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ON GENERALIZATION OF MOSER'S THEOREM IN THE CRITICAL CASE

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Abstract. Let Ω be an open bounded set in $\mathbb{R}^n$, $n \geqslant 2$. In paper [13] Moser proved that for every $K \geqslant K_0 = n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}}$ we have

$$\sup\Big\{\int_\Omega \exp\Big(\Big(\frac{f(x)}{K}\Big)^{\frac{n}{n-1}}\Big) : f \in W_0^{1,n}(\Omega), ||\nabla f||_{L^n} \leqslant 1\Big\} < \infty,$$

but for $K < K_0$ the supremum is not finite.

In this paper we study the critical case $K = K_0$ for arbitrary Orlicz-Sobolev spaces with Young functions that behave like t^n close to ∞. We show that for functions like $t^n(1-\log^{-a} t)$ the supremum is finite for $a > 1$ but infinite for $0 < a < 1$.

1. Introduction

Throughout the paper Ω denotes an open bounded set in $\mathbb{R}^n$, $n \geqslant 2$. We write $n' = \frac{n}{n-1}$ (i.e. $\frac{1}{n} + \frac{1}{n'} = 1$) and ω_{n-1} is the measure of the surface of the unit sphere in $\mathbb{R}^n$.

The classical Sobolev embedding theorem states that $W_0^{1,p}(\Omega)$ is continuously embedded into $L^{p^*}(\Omega)$ if $1 \leqslant p < n$ and $p^* = \frac{pn}{n-p}$. If $p > n$ then every function from $W_0^{1,p}(\Omega)$ is bounded (i.e. belongs to $L^\infty(\Omega)$) and in the limiting case $p = n$ it is known that every function from $W_0^{1,n}(\Omega)$ belongs to $L^q(\Omega)$ for every $1 \leqslant q < \infty$ but not necessarily to $L^\infty(\Omega)$.

A famous result by Trudinger (see [11], [16], [18] and [19]) implies that the first-order Sobolev space $W_0^{1,n}(\Omega)$ may be continuously embedded in the Orlicz space $L^\Phi(\Omega)$ with the Young function of the exponential type $\Phi(t) = \exp t^{n'} - 1, t > 0$. These results were later generalized to many function spaces close to $W^{1,n}$ (see e.g. [2], [3], [9]).

In [13] Moser proved that for $K \geqslant K_0 = n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}}$ we have

$$\sup\Big\{\int_\Omega \exp\Big(\Big(\frac{f(x)}{K}\Big)^{n'}\Big) : f \in W_0^{1,n}(\Omega), ||\nabla f||_{L^n} \leqslant 1\Big\} < \infty \qquad (1)$$

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but that for $K < K_0$ the integral $\int_\Omega \exp((\frac{f(x)}{K})^{n'})$ can be made arbitrarily large by an appropriate choice of $f \in W_0^{1,n}(\Omega)$, $||\nabla f||_{L^n} \leqslant 1$. We would like to explore this phenomenon in detail. In fact, we show that this result is no longer true in the critical case $K = K_0$ if we replace the function t^n in the definition of the L^n-space by an arbitrary Young function that behaves like t^n close to ∞. Moreover we give sharp bounds on the Young function for the validity of the results in the case $K = K_0$.

By ∇f we denote the generalized derivative of f while $W_0^{1,n}(\Omega)$ and $L_0^\Phi(\Omega)$ stand for the closure of $C_0^\infty(\Omega)$ in $W^{1,n}(\Omega)$ and $L^\Phi(\Omega)$, respectively. For the definition of the norm in $L^\Phi(\Omega)$ see the Preliminaries. By $WL^\Phi(\Omega)$ we denote the set of all functions f such that $f, |\nabla f| \in L^\Phi(\Omega)$. This space is equipped with the norm

$$||f||_{WL^\Phi(\Omega)} = ||f||_{L^\Phi(\Omega)} + ||\,|\nabla f|\,||_{L^\Phi(\Omega)} \,.$$

By $W_0L^\Phi(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $WL^\Phi(\Omega)$.

Let $a > 1$. Our positive result states that the conclusion of (1) is true in the critical case $K = K_0$ for functions that behave like $t^n(1 - \log^{-a} t)$ for t close to ∞.

THEOREM 1.1. *Let Ω be an open bounded set, $a > 1$ and $K = n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}}$. Then there is $A > 1$ such that every Young function Φ satisfying*

$$\begin{aligned} \Phi(t) &= t^n && \textit{for} \quad t \in [0,A] \\ \Phi(t) &\geqslant t^n\Big(1 - \log^{-a}(t)\Big) && \textit{for} \quad t \in [A,\infty) \end{aligned}$$

has the following property:

For every $f \in W_0L^\Phi(\Omega)$ such that $\int_\Omega \Phi(|\nabla f(x)|)\,dx \leqslant 1$ we have

$$\int_\Omega \exp\Big(\Big(\frac{f(x)}{K}\Big)^{\frac{n}{n-1}}\Big)\,dx \leqslant C\,,$$

where C depends on n, $\mathscr{L}_n(\Omega)$ and a only.

Conversely, if the power of the logarithm satisfies $0 < a < 1$, then the supremum is not finite.

THEOREM 1.2. *Assume $0 < a < 1$, $K = n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}}$. Let Φ be a Young function such that there are $L_1 \geqslant 1$ and $L_2 > e$ satisfying*

$$\Phi(t) \leqslant \begin{cases} L_1 t^n & \textit{for } t \in [0,\infty) \\ \Big(1 - \log^{-a}(t)\Big)t^n & \textit{for } t \in [L_2,\infty)\,. \end{cases} \tag{2}$$

Then for every $m \in \mathbb{N}$ there is $f \in W_0L^\Phi(B(0,1))$ such that $\int_{B(0,1)} \Phi(|\nabla f(x)|)\,dx \leqslant 1$ but

$$\int_{B(0,1)} \exp\Big(\Big(\frac{f(x)}{K}\Big)^{\frac{n}{n-1}}\Big)\,dx > m\,.$$

Now let us recall what was known before we started our research. It is known that the space $W_0L^n\log^\alpha L(\Omega)$ (i.e. the space $W_0L^\Phi(\Omega)$ with the Young function Φ satisfying $\Phi(t)=t^n\log^\alpha(t)$ on $[t_0,\infty)$ for some $t_0>0$), $\alpha<n-1$, is continuously embedded into the Orlicz space with the Young function $\exp(t^\gamma)-1$, where

$$\gamma=\frac{n}{n-1-\alpha}>0\,.$$

These results are due to Fusco, Lions, Sbordone [9] for $\alpha<0$ and Edmunds, Gurka, Opic [3] in general. For other results concerning these spaces we refer the reader to [4], [5], [6], [7], [8] and [14].

Analogy of Moser's result in this setting was shown by Hencl [12]. It is shown there that for every Young function Φ such that

$$\lim_{t\to\infty}\frac{\Phi(t)}{t^n\log^\alpha t}=1 \text{ and for every } K>K_\alpha=\Big(1-\frac{\alpha}{n-1}\Big)^{-\frac{n-1}{n}}n^{-\frac{1}{\gamma}}\omega_{n-1}^{-\frac{1}{n}}$$

we have

$$\sup\Big\{\int_\Omega\exp\Big(\Big(\frac{f(x)}{K}\Big)^\gamma\Big):f\in W_0L^\Phi(\Omega),\int_\Omega\Phi(|\nabla f|)\,dx\leqslant 1\Big\}<\infty\,. \tag{3}$$

Conversely, if $K<K_\alpha$ then the supremum is not finite.

