frac{a_3}{2} & \varepsilon_1 & \dots & \varepsilon_{n-3} & \varepsilon_{n-2} \\ \varepsilon_1 & \frac{a_3}{2} & a_4 & \frac{a_5}{2} & \dots & \varepsilon_{n-4} & \varepsilon_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \varepsilon_{n-2} & \varepsilon_{n-3} & \dots & \varepsilon_1 & \frac{a_{2n-3}}{2} & a_{2n-2} & \frac{a_{2n-1}}{2} \\ \varepsilon_{n-1} & \varepsilon_{n-2} & \varepsilon_{n-3} & \dots & \varepsilon_1 & \frac{a_{2n-1}}{2} & a_{2n} \end{array} \right\| , \tag{5}$$

where $\varepsilon_1 > \varepsilon_2 > \dots > \varepsilon_{n-1} > 0$ will be chosen in such a way that the matrix $T(\varepsilon_1,\dots,\varepsilon_{n-1})$ will satisfy the assumptions of Theorem A. At first we will choose ε_1 such that

$$\frac{a_{2j-1}a_{2j+1}}{4} > 4\cos^2\frac{\pi}{n+2}\, a_{2j}\varepsilon_1, \quad j = 1,2,\dots,n-1.$$

After that we will choose ε_2 such that

$$\varepsilon_1^2 > 4\cos^2\frac{\pi}{n+2}\, a_{2j+1}\varepsilon_2, \quad j = 1,2,\dots,n-2.$$

Then we will choose $\varepsilon_3 > \varepsilon_4 > \cdots > \varepsilon_{n-1} > 0$ one after another such that

$$\varepsilon_j^2 > 4\cos^2 \frac{\pi}{n+2} \varepsilon_{j-1}\varepsilon_{j+1}, \quad j = 2,3,\ldots,n-2.$$

For our choice of $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_{n-1} > 0$ and under the assumptions on $a_0, a_1, \ldots, a_{2n}$, the matrix $T(\varepsilon_1,\ldots,\varepsilon_{n-1})$ satisfies the assumptions of Theorem A. So by Theorem A all minors of $T(\varepsilon_1,\ldots,\varepsilon_{n-1})$ are positive. Letting $\varepsilon_{n-1} \to 0,\ \varepsilon_{n-2} \to 0, \ldots,\ \varepsilon_1 \to 0$, we obtain that all minors of M_{Q_p} are nonnegative. It remains to prove that all leading principal minors of the matrix M_{Q_p}

$$\Delta_1(M_{Q_p}) = a_0,\ \Delta_2(M_{Q_p}) = \det\begin{pmatrix} a_0 & \frac{a_1}{2} \\ \frac{a_1}{2} & a_2 \end{pmatrix}, \ldots,\ \Delta_{n+1}(M_{Q_p}) = \det M_{Q_p}$$

are positive.

Suppose there is a leading principal minor of M_{Q_p} which is equal to zero. Denote by j the smallest order of a leading principal minor which is equal to zero. Since $\Delta_1(M_{Q_p}) = a_0 > 0$, we have $j \geqslant 2$ and $\Delta_{j-1}(M_{Q_p}) > 0, \Delta_j(M_{Q_p}) = 0$. Let us consider a polynomial $P_\varepsilon(x) = P(x) - \varepsilon x^{2j-2}$ where $\varepsilon > 0$ is so small that $P_\varepsilon(x)$ satisfies the assumptions of Theorem 1. As we have proved it implies that all minors of a corresponding matrix $M_{Q_{p\varepsilon}}$ are nonnegative, in particular

$$\Delta_j(M_{Q_{p\varepsilon}}) = \det\begin{pmatrix} a_0 & \frac{a_1}{2} & 0 & 0 & \ldots & 0 & 0 \\ \frac{a_1}{2} & a_2 & \frac{a_3}{2} & 0 & \ldots & 0 & 0 \\ 0 & \frac{a_3}{2} & a_4 & \frac{a_5}{2} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \frac{a_{2j-5}}{2} & a_{2j-4} & \frac{a_{2j-3}}{2} \\ 0 & 0 & 0 & \ldots & 0 & \frac{a_{2j-3}}{2} & a_{2j-2}-\varepsilon \end{pmatrix} \geqslant 0.$$

Since $\Delta_{j-1}(M_{Q_p}) > 0$ we conclude that the determinant $\Delta_j(M_{Q_{p\varepsilon}})$ is strictly decreasing in $\varepsilon > 0$ and so $\Delta_j(M_{Q_p})$ (which is equal to $\Delta_j(M_{Q_{p\varepsilon}})$ for $\varepsilon = 0$) is strictly positive. Thus there are no zero leading principal minors of M_{Q_p}, all leading principal minors of M_{Q_p} are positive. By Sylvester's Criterion it means that the quadratic form Q_P is positive-definite, and in particular from (3) we obtain that $P(x) > 0$ for every $x \in \mathbb{R}$.

