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HILBERT–TYPE INEQUALITIES AND RELATED OPERATORS WITH HOMOGENEOUS KERNEL OF DEGREE 0

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Abstract. In this paper we provide an unified approach to the Hilbert-type inequalities with homogeneous kernel of degree 0 and certain weight functions. As an application, we define the related Hilbert-type operators and analyze their norms. In the case of conjugate exponents, we obtain the best possible constants involved in the right-hand sides of derived inequalities, and norms of the Hilbert-type operators as well. Finally, we consider some special choices of homogeneous kernels and parameters.

1. Introduction

Let p and q be conjugate exponents, that is, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. One of the earliest versions of the classical Hilbert inequality, that holds for all non-negative functions $f\in L^p(\mathbb{R}_+)$ and $g\in L^q(\mathbb{R}_+)$, is

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \leqslant \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible in the sense that it can not be replaced with a smaller constant (see [10]).

In 1934, Hardy et. al. (see [11]) extended (1) to the case of non-homogeneous kernel $k_1(x,y)$ of degree -1,

$$\int_0^\infty\int_0^\infty k_1(x,y)f(x)g(y)dxdy \leqslant c_1(p)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)}, \tag{2}$$

where $c_1(p)=\int_0^\infty k_1(u,1)u^{-\frac{1}{p}}du\in\mathbb{R}$ and p,q are conjugate exponents with $p>1$. Recall that homogeneous function $k_\alpha:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ has degree $-\alpha$, $\alpha>0$, if $k_\alpha(tx,ty)=t^{-\alpha}k_\alpha(x,y)$ for all $t,x,y\in\mathbb{R}_+$.

The Hilbert inequality is very important in mathematical analysis and its applications and, although classical, is still a field of interest of numerous mathematicians. During decades, it was generalized in many different directions, such as different choices

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of kernels, sets of integration etc. One of the most interesting problems concerning the Hilbert inequality is the problem of the best possible constant factors. That problem will be investigated in this paper.

In paper [2], Y. Bicheng obtained generalization of (1) considering the kernel $\frac{1}{x^s+y^s}$, $s>0$, and introducing two pairs of conjugate parameters. Also, Y. Bicheng established in [4] extension of (2) by introducing the non-homogeneous kernel. In both papers, the best possible constants are obtained.

On the other hand, M. Krnić and J. Pečarić provided in [12] an unified treatment of the Hilbert-type inequalities. As an application, they obtained appropriate results for homogeneous kernels of negative degree and power weights. Further, M. Krnić et. al. obtained in [13] the inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^l} dxdy \leqslant B(1-A_2p, l-1+A_2p) \times \\ \times \left[\int_0^\infty x^{pqA_1-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty x^{pqA_2-1} g^q(y)dy\right]^{\frac{1}{q}}, \quad (3)$$

with the best possible constant factor on the right-hand side of the inequality, where $l>0$, $A_1 \in \left\langle \frac{1}{q}, \frac{1-l}{q} \right\rangle$, $A_2 \in \left\langle \frac{1-l}{p}, \frac{1}{p} \right\rangle$, $pA_2+qA_1=2-l$, and $B(\cdot,\cdot)$ is the usual Beta function. Moreover, M. Krnić obtained in [15] the best constant factors expressed in terms of Gaussian hypergeometric functions.

Very recently, Y. Bicheng obtained in [7] the best possible constants for arbitrary homogeneous kernel of negative degree and some special power weights. Namely, he considered non-negative homogeneous kernel $k_l : \mathbb{R}_+^2 \to \mathbb{R}$ of degree $-l$, $l>0$, two pairs p,q and r,s of conjugate exponents, and power weights $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}$ defined by $\varphi(x) = x^{p\left(1-\frac{l}{r}\right)-1}$, $\psi(y) = y^{q\left(1-\frac{l}{s}\right)-1}$. He obtained the Hilbert-type inequality

$$\int_0^\infty \int_0^\infty k_l(x,y)f(x)g(y)\,dxdy \leqslant c_l(r)\|f\|_{L^p_\varphi(\mathbb{R}_+)}\|g\|_{L^q_\psi(\mathbb{R}_+)}, \quad (4)$$

where $c_l(r) = \int_0^\infty k_l(t,1)t^{\frac{l}{r}-1}dt$ and $f,g : \mathbb{R}_+ \to \mathbb{R}$ are non-negative measurable functions with respect to the norm

$$\|f\|_{L^r_\Omega\langle a,b\rangle} = \left[\int_a^b \Omega(x)f^r(x)\,dx\right]^{\frac{1}{r}}, \quad (5)$$

for $r>0$ and non-negative measurable weight function $\Omega : \langle a,b\rangle \to \mathbb{R}_+$ Moreover, the related Hilbert-type operator $\mathscr{T}_l : L^p_\varphi(\mathbb{R}_+) \mapsto L^p_{\psi^{1-p}}(\mathbb{R}_+)$, defined by $(\mathscr{T}_l f)(y) = \int_0^\infty k_l(x,y)f(x)\,dx$ is also studied. Since $c_l(r)$ is the best possible constant in (4), one obtains that the norm of the Hilbert-type operator $\mathscr{T}_l$ is $\|\mathscr{T}_l\| = c_l(r)$.