In the critical case $K=K_\alpha$ it is shown in [12] that there are Young functions Φ_1 and Φ_2 that behave like $t^n\log^\alpha(t)$ close to ∞, the conclusion of (3) is valid for Φ_1, but the supremum is infinite for Φ_2. However, there is a huge gap between the functions Φ_1 and Φ_2, because

$$\Phi_2(t)\sim t^n\log^\alpha(t)\Big(1-\frac{1}{\sqrt{\log(\log(t))}}\Big)\qquad\text{and}\qquad\Phi_1(t)\sim t^n\log^\alpha(t)\Big(1+\log^{-a}(t)\Big)$$

for some $a\in\big(0,\min(1,\frac{1}{\gamma})\big)$. In fact, even in the simplest Moser's case $\alpha=0$ and $K=K_0$ these results are not sharp enough to say anything about the Young function $\Phi(t)=t^n$. One of the motivations for our research was to close this gap and we have found sharp criteria in the simplest model case $\alpha=0$.

The proof of Theorem 1.2 is given in the third section. We modify the construction from [12] and we use more careful estimates. Section 4 contains the proof of Theorem 1.1. The basic outline of the proof is similar to the one in [12], but our result requires new ideas and a finer technique.

2. Preliminaries

We denote the n-dimensional Lebesgue measure by $\mathscr{L}_n$.

By $B(0,R)$ we denote an open Euclidean ball in $\mathbb{R}^n$ centered at the origin with the radius $R>0$.

A function $\Phi:\mathbb{R}^+\to\mathbb{R}^+$ is a Young function if $\Phi(0)=0$, Φ is increasing and convex.

We denote the Orlicz space corresponding to a Young function Φ on a set A with a measure μ by $L^{\Phi}(A,d\mu)$. This space is equipped with the norm

$$||f||_{L^{\Phi}(A,d\mu)} = \inf\Big\{\lambda > 0 : \int_A \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\, d\mu(x) \leqslant \Phi(1)\Big\} . \tag{4}$$

Note that this definition is slightly different from the usual definition where the condition $\int_A \Phi(\frac{|f(x)|}{\lambda})\, d\mu(x) \leqslant \Phi(1)$ is replaced by $\int_A \Phi(\frac{|f(x)|}{\lambda})\, d\mu(x) \leqslant 1$. We use (4) to have the generalized Hölder inequality with the multiplicative constant 1 (see (8) bellow). If $||f||_{L^{\Phi}(A,d\mu)} > 0$ then it follows from Fatou's lemma that

$$\int_A \Phi\Big(\frac{|f(x)|}{||f||_{L^{\Phi}(A,d\mu)}}\Big)\, d\mu(x) \leqslant \Phi(1) . \tag{5}$$

Further from (4) we have

$$\Phi_0(t) = \alpha\Phi(t) \quad \text{on } [0,\infty) \quad \text{with } \alpha > 0 \quad \Rightarrow \quad ||f||_{L^{\Phi_0}(A,d\mu)} = ||f||_{L^{\Phi}(A,d\mu)} . \tag{6}$$

For any given differentiable Young function Φ we can define a generalized inverse function to $\phi(u) = \Phi'(u)$ by

$$\psi(s) = \inf\{u : \phi(u) > s\} \qquad \text{for} \quad s > 0$$

and further we define its associated Young function Ψ by

$$\Psi(t) = \int_0^t \psi(s)\, ds \qquad \text{for} \quad t \geqslant 0 .$$

If Φ_0 and Φ_1 are Young functions and Ψ_0 and Ψ_1 are their associated Young functions, respectively, then we have

$$\Phi_0(t) \geqslant \Phi_1(t) \quad \text{on} \quad [0,\infty) \quad \Rightarrow \quad \Psi_0(t) \leqslant \Psi_1(t) \quad \text{on} \quad [0,\infty) . \tag{7}$$

The dual space of $L^{\Phi}(A,d\mu)$ can be identified as the Orlicz space $L^{\Psi}(A,d\mu)$. If in addition we have $\Phi(1) + \Psi(1) = 1$ then the following generalization of the Hölder inequality is valid (see [15] page 58 for the proof)

$$\int_A |f(y)g(y)|\, d\mu(y) \leqslant ||f||_{L^{\Phi}(A,d\mu)} ||g||_{L^{\Psi}(A,d\mu)} . \tag{8}$$

We use this inequality for a measurable set $A \subset \mathbb{R}$ and the measure $d\mu(y) = \omega_{n-1} y^{n-1} dy$. For an introduction to Orlicz spaces see e.g. [15].

The non-increasing rearrangement f^* of a measurable function f on Ω is defined by

$$f^*(t) = \inf\big\{s > 0 : \mathscr{L}_n(\{x \in \Omega : |f(x)| > s\}) \leqslant t\big\}, \quad t > 0 .$$

We also define the non-increasing radially symmetric rearrangement $f^{\#}$ by

$$f^{\#}(x) = f^*\Big(\frac{\omega_{n-1}}{n}|x|^n\Big) \qquad \text{for} \quad x \in B(0,R),\ \mathscr{L}_n(B(0,R)) = \mathscr{L}_n(\Omega) .$$

For an introduction to these rearrangements see e.g. [17]. We need the fact that for every Young function Φ and for every measurable function $f:\Omega\to\mathbb{R}$ we have

$$\int_\Omega \Phi(|f(x)|)\,dx=\int_{B(0,R)}\Phi(|f^{\#}(x)|)\,dx=\int_0^{\mathscr{L}_n(\Omega)}\Phi(|f^{*}(y)|)\,dy\,.$$

We also use the Polya-Szegö principle (see e.g. Talenti [17] for the proof).

THEOREM 2.1. *Let Ω be an open bounded set and let $R>0$ be such that $\mathscr{L}_n(B(0,R))=\mathscr{L}_n(\Omega)$. Let Φ be a Young function. Suppose that the function $f:\Omega\to\mathbb{R}$ is Lipschitz continuous, $\int_\Omega\Phi(|\nabla f|)<\infty$ and $f\in W_0L^\Phi(\Omega)$. Then f^* is locally absolutely continuous and*

$$\int_\Omega \Phi(|\nabla f(x)|)\,dx\geqslant\int_{B(0,R)}\Phi(|\nabla f^{\#}(x)|)\,dx\,.$$

By C we denote a generic positive constant which may depend on n, $\mathscr{L}_n(\Omega)$ and Φ. This constant may vary from expression to expression. Some lemmata state that for every $a>0$ or $b>0$ something is true. Then this constant C in the proof of such a lemma may depend also on fixed $a>0$ or $b>0$.

Sometimes it is convenient for us to write C_1, C_2, etc.

3. Counterexample

Proof of Theorem 1.2. For $s>e$ we define $f_s(x)=g_s(|x|)$ where

$$g_s(y)=\begin{cases}(-2y+2)Kn\log(2)s^{-\frac1n}\left(1+\frac{\log(s)}{s}\right)^{\frac{n-1}{n}} & \text{for } y\in\left[\frac12,1\right]\\ Kn\log\left(\frac1y\right)s^{-\frac1n}\left(1+\frac{\log(s)}{s}\right)^{\frac{n-1}{n}} & \text{for } y\in\left[e^{-\frac sn},\frac12\right]\\ Ks^{\frac{n-1}{n}}\left(1+\frac{\log(s)}{s}\right)^{\frac{n-1}{n}} & \text{for } y\in\left[0,e^{-\frac sn}\right].\end{cases}$$

An easy computation gives us

$$\begin{aligned}\int_{B(0,1)}\exp\left(\left(\frac{f_s(x)}{K}\right)^{\frac{n}{n-1}}\right)dx&\geqslant\int_{B(0,e^{-\frac sn})}\exp\left(\left(\frac{f_s(x)}{K}\right)^{\frac{n}{n-1}}\right)dx\\&\geqslant Ce^{-s}e^{(1+\frac{\log(s)}{s})s}\underset{s\to\infty}{\longrightarrow}\infty\,.\end{aligned}$$

It remains to prove that $\int_{B(0,1)}\Phi(|\nabla f_s|)\leqslant 1$ for s large enough.