Theorem 1 is proved. □

3. Proof of Theorems 2 and 3

Proof of Theorem 2. Let us fix an arbitrary $n \in \mathbb{N}$ and denote by $\alpha := \frac{\pi}{n+2}$. Consider the following polynomial

$$Q(x) := \sum_{k=1}^{n} \sin k\alpha \sin(k+1)\alpha\ (1+x)^2 x^{2k-2}.$$

Obviously $Q(x)$ is a polynomial with positive coefficients of degree $2n$ and -1 is a root of Q of multiplicity not less than 2. We have

$$\begin{aligned}
Q(x) &= \sum_{k=1}^{n} \sin k\alpha \sin(k+1)\alpha \,(x^{2k-2}+x^{2k}) + 2\sum_{k=1}^{n} \sin k\alpha \sin(k+1)\alpha \, x^{2k-1} \qquad (6)\\
&= \sum_{k=0}^{n-1} \sin(k+1)\alpha \sin(k+2)\alpha \, x^{2k} + \sum_{k=1}^{n} \sin k\alpha \sin(k+1)\alpha \, x^{2k}\\
&\quad +2\sum_{k=1}^{n} \sin k\alpha \sin(k+1)\alpha \, x^{2k-1}\\
&= \sin\alpha \sin 2\alpha + \sum_{k=1}^{n-1} \sin(k+1)\alpha(\sin(k+2)\alpha + \sin k\alpha)x^{2k}\\
&\quad + \sin(n+1)\alpha(\sin n\alpha + \sin(n+2)\alpha)x^{2n} + 2\sum_{k=1}^{n} \sin k\alpha \sin(k+1)\alpha \, x^{2k-1}\\
&= 2\sin^2\alpha\cos\alpha + 2\sum_{k=1}^{n-1} \sin^2(k+1)\alpha\cos\alpha \, x^{2k} + 2\sin^2(n+1)\alpha\cos\alpha \, x^{2n}\\
&\quad +2\sum_{k=1}^{n} \sin k\alpha \sin(k+1)\alpha \, x^{2k-1}\\
&= 2\sum_{k=0}^{n} \sin^2(k+1)\alpha\cos\alpha \, x^{2k} + 2\sum_{k=1}^{n} \sin k\alpha \sin(k+1)\alpha \, x^{2k-1}
\end{aligned}$$

(we use the fact that $\sin(n+2)\alpha = 0$). So if we define by a_j, $j = 0,1,\ldots,2n$, the coefficients of Q then

$$a_{2k} = 2\sin^2(k+1)\alpha\cos\alpha, \; a_{2k-1} = 2\sin k\alpha \sin(k+1)\alpha.$$

We have

$$\frac{a_{2k+1}^2}{a_{2k}a_{2k+2}} = \frac{4\sin^2(k+1)\alpha\sin^2(k+2)\alpha}{2\sin^2(k+1)\alpha\cos\alpha\cdot 2\sin^2(k+2)\alpha\cos\alpha} = \frac{1}{\cos^2\alpha}$$

for $k = 0,1,\ldots,n-1$.

Theorem 2 is proved. □

Proof of Theorem 3. Let us fix an arbitrary $n\in\mathbb{N}$ and denote by $\alpha := \frac{\pi}{n+2}$. Let us consider the following primitive of the polynomial $Q(x)$ constructed in the proof of Theorem 2:

$$H(x) = \sum_{k=1}^{n} \sin k\alpha \sin(k+1)\alpha \left(\frac{x^{2k-1}}{2k-1} + 2\frac{x^{2k}}{2k} + \frac{x^{2k+1}}{2k+1}\right).$$

We have

$$H(-1) = \sum_{k=1}^{n} \sin k\alpha \sin(k+1)\alpha \left(-\frac{1}{2k-1} + \frac{1}{k} - \frac{1}{2k+1}\right)$$
$$= \sum_{k=1}^{n} \sin k\alpha \sin(k+1)\alpha \left(\frac{-1}{k(2k-1)(2k+1)}\right) < 0.$$

So the following polynomial

$$S(x) = H(x) - H(-1)$$

is a polynomial with positive coefficients of degree $2n+1$ and -1 is a root of S of multiplicity not less than 3. Using (6) we can rewrite $S(x)$ in the form

$$S(x) = -H(-1) + 2\sum_{k=0}^{n} \sin^2(k+1)\alpha \cos\alpha \, \frac{x^{2k+1}}{2k+1} + 2\sum_{k=1}^{n} \sin k\alpha \sin(k+1)\alpha \, \frac{x^{2k}}{2k}.$$

So if we define by $b_j,\ j = 0, 1, \ldots, 2n+1,$ the coefficients of Q then $b_0 = -H(-1)$ and

$$b_{2k+1} = \frac{2\sin^2(k+1)\alpha \cos\alpha}{2k+1}, \quad k = 0, 1, \ldots, n;$$
$$b_{2k} = \frac{2\sin k\alpha \sin(k+1)\alpha}{2k}, \quad k = 1, 2 \ldots, n.$$

We have

$$\frac{b_{2k}^2}{b_{2k-1}b_{2k+1}} = \frac{4\sin^2 k\alpha \sin^2(k+1)\alpha}{4k^2} \cdot \frac{2k-1}{2\sin^2 k\alpha \cos\alpha} \cdot \frac{2k+1}{2\sin^2(k+1)\alpha \cos\alpha}$$
$$= \frac{4k^2-1}{4k^2} \cdot \frac{1}{\cos^2\alpha}, \quad k = 1, 2 \ldots, n.$$

Theorem 3 is proved. □

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