Set $M=M(s)=\frac{1}{s^{\log(s)}}$. Plainly there is $s_1>e$ such that for $s>s_1$ we have $\exp(-\frac sn)<M<\frac12$ and therefore

$$\int_0^1\Phi(|g_s'(y)|)y^{n-1}dy=\int_{e^{-\frac sn}}^M+\int_M^1=I_1+I_2\,. \tag{9}$$

Obviously $|g_s'(y)| \leqslant Cs^{-\frac{1}{n}} \leqslant Cs^{-\frac{1}{n}}\frac{1}{y}$ for $y \in (\frac{1}{2},1)$ and $|g_s'(y)| \leqslant Cs^{-\frac{1}{n}}\frac{1}{y}$ also on $(M,\frac{1}{2})$. It follows from (2) and $\log(\frac{1}{M}) = \log^2(s)$ that

$$\begin{aligned} I_2 &\leqslant C\int_M^1 |g'(y)|^n y^{n-1}\,dy \leqslant Cs^{-1}\int_M^1 \frac{dy}{y} = Cs^{-1}\log\Big(\frac{1}{M}\Big) \\ &\leqslant Cs^{-1}\log^2(s)\ . \end{aligned} \tag{10}$$

Since $M = \frac{1}{s^{\log(s)}}$ we can find $s_2 > s_1$ such that if $s > s_2$ then

$$y \in (e^{-\frac{s}{n}},M) \Rightarrow |g_s'(y)| = Kn\frac{1}{y}s^{-\frac{1}{n}}\Big(1+\frac{\log(s)}{s}\Big)^{\frac{n-1}{n}} \geqslant Kn\frac{1}{M}s^{-\frac{1}{n}} > L_2\ .$$

Plainly there is $s_3 > s_2$ such that for $s > s_3$ we have

$$\sup_{y\in(e^{-\frac{s}{n}},M)} |g_s'(y)| = \lim_{y\to(e^{-\frac{s}{n}})_+} |g_s'(y)| = Kne^{\frac{s}{n}}s^{-\frac{1}{n}}\Big(1+\frac{\log(s)}{s}\Big)^{\frac{n-1}{n}} \leqslant e^s\ .$$

Thus (2) gives us for $y \in (e^{-\frac{s}{n}},M)$ that

$$\Phi(|g_s'(y)|) \leqslant \Big(1-\frac{1}{s^a}\Big)|g_s'(y)|^n\ . \tag{11}$$

From (11), $K = n^{-\frac{n-1}{n}}\omega_{n-1}^{-\frac{1}{n}}$ and $\log(\frac{1}{M}) = \log^2(s) > 0$ for $s > s_3$ we have

$$\begin{aligned} I_1 &\leqslant \Big(1-\frac{1}{s^a}\Big)\int_{e^{-\frac{s}{n}}}^M |g_s'(y)|^n y^{n-1}dy \\ &= \Big(1-\frac{1}{s^a}\Big)\Big(1+\frac{\log s}{s}\Big)^{n-1} K^n n^n s^{-1}\int_{e^{-\frac{s}{n}}}^M \frac{dy}{y} \\ &= \Big(1-\frac{1}{s^a}\Big)\Big(1+\frac{\log s}{s}\Big)^{n-1} K^n n^n s^{-1}\Big(\frac{s}{n}-\log\Big(\frac{1}{M}\Big)\Big) \\ &\leqslant \frac{1}{\omega_{n-1}}\Big(1-\frac{1}{s^a}\Big)\Big(1+\frac{\log s}{s}\Big)^{n-1}\ . \end{aligned} \tag{12}$$

Since $0 < a < 1$, using (10) and (12) for s large enough we finally obtain

$$\begin{aligned} \int_{B(0,1)} \Phi(|\nabla f_s(x)|)dx &= \omega_{n-1}\int_0^1 \Phi(|g_s'(y)|)y^{n-1}\,dy = \omega_{n-1}(I_1+I_2) \\ &\leqslant \Big(1-\frac{1}{s^a}\Big)\Big(1+\frac{\log s}{s}\Big)^{n-1} + Cs^{-1}\log^2(s) \leqslant 1\ . \qquad \square \end{aligned}$$

4. Bounded norms

Most of this section is devoted to the proof of the following proposition which is a weaker version of Theorem 1.1 with the additional technical assumption:

$$\begin{gathered}\text{There are } \omega > 0 \text{ and } \tau \in \left(0, \frac{1}{\omega}\right) \text{ such that } \phi(t) := \frac{\Phi(t)}{t^n} \text{ satisfies} \\ \frac{\phi(\frac{t}{1-\delta})}{\phi(t)} \leqslant \frac{1}{1-\omega\delta} \quad \text{whenever} \quad \delta \in (0,\tau) \quad \text{and} \quad t \in (0,\infty)\,. \end{gathered} \tag{13}$$

This assumption is removed at the end of this section.

PROPOSITION 4.1. *Let Ω be an open bounded set, $a > 1$ and $K = n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}}$. Then there is $A_0 > 1$ such that every Young function Φ satisfying*

$$\begin{aligned} \Phi(t) &= t^n && \text{for} \quad t \in [0, A_0] \\ \Phi(t) &\geqslant t^n \left(1 - \log^{-a}(t)\right) && \text{for} \quad t \in [A_0, \infty) \end{aligned} \tag{14}$$

and condition (13) *has the following property:*

For every $f \in W_0 L^{\Phi}(\Omega)$ such that $\int_\Omega \Phi(|\nabla f(x)|)\, dx \leqslant 1$ we have

$$\int_\Omega \exp\left(\left(\frac{f(x)}{K}\right)^{\frac{n}{n-1}}\right) dx \leqslant C\,,$$

where C depends on n, $\mathscr{L}_n(\Omega)$ and Φ only.

The proof of Proposition 4.1 is based on the generalized Hölder inequality. Therefore it is convenient for us to deal with the Young function $\Phi_0 = \frac{1}{n}\Phi$ instead of Φ (note that Young functions Φ and Φ_0 give us the same norm by (4)). First we need some estimates of the Young function associated to Φ_0.

LEMMA 4.2. *Let $A_0 \geqslant 1$, $1 < b < a$ and suppose that the Young function Φ_0 satisfies*

$$\Phi_0(t) = \frac{1}{n} t^n \qquad \text{for} \quad t \in [0, A_0]\,. \tag{15}$$

Let Ψ_0 be the Young function associated to Φ_0. Then

$$\Psi_0(t) = \frac{n-1}{n} t^{\frac{n}{n-1}} \qquad \text{for} \quad t \in [0, A_0]\,. \tag{16}$$

Moreover, there is $A_1 > 1$ such that if $A_0 \geqslant A_1$ and Φ_0 satisfies in addition

$$\Phi_0(t) \geqslant \Phi_1(t) := \frac{1}{n} t^n \left(1 - \log^{-a}(t)\right) \qquad \text{for} \quad t \in [A_0, \infty)\,, \tag{17}$$

then

$$\Psi_0(t) \leqslant \tilde{\Psi}(t) := \frac{n-1}{n} t^{\frac{n}{n-1}} \left(1 + \log^{-b}(t)\right) \qquad \text{for} \quad t \in [A_0, \infty)\,. \tag{18}$$

Proof. Property (16) immediately follows from the definition of the associated Young function and the fact that $A_0 \geqslant 1$.

Let us prove the second part of the lemma. If $B_1 > 1$ is sufficiently large and $A_0 \geqslant B_1$, then the function Φ_1 can be extended from $[A_0,\infty)$ to a Young function defined on $[0,\infty)$ such that $\Phi_1(t) \leqslant \frac{1}{n}t^n$ for $t \in [0,\infty)$. Thus for every Young function Φ_0 satisfying (15) and (17) with this A_0 we have

$$\Phi_0(t) \geqslant \Phi_1(t) \qquad \text{for} \quad t \in [0,\infty) . \tag{19}$$

Let us fix $b_1 \in (b,a)$ and put

$$\tilde{\Psi}_1(t) = \frac{n-1}{n} t^{\frac{n}{n-1}} \Big(1 + \log^{-b_1}(t)\Big) \qquad \text{for} \quad t \in (0,\infty) .$$

Denote $\phi_1 = \Phi_1'$, $\psi_1 = \phi_1^{-1}$, $\Psi_1(t) = \int_0^t \psi_1$ and $\tilde{\psi}_1 = \tilde{\Psi}_1'$. Clearly as a is nonnegative and $B_1 > 1$ for every $t > B_1$ we have

$$\phi_1(t) = t^{n-1}\Big(1 - \log^{-a}(t) + \frac{a}{n}\log^{-a-1}(t)\Big) \geqslant t^{n-1}\Big(1 - \log^{-a}(t)\Big) = \tilde{\phi}(t) .$$

Further there is $B_2 > B_1$ such that for every $t > B_2$

$$\begin{aligned}\tilde{\psi}_1(t) &= t^{\frac{1}{n-1}}\Big(1 + \log^{-b_1}(t) - b_1\frac{n-1}{n}\log^{-b_1-1}(t)\Big)\\ &\geqslant t^{\frac{1}{n-1}}\Big(1 + \frac{1}{2}\log^{-b_1}(t)\Big) = \tilde{\psi}(t) .\end{aligned}$$

Clearly, as $0 < b_1 < a$, there is $B_3 > B_2$ such that for $t > B_3$ we have

$$\Big(1 + \frac{1}{2}\log^{-b_1}(t)\Big)^{n-1}\Big(1 - \log^{-a}\big(t^{\frac{1}{n-1}}\big)\Big) \geqslant 1 .$$

Moreover $\tilde{\psi}(t) \geqslant t^{\frac{1}{n-1}}$ for $t \geqslant 1$, which implies

$$1 - \log^{-a}\big(\tilde{\psi}(t)\big) \geqslant 1 - \log^{-a}\big(t^{\frac{1}{n-1}}\big) .$$

Therefore there is $B_4 > B_3$ such that for all $t > B_4$ we have

$$\begin{aligned}\tilde{\phi}(\tilde{\psi}(t)) &= \tilde{\psi}^{n-1}(t)\Big(1 - \log^{-a}(\tilde{\psi}(t))\Big)\\ &\geqslant t\Big(1 + \frac{1}{2}\log^{-b_1}(t)\Big)^{n-1}\Big(1 - \log^{-a}\big(t^{\frac{1}{n-1}}\big)\Big) \geqslant t .\end{aligned}$$

It follows that $\phi_1(\tilde{\psi}_1(t)) > t$ for $t > B_4$ and thus

$$\tilde{\psi}_1(t) > \phi_1^{-1}(t) = \psi_1(t) . \tag{20}$$

Hence $\Psi_1'(t) < \tilde{\Psi}_1'(t)$ for $t > B_4$ and we can find $C > 0$ so that for $t > B_4$ we have

$$\Psi_1(t) < \tilde{\Psi}_1(t) + C .$$

Together with $b < b_1$ this implies that there is $A_1 > B_4$ such that for all $t > A_1$ we have

$$\Psi_1(t) < \tilde{\Psi}(t)\ .$$

Thus if $A_0 \geqslant A_1$, then (7) and (19) imply

$$\Psi_0(t) \leqslant \Psi_1(t) < \tilde{\Psi}(t) \qquad \text{on} \quad [A_0,\infty)\ . \qquad \square$$

Now we need to estimate the term $||\frac{1}{y^{n-1}}||_{L^{\Psi_0}((t,s),\omega_{n-1}y^{n-1}dy)}$.

LEMMA 4.3. *Let $b > 1$. Then there is $A_2 = A_2(b) > 1$ such that if the Young function Ψ_0 satisfies*

$$\Psi_0(1) = \frac{n-1}{n} \tag{21}$$

and

$$\Psi_0(t) \leqslant \begin{cases} \tilde{\Psi}_1(t) := \frac{n-1}{n} t^{\frac{n}{n-1}} & \text{for} \quad t \in [0,A_2] \\ \tilde{\Psi}_2(t) := \frac{n-1}{n} t^{\frac{n}{n-1}} \Big(1+\log^{-b}(t)\Big) & \text{for} \quad t \in [A_2,\infty)\ , \end{cases} \tag{22}$$

then for $0 < t < s < \infty$ we have

$$\Big|\Big|\frac{1}{y^{n-1}}\Big|\Big|_{L^{\Psi_0}((t,s),\omega_{n-1}y^{n-1}dy)} \leqslant \omega_{n-1}^{\frac{n-1}{n}} \Big(\log\Big(\frac{s}{t}\Big)+1\Big)^{\frac{n-1}{n}}\ . \tag{23}$$

Proof. We want to prove that for

$$\lambda = \omega_{n-1}^{\frac{n-1}{n}} \Big(\log\Big(\frac{s}{t}\Big)+1\Big)^{\frac{n-1}{n}}$$

we have

$$I = \int_t^s \Psi_0\Big(\frac{1}{\lambda y^{n-1}}\Big)\omega_{n-1}y^{n-1}\,dy \leqslant \Psi_0(1) = \frac{n-1}{n}\ .$$

Recall $b > 1$. Let us find $A_2 = A_2(b) > 1$ large enough so that

$$\frac{\log^{1-b}(A_2)}{(n-1)(b-1)} < 1\ . \tag{24}$$

Set $M = (A_2\lambda)^{-\frac{1}{n-1}} \in (0,\infty)$. We distinguish three cases. If $0 < M \leqslant t$, then we have $\frac{1}{\lambda y^{n-1}} \in [0,A_2]$ for all $y \in (t,s)$ and from (22) we obtain

$$\begin{aligned} I &\leqslant \int_t^s \tilde{\Psi}_1\Big(\frac{1}{\lambda y^{n-1}}\Big)\omega_{n-1}y^{n-1}\,dy = \frac{n-1}{n}\frac{\omega_{n-1}}{\lambda^{\frac{n}{n-1}}}\int_t^s \frac{dy}{y} \\ &= \frac{n-1}{n}\frac{\log(s)-\log(t)}{\log(\frac{s}{t})+1} = \frac{n-1}{n}\frac{\log(\frac{s}{t})}{\log(\frac{s}{t})+1} \leqslant \frac{n-1}{n} \end{aligned}$$

and we are done. If $s \leqslant M$, then $A_2 \leqslant \frac{1}{\lambda s^{n-1}}$. Hence from the estimate $\Psi_0(t) \leqslant \tilde{\Psi}_1(t) \leqslant \tilde{\Psi}_2(t)$, $t > 0$, and the fact that

$$\Big(\frac{1}{(n-1)(b-1)}\log^{1-b}\Big(\frac{1}{\lambda y^{n-1}}\Big)\Big)' = \log^{-b}\Big(\frac{1}{\lambda y^{n-1}}\Big)\frac{1}{y} \tag{25}$$

we obtain from (24)

$$\begin{aligned} I &\leqslant \frac{n-1}{n} \frac{\omega_{n-1}}{\lambda^{\frac{n}{n-1}}} \int_t^s \Big(1+\log^{-b}\Big(\frac{1}{\lambda y^{n-1}}\Big)\Big)\frac{dy}{y} \\ &= \frac{n-1}{n} \frac{\log(\frac{s}{t}) + \frac{1}{(n-1)(b-1)}(\log^{1-b}(\frac{1}{\lambda s^{n-1}}) - \log^{1-b}(\frac{1}{\lambda t^{n-1}}))}{\log(\frac{s}{t})+1} \\ &\leqslant \frac{n-1}{n} \frac{\log(\frac{s}{t}) + \frac{1}{(n-1)(b-1)}\log^{1-b}(A_2)}{\log(\frac{s}{t})+1} \\ &\leqslant \frac{n-1}{n}\,. \end{aligned}$$

In the remaining case $t<M<s$ using (22) we have

$$I \leqslant \int_t^M \tilde{\Psi}_2\Big(\frac{1}{\lambda y^{n-1}}\Big)\omega_{n-1}y^{n-1}\,dy + \int_M^s \tilde{\Psi}_1\Big(\frac{1}{\lambda y^{n-1}}\Big)\omega_{n-1}y^{n-1}\,dy = I_1 + I_2\,.$$

Further

$$I_2 = \frac{n-1}{n}\frac{\omega_{n-1}}{\lambda^{\frac{n}{n-1}}}\int_M^s \frac{dy}{y} = \frac{n-1}{n}\frac{\omega_{n-1}}{\lambda^{\frac{n}{n-1}}}\log\Big(\frac{s}{M}\Big)\,.$$

Using (22) and (25) we obtain

$$\begin{aligned} I_1 &= \frac{n-1}{n} \frac{\omega_{n-1}}{\lambda^{\frac{n}{n-1}}} \int_t^M \Big(1+\log^{-b}\Big(\frac{1}{\lambda y^{n-1}}\Big)\Big)\frac{dy}{y} \\ &= \frac{n-1}{n} \frac{\omega_{n-1}}{\lambda^{\frac{n}{n-1}}} \Big(\log\Big(\frac{M}{t}\Big) + \frac{1}{(n-1)(b-1)}\Big(\log^{1-b}\Big(\frac{1}{\lambda M^{n-1}}\Big) - \log^{1-b}\Big(\frac{1}{\lambda t^{n-1}}\Big)\Big)\Big) \\ &\leqslant \frac{n-1}{n} \frac{\omega_{n-1}}{\lambda^{\frac{n}{n-1}}} \Big(\log\Big(\frac{M}{t}\Big) + \frac{1}{(n-1)(b-1)}\log^{1-b}(A_2)\Big)\,. \end{aligned}$$

Therefore from (24) we have

$$I \leqslant I_1 + I_2 \leqslant \frac{n-1}{n}\frac{1}{\log(\frac{s}{t})+1}\Big(\log\Big(\frac{s}{t}\Big) + \frac{1}{(n-1)(b-1)}\log^{1-b}(A_2)\Big) \leqslant \frac{n-1}{n}\,. \quad \square$$

If we consider the function $\tilde{\Phi}(t) = t^n$ then the norm corresponding to this Young function has the following property:

If $||f||_{L^{\tilde{\Phi}}((\alpha,\beta),d\mu)} = 1$, $\gamma \in (\alpha,\beta)$ and $||f||_{L^{\tilde{\Phi}}((\alpha,\gamma),d\mu)} = L$, then $||f||_{L^{\tilde{\Phi}}((\gamma,\beta),d\mu)} = (1-L^n)^{\frac{1}{n}}$. In the proof of Proposition 4.1 we need a similar property of the norm given by the Young function Φ.

LEMMA 4.4. *Let $\Phi(t) = t^n\phi(t)$ be a Young function satisfying $\phi(1)=1$, $\phi(t) \in [\frac{1}{2},1]$ on $(0,\infty)$ and (13). Then there is $\eta \in (0,1]$ with the following property:*

If $\alpha<\gamma<\beta$, $||f||_{L^{\Phi}((\alpha,\beta),d\mu)} \leqslant 1$ and $||f||_{L^{\Phi}((\alpha,\gamma),d\mu)} = L$, then $||f||_{L^{\Phi}((\gamma,\beta),d\mu)} \leqslant 1-\eta L^n$.

Proof. Since $||f||_{L^\Phi((\alpha,\beta),d\mu)} \leqslant 1$, by (5) we have

$$\int_\alpha^\beta f^n(x)\phi(f(x))\,d\mu(x) = \int_\alpha^\beta \Phi(f(x))\,d\mu(x) \leqslant \Phi(1) = 1\,. \tag{26}$$

From $||f||_{L^\Phi((\alpha,\gamma),d\mu)} = L > \tilde{L} := \frac{1}{2^{\frac{1}{n}}}L$ and (4) we obtain

$$\int_\alpha^\gamma \Big(\frac{f(x)}{\tilde{L}}\Big)^n \phi\Big(\frac{f(x)}{\tilde{L}}\Big)\,d\mu(x) = \int_\alpha^\gamma \Phi\Big(\frac{f(x)}{\tilde{L}}\Big)\,d\mu(x) > \Phi(1) = 1$$

and thus the assumption $\phi(t) \in [\frac{1}{2},1]$ implies

$$\int_\alpha^\gamma f^n(x)\phi(f(x))\,d\mu(x) \geqslant \int_\alpha^\gamma \tilde{L}^n\Big(\frac{f(x)}{\tilde{L}}\Big)^n\frac{1}{2}\phi\Big(\frac{f(x)}{\tilde{L}}\Big)\,d\mu(x) \geqslant \frac{1}{2}\tilde{L} = \frac{1}{4}L^n\,. \tag{27}$$

Therefore from (26) and (27) we observe

$$\int_\gamma^\beta f^n(x)\phi(f(x))\,d\mu(x) \leqslant 1 - \frac{1}{4}L^n\,. \tag{28}$$

Let us find $\eta = \eta(\omega,\tau,n) \in (0,\tau)$ (where ω and τ come from (13)) small enough so that for every $t \in [0,1]$ we have

$$\Big(\frac{1}{1-\eta t}\Big)^n\Big(\frac{1}{1-\omega\eta t}\Big) \leqslant \frac{1}{1-\frac{1}{4}t}\,. \tag{29}$$

Hence from (13), (28) and (29) we obtain

$$\begin{aligned}\int_\gamma^\beta \Phi\Big(\frac{f(x)}{1-\eta L^n}\Big)\,d\mu(x) &= \int_\gamma^\beta \Big(\frac{f(x)}{1-\eta L^n}\Big)^n\phi\Big(\frac{f(x)}{1-\eta L^n}\Big)\,d\mu(x)\\ &\leqslant \int_\gamma^\beta \Big(\frac{1}{1-\eta L^n}\Big)^n f^n(x)\Big(\frac{1}{1-\omega\eta L^n}\Big)\phi(f(x))\,d\mu(x)\\ &\leqslant \frac{1}{1-\frac{1}{4}L^n}\int_\gamma^\beta f^n(x)\phi(f(x))\,d\mu(x) \leqslant 1 = \Phi(1)\,. \qquad \square\end{aligned}$$

Now we can prove Proposition 4.1. The proof uses some ideas of Garsia [10] and also the technique used in the proof of Lemma 3.2.2 in [1].

Proof of Proposition 4.1. Let $A_0 \geqslant \max(A_1,A_2)$ (A_1 and A_2 are given by Lemmata 4.2 and 4.3) be large enough so that $(1-\log^{-a}(t)) > \frac{1}{2}$ for $t \geqslant A_0$. Further we can suppose that $\Phi(t) \leqslant t^n$ on $[0,\infty)$ (the smaller Φ is the worse). Hence $\phi(t) = \frac{\Phi(t)}{t^n} \in [\frac{1}{2},1]$ on $(0,\infty)$ and we can use Lemma 4.4.

As $A_0 \geqslant \max(A_1,A_2)$, Lemma 4.2 and Lemma 4.3 give us estimate (23) for any couple $0 < t < s < \infty$.

Since the $C_0^\infty(\Omega)$ functions are dense in $W_0L^\Phi(\Omega)$ (by the definition) we can suppose without loss of generality that f is Lipschitz continuous. Find $R > 0$ such that

$\mathscr{L}_n(\Omega) = \mathscr{L}_n(B(0,R))$. From the basic properties of the radially symmetric rearrangement we obtain

$$\int_\Omega \exp\Big(\Big(\frac{f(x)}{K}\Big)^{\frac{n}{n-1}}\Big)\,dx = \int_{B(0,R)} \exp\Big(\Big(\frac{f^\#(x)}{K}\Big)^{\frac{n}{n-1}}\Big)\,dx$$

and the Polya-Szegö principle (Theorem 2.1) gives us

$$\int_{B(0,R)} \Phi(|\nabla f^\#(x)|)\,dx \leqslant \int_\Omega \Phi(|\nabla f(x)|)\,dx \leqslant 1\,. \tag{30}$$

Hence we can suppose without loss of generality that $f(x) = g(|x|)$, g is non-increasing, classically differentiable almost everywhere and moreover $\Omega = B(0,R)$.

Since $f \in W_0L^\Phi(\Omega)$ we have $g(R) = 0$. Put $d\mu(y) = \omega_{n-1}y^{n-1}dy$. From the assumption $\int_{B(0,R)} \Phi(|\nabla f(x)|)\,dx \leqslant 1$, we obtain

$$\begin{aligned}\int_0^R \Phi_0(|g'(y)|)\,d\mu &= \int_0^R \frac{1}{n}\Phi(|g'(y)|)\omega_{n-1}y^{n-1}\,dy = \frac{1}{n}\int_{B(0,R)} \Phi(|\nabla f(x)|)\,dx \\ &\leqslant \frac{1}{n}\Phi(1) = \frac{1}{n} = \Phi_0(1)\end{aligned}$$

and thus

$$||g'(y)||_{L^{\Phi_0}((0,R),d\mu)} \leqslant 1\,. \tag{31}$$

Further (8) (recall $\Phi_0(1) + \Psi_0(1) = \frac{1}{n} + \frac{n-1}{n} = 1$) and (23) give us for $0 < t \leqslant s \leqslant R$ that

$$\begin{aligned} g(t) - g(s) &\leqslant \int_t^s |g'(y)|\,dy = \int_{y\in(t,s)} |g'(y)|\frac{1}{\omega_{n-1}y^{n-1}}\,d\mu(y) \\ &\leqslant \frac{1}{\omega_{n-1}}||g'(y)||_{L^{\Phi_0}((t,s),d\mu)}\Big|\Big|\frac{1}{y^{n-1}}\Big|\Big|_{L^{\Psi_0}((t,s),d\mu)} \\ &\leqslant \omega_{n-1}^{-\frac{1}{n}}||g'(y)||_{L^{\Phi_0}((t,s),d\mu)}\Big(\log\Big(\frac{s}{t}\Big)+1\Big)^{\frac{n-1}{n}}\,. \end{aligned} \tag{32}$$

Since $g(R) = 0$, estimate (32) in the case $s = R$ reads

$$g(t) \leqslant \omega_{n-1}^{-\frac{1}{n}}||g'(y)||_{L^{\Phi_0}((t,R),d\mu)}\Big(\log\Big(\frac{R}{t}\Big)+1\Big)^{\frac{n-1}{n}}\,. \tag{33}$$

Set

$$G(z) = \Big(\frac{g(Re^{-\frac{z}{n}})}{K}\Big)^{\frac{n}{n-1}} - z\,. \tag{34}$$

The substitution $y = Re^{-\frac{z}{n}}$ gives us

$$\begin{aligned}\int_{B(0,R)} \exp\Big(\Big(\frac{f(x)}{K}\Big)^{\frac{n}{n-1}}\Big)\,dx &= \omega_{n-1}\int_0^R \exp\Big(\Big(\frac{g(y)}{K}\Big)^{\frac{n}{n-1}}\Big)y^{n-1}\,dy \\ &= C\int_0^\infty \exp(G(z))\,dz = C\int_0^\infty\int_{-G(z)}^\infty e^{-\lambda}\,d\lambda\,dz \\ &= C\int_{-\infty}^\infty |E_\lambda|e^{-\lambda}\,d\lambda \end{aligned} \tag{35}$$

where

$$E_\lambda = \left\{ z \in (0,\infty) :\ -G(z) < \lambda \right\} .$$

We want to estimate $|E_\lambda|$ for every $\lambda \in \mathbb{R}$. For $z \in (0,\infty)$ let us set

$$L(z) = ||g'(y)||_{L^{\Phi_0}((0,Re^{-\frac{z}{n}}),d\mu)} .$$

From (31) we see that we always have $0 \leqslant L(z) \leqslant 1$.

If $z \in E_\lambda$, then from $K = n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}}$, (33) and Lemma 4.4 together with (6) and (31) we obtain

$$\begin{aligned}
z - \lambda &\leqslant \left(\frac{g(Re^{-\frac{z}{n}})}{K} \right)^{\frac{n}{n-1}} \\
&\leqslant \left(\frac{\omega_{n-1}^{-\frac{1}{n}}}{K} \left(\log\left(e^{\frac{z}{n}}\right) + 1 \right)^{\frac{n-1}{n}} ||g'(y)||_{L^{\Phi_0}((Re^{-\frac{z}{n}},R),d\mu)} \right)^{\frac{n}{n-1}} \\
&\leqslant \left(n^{\frac{n-1}{n}} \left(\frac{z}{n} + 1 \right)^{\frac{n-1}{n}} \right)^{\frac{n}{n-1}} \left(1 - \eta L^n(z) \right)^{\frac{n}{n-1}} \\
&\leqslant (z+n) \left(1 - \eta L^n(z) \right) \\
&\leqslant z - \eta z L^n(z) + n .
\end{aligned}$$

And thus

$$\lambda \geqslant \eta z L^n(z) - n .$$

This immediately implies

$$E_\lambda = \emptyset \qquad \text{whenever} \quad \lambda < -n \tag{36}$$

and

$$z L^n(z) \leqslant C_1 \lambda + C_2 \qquad \text{for} \quad \lambda \geqslant -n, z \in E_\lambda . \tag{37}$$

We want to show that $|E_\lambda|$ is small also for $\lambda \geqslant -n$. Suppose $z_1, z_2 \in E_\lambda$, $z_1 \geqslant n$ and $z_2 - z_1 \geqslant n$. Let us set $\delta = (z_2 - z_1)^{\frac{n-1}{n}}$. Using $K = n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}}$, (6), (32), (33) and $\frac{z_2 - z_1}{n} + 1 \leqslant 2(z_2 - z_1) = 2\delta^{\frac{n}{n-1}}$ we obtain

$$\begin{aligned}
z_1 + \delta^{\frac{n}{n-1}} - \lambda = z_2 - \lambda &\leqslant \left(\frac{g(Re^{-\frac{z_2}{n}})}{K} \right)^{\frac{n}{n-1}} \\
&\leqslant \left(\frac{g(Re^{-\frac{z_1}{n}})}{K} + \frac{g(Re^{-\frac{z_2}{n}}) - g(Re^{-\frac{z_1}{n}})}{K} \right)^{\frac{n}{n-1}} \\
&\leqslant \Bigg(n^{\frac{n-1}{n}} \left(\log\left(e^{\frac{z_1}{n}}\right) + 1 \right)^{\frac{n-1}{n}} ||g'(y)||_{L^{\Phi_0}((Re^{-\frac{z_1}{n}},R),d\mu)} \\
&\qquad + n^{\frac{n-1}{n}} \left(\log\left(e^{\frac{z_2}{n} - \frac{z_1}{n}}\right) + 1 \right)^{\frac{n-1}{n}} ||g'(y)||_{L^{\Phi_0}((Re^{-\frac{z_2}{n}},Re^{-\frac{z_1}{n}}),d\mu)} \Bigg)^{\frac{n}{n-1}} \\
&\leqslant \left(n^{\frac{n-1}{n}} \left(\frac{z_1}{n} + 1 \right)^{\frac{n-1}{n}} + n^{\frac{n-1}{n}} \left(\frac{z_2 - z_1}{n} + 1 \right)^{\frac{n-1}{n}} L(z_1) \right)^{\frac{n}{n-1}} \\
&\leqslant \left((z_1 + n)^{\frac{n-1}{n}} + 2^{\frac{n-1}{n}} \delta L(z_1) \right)^{\frac{n}{n-1}} .
\end{aligned}$$

Further the inequality

$$(\alpha+\beta)^{1+\theta} \leqslant \alpha^{1+\theta} + \beta \max_{x\in[0,\beta]} \frac{d}{dx}\Big((\alpha+x)^{1+\theta}\Big)$$
$$= \alpha^{1+\theta} + \beta(1+\theta)(\alpha+\beta)^{\theta} \leqslant \alpha^{1+\theta} + (1+\theta)2^{\theta}(\alpha^{\theta}\beta + \beta^{1+\theta}) ,$$

which is satisfied for $\alpha, \beta, \theta \geqslant 0$, implies

$$z_1 + \delta^{\frac{n}{n-1}} - \lambda \leqslant z_1 + n + C_3\Big((z_1+n)^{\frac{1}{n}}\delta L(z_1) + \delta^{\frac{n}{n-1}} L^{\frac{n}{n-1}}(z_1)\Big)$$

hence using (37) and the estimate $(z_1+n)^{\frac{1}{n}} \leqslant 2z_1^{\frac{1}{n}}$ (recall $z_1 \geqslant n$) we obtain

$$\delta^{\frac{n}{n-1}} \leqslant \lambda + n + 2C_3(C_1\lambda + C_2)^{\frac{1}{n}}\delta + C_3 L^{\frac{n}{n-1}}(z_1)\delta^{\frac{n}{n-1}} . \tag{38}$$

Further the Young inequality

$$\alpha\beta \leqslant \frac{\varepsilon^n}{n}\alpha^n + \frac{1}{\frac{n}{n-1}\varepsilon^{\frac{n}{n-1}}}\beta^{\frac{n}{n-1}}$$

implies (with suitable $\varepsilon > 0$)

$$2C_3(C_1\lambda + C_2)^{\frac{1}{n}}\delta \leqslant C_4\lambda + C_5 + \frac{1}{3}\delta^{\frac{n}{n-1}} . \tag{39}$$

From (37) we see that there are $C_6, C_7 > 0$ such that if $z_1 \geqslant C_6\lambda + C_7$, then

$$C_3 L^{\frac{n-1}{n}}(z_1) \leqslant \frac{1}{3} . \tag{40}$$

Further we can assume that $C_6\lambda + C_7 \geqslant n$ for any $\lambda \geqslant -n$. Therefore from (38), (39) and (40) we obtain for $z_1 \geqslant C_6\lambda + C_7$ and $z_2 \geqslant z_1 + n$

$$\frac{1}{3}(z_2 - z_1) = \frac{1}{3}\delta^{\frac{n}{n-1}} \leqslant \lambda + n + C_4\lambda + C_5 \leqslant C + C\lambda .$$

Hence we see that $|E_\lambda| = |E_\lambda \cap (-n, C_6\lambda + C_7)| + |E_\lambda \cap [C_6\lambda + C_7, \infty)| \leqslant C + C\lambda$. Using this estimate and (35) we conclude the proof with

$$\int_\Omega \exp\Big(\Big(\frac{f(x)}{K}\Big)^{\frac{n}{n-1}}\Big)\,dx \leqslant C\int_{-\infty}^{\infty} |E_\lambda| e^{-\lambda}\,d\lambda \leqslant \int_{-n}^{\infty} (C + C\lambda)e^{-\lambda}\,d\lambda \leqslant C . \qquad \square$$

In the following lemma we show that there is a suitable Young function satisfying assumptions of Proposition 4.1. After this we use such a Young function to remove assumption (13) from Proposition 4.1 and we obtain Theorem 1.1.

LEMMA 4.5. *For every $a > 1$ there are $\tilde{A}_0 > 1$ and $\tilde{A}_1 > \tilde{A}_0$ such that*

$$\tilde{\Phi}(t) = \begin{cases} t^n & t \in [0, \tilde{A}_0] \\ \tilde{A}_0^n + \int_{\tilde{A}_0}^t \max\Big(n\tilde{A}_0^{n-1}, \frac{d}{ds}\Big(s^n\Big(1 - \log^{-\frac{a+1}{2}}(s)\Big)\Big)\Big)\,ds & t \in [\tilde{A}_0, \tilde{A}_1] \\ t^n\Big(1 - \log^{-a}(t)\Big) & t \in [\tilde{A}_1, \infty) \end{cases} \tag{41}$$

is a Young function satisfying assumptions of Proposition 4.1.

Proof. If $\tilde{A}_0 > A_0$ (A_0 is given by Proposition 4.1) is sufficiently large, then there is $\tilde{A}_1 > \tilde{A}_0$ such that Φ is continuous. It is not difficult to see that $\tilde{\Phi}$ is also increasing and convex provided $\tilde{A}_0$ is large enough. Property (14) follows from (41).

It remains to check condition (13). In the sequel, we say that (13) is satisfied on the interval $I \subset (0,\infty)$ if the inequality in (13) is satisfied for couples $t, \frac{t}{1-\delta} \in I$.

Step 1. (Properties of condition (13) restricted to an interval)

First, we claim that if a nonnegative nondecreasing function Φ satisfies (13) on I then also $\Phi_C := \Phi + C,\ C \geqslant 0$, satisfies (13) on I (with the same parameters ω and τ). Indeed, (13) applied to Φ on I implies (recall that we denote $\phi(t) = \frac{\Phi(t)}{t^n}$)

$$\frac{\Phi(\frac{t}{1-\delta})}{\Phi(t)}(1-\delta)^n = \frac{\Phi(\frac{t}{1-\delta})}{\Phi(t)}\frac{t^n}{(\frac{t}{1-\delta})^n} = \frac{\phi(\frac{t}{1-\delta})}{\phi(t)} \leqslant \frac{1}{1-\omega\delta}\,.$$

Hence, if we define $\phi_C(t) := \frac{\Phi_C(t)}{t^n}$, $t \in (0,\infty)$, then we conclude

$$\begin{aligned}\frac{\phi_C(\frac{t}{1-\delta})}{\phi_C(t)} &= \frac{\Phi_C(\frac{t}{1-\delta})}{\Phi_C(t)}(1-\delta)^n = \frac{\Phi(\frac{t}{1-\delta})+C}{\Phi(t)+C}(1-\delta)^n \\ &= \Big[\frac{\Phi(\frac{t}{1-\delta})}{\Phi(t)} + \frac{C\Phi(t)-C\Phi(\frac{t}{1-\delta})}{\Phi(t)(\Phi(t)+C)}\Big](1-\delta)^n \leqslant \frac{\Phi(\frac{t}{1-\delta})}{\Phi(t)}(1-\delta)^n \leqslant \frac{1}{1-\omega\delta}\,.\end{aligned}$$

The second claim is that if $0 \leqslant \alpha < \beta < \gamma \leqslant \infty$ and Φ satisfies (13) on $(\alpha,\beta]$ (with the parameters ω_1 and τ_1) and on $[\beta,\gamma)$ (with ω_2 and τ_2), then Φ satisfies (13) on (α,γ) (with $\omega = \omega_1+\omega_2$, $\tau = \frac{1}{2}\min(\tau_1,\tau_2)$). Let us prove this claim. If either $t, \frac{t}{1-\delta} \in (\alpha,\beta]$ or $t, \frac{t}{1-\delta} \in [\beta,\gamma)$, then the proof is obvious. Therefore let us suppose that $t \in (\alpha,\beta)$ and $\frac{t}{1-\delta} \in (\beta,\gamma)$. Hence (13) on $(\alpha,\beta]$ and on $[\beta,\gamma)$ gives

$$\frac{\phi(\frac{t}{1-\delta})}{\phi(t)} = \frac{\phi(\frac{t}{1-\delta})}{\phi(\beta)}\frac{\phi(\beta)}{\phi(t)} \leqslant \frac{1}{1-\omega_2\delta}\frac{1}{1-\omega_1\delta} \leqslant \frac{1}{1-(\omega_1+\omega_2)\delta}\,.$$

Step 2. (We check (13) for each part of the function $\tilde{\Phi}$)

First, on $(0,\tilde{A}_0]$ we have $\tilde{\phi}(t) := \frac{\tilde{\Phi}(t)}{t^n} = \frac{t^n}{t^n} \equiv 1$. Hence (13) is satisfied on $(0,\tilde{A}_0]$.

Next we claim that for every $b > 0$ there is $B \geqslant 1$ such that the function $\Phi_b(t) = t^n\phi_b(t) = t^n\Big(1-\log^{-b}(t)\Big)$ satisfies (13) on $[B,\infty)$. To show this, let $B > 1$ be large enough so that $\log^{-b}(B) \leqslant \frac{1}{2}$ and $\frac{\log(1-\delta)}{\log(B)} \geqslant -\delta$ for every $\delta \in (0,\frac{1}{2}]$. Then there is

$\omega > 0$ such that for every $\delta \in (0, \frac{1}{2}]$ and $t \geqslant B$ we have

$$\begin{aligned}
\frac{\phi_b(\frac{t}{1-\delta})}{\phi_b(t)} &= \frac{1-\log^{-b}(\frac{t}{1-\delta})}{1-\log^{-b}(t)} \\
&= 1 + \frac{\log^{-b}(t) - \log^{-b}(\frac{t}{1-\delta})}{1-\log^{-b}(t)} \\
&= 1 + \frac{\log^{-b}(t)}{1-\log^{-b}(t)}\Big(1 - \frac{\log^{-b}(\frac{t}{1-\delta})}{\log^{-b}(t)}\Big) \\
&= 1 + \frac{\log^{-b}(t)}{1-\log^{-b}(t)}\Big(1 - \Big(\frac{\log(t)-\log(1-\delta)}{\log(t)}\Big)^{-b}\Big) \\
&= 1 + \frac{\log^{-b}(t)}{1-\log^{-b}(t)}\Big(1 - \Big(1 - \frac{\log(1-\delta)}{\log(t)}\Big)^{-b}\Big) \\
&\leqslant 1 + \frac{\frac{1}{2}}{\frac{1}{2}}\Big(1 - (1-(-\delta))^{-b}\Big) \\
&= 1 + 1 - (1+\delta)^{-b} = \frac{2(1+\delta)^b - 1}{(1+\delta)^b} \\
&\leqslant 2(1+\delta)^b - 1 \leqslant 1 + \omega\delta \leqslant \frac{1}{1-\omega\delta} \ .
\end{aligned}$$

Hence we are also done on $[\tilde{A}_1, \infty)$ provided $\tilde{A}_1$ is sufficiently large (take $\tilde{A}_0$ large).

From (41) we see that there is $\tilde{A}_2 \in [\tilde{A}_0, \tilde{A}_1]$ such that $\Phi(t) = \tilde{A}_0^n + n\tilde{A}_0^{n-1}(t - \tilde{A}_0)$ on $[\tilde{A}_0, \tilde{A}_2]$ and $\Phi(t) = t^n(1 - \log^{-\frac{a+1}{2}}(t)) + C$ on $[\tilde{A}_2, \tilde{A}_1]$, with $C > 0$. Therefore the first claim from Step 1. and the above claim concerning the function $t^n(1 - \log^{-b}(t))$, $b := \frac{a+1}{2}$, imply that we are also done on $[\tilde{A}_2, \tilde{A}_1]$.

Finally, as the function $t \mapsto \frac{\tilde{A}_0 + n(\frac{t}{1-\delta} - \tilde{A}_0)}{\tilde{A}_0 + n(t - \tilde{A}_0)}$ is decreasing on $[\tilde{A}_0, \infty)$ (the derivative is $\frac{\tilde{A}_0 \delta n(1-n)}{(1-\delta)(\tilde{A}_0 + n(t-\tilde{A}_0))^2}$), it attains its maximum at $\tilde{A}_0$. Hence on $[\tilde{A}_0, \tilde{A}_2]$ we conclude

$$\begin{aligned}
\frac{\tilde{\phi}(\frac{t}{1-\delta})}{\tilde{\phi}(t)} &= \frac{\tilde{\Phi}(\frac{t}{1-\delta})}{\tilde{\Phi}(t)}(1-\delta)^n \\
&= \frac{\tilde{A}_0^n + n\tilde{A}_0^{n-1}(\frac{t}{1-\delta} - \tilde{A}_0)}{\tilde{A}_0^n + n\tilde{A}_0^{n-1}(t - \tilde{A}_0)}(1-\delta)^n \\
&\leqslant \frac{\tilde{A}_0 + n(\frac{t}{1-\delta} - \tilde{A}_0)}{\tilde{A}_0 + n(t - \tilde{A}_0)} \\
&\leqslant \frac{\tilde{A}_0 + n(\frac{\tilde{A}_0}{1-\delta} - \tilde{A}_0)}{\tilde{A}_0} \\
&= \frac{1 + (n-1)\delta}{1-\delta} \leqslant \frac{1}{1-n\delta} \ .
\end{aligned}$$

Step 3. (We prove (13))

Since all four parts of the function $\tilde{\Phi}$ satisfy (13) on the corresponding intervals by Step 2., applying three times the second claim from Step 1. we obtain that $\tilde{\Phi}$ satisfies (13) on $(0,\infty)$. □

Proof of Theorem 1.1. Pick $\tilde{\Phi}$ from Lemma 4.5 and set $A = \tilde{A}_1$. If a Young function Φ satisfies assumptions of Theorem 1.1, then we have $\Phi(t) \geqslant \tilde{\Phi}(t)$ on $[0,\infty)$. Hence every function $f \in W_0L^{\Phi}(\Omega)$ such that $\int_\Omega \Phi(|\nabla f(x)|)\,dx \leqslant 1$ satisfies also $\int_\Omega \tilde{\Phi}(|\nabla f(x)|)\,dx \leqslant 1$ and Proposition 4.1 together with Lemma 4.5 conclude the proof. □

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