The best possible constant factors were also studied in extensions of the Hilbert inequality to a multidimensional case. See, for example, papers [1], [3] and [14].

In 2007, Y. Bicheng obtained in [5] the best possible constant factor for the homogeneous kernel of degree 0 and non-negative measurable functions $f,g : \mathbb{R}_+ \to \mathbb{R}$:

$$\int_0^\infty \int_0^\infty \frac{\min\{x,y\}}{\max\{x,y\}} f(x)g(y)\,dxdy \leqslant 2\left[\int_0^\infty xf^2(x)\,dx\right]^{\frac{1}{2}} \left[\int_0^\infty yg^2(y)\,dy\right]^{\frac{1}{2}}. \quad (6)$$

The main objective of this paper is an investigation of the Hilbert-type inequalities with homogeneous kernels of degree 0. We provide an unified treatment to such inequalities and also obtain the best possible constant factors in some general cases. As an application, we define the related Hilbert-type operators and analyze their norms. Further, we regard the Hilbert-type inequalities in more general manner, that is, in the setting of non-conjugate exponents. Basic definitions and results about non-conjugate exponents will be introduced in the next section.

Techniques that will be used in the proofs are mainly based on the classical real analysis. Also, throughout this paper we suppose that all functions are non-negative and measurable, so that all integrals converge.

2. Non-conjugate exponents

Suppose p and q are real parameters, such that

$$p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}\geqslant 1, \tag{7}$$

and let $p'=\frac{p}{p-1}$ and $q'=\frac{q}{q-1}$ respectively be their conjugate exponents, that is, $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Further, define

$$\lambda=\frac{1}{p'}+\frac{1}{q'} \tag{8}$$

and observe that $0<\lambda\leqslant 1$ holds for all p and q as in (7). In particular, equality $\lambda=1$ holds in (8) if and only if $q=p'$, that is, only if p and q are mutually conjugate. Otherwise, we have $0<\lambda<1$, and such parameters p and q will be referred to as non-conjugate exponents.

Considering p, q, and λ as in (7) and (8), Hardy, Littlewood, and Pólya, [11], obtained an extension of (1). However, the original proof did not bring any information about the value of the constant included in the right-hand side of the inequality. That drawback was improved by Levin, [17], who obtained an explicit upper bound for the constant, that, in conjugate case, reduces to the previously known sharp constant $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$. A simpler proof of that extension, based on a single application of the Hölder inequality, was given later by F. F. Bonsall, [8].

Very recently, A. Čižmešija et. al. developed in paper [9] an unified treatment of the general Hilbert-type inequalities extended to the case of non-conjugate exponents. Their main result, based on one Bonsall's idea from paper [8], is the following.

THEOREM 1. *Let real parameters p, q, and λ be as in* (7) *and* (8), *and let X and Y be measure spaces with positive σ-finite measures μ_1 and μ_2 respectively. Let K be a non-negative measurable function on $X\times Y$, φ a measurable, a.e. positive function on X, and ψ a measurable, a.e. positive function on Y. If the functions F on X and G on Y are defined by*

$$F(x)=\left[\int_Y K(x,y)\psi^{-q'}(y)\,d\mu_2(y)\right]^{\frac{1}{q'}},\ \ x\in X, \tag{9}$$

and

$$G(y)=\left[\int_X K(x,y)\varphi^{-p'}(x)\,d\mu_1(x)\right]^{\frac{1}{p'}},\ \ y\in Y, \tag{10}$$

then for all non-negative measurable functions f on X and g on Y the inequalities

$$\int_X\int_Y K^{\lambda}(x,y)f(x)g(y)\,d\mu_1(x)d\mu_2(y)\leqslant \|\varphi Ff\|_{L^p(\mu_1)}\|\psi Gg\|_{L^q(\mu_2)} \tag{11}$$

and

$$\left\{\int_Y\left[(\psi G)^{-1}(y)\int_X K^{\lambda}(x,y)f(x)\,d\mu_1(x)\right]^{q'}d\mu_2(y)\right\}^{\frac{1}{q'}}\leqslant \|\varphi Ff\|_{L^p(\mu_1)} \tag{12}$$

hold and are equivalent.

Clearly, inequalities (11) and (12) are further generalizations of the classical Hilbert and Hardy inequality. So, inequalities deduced from (11) will be referred to as the Hilbert-type inequalities and inequalities deduced from (12), as the Hardy-Hilbert type inequalities.

Further, in paper [9] it was shown that necessary and sufficient conditions for equalities in (11) and (12) are described with relations

$$f=\alpha\varphi^{-p'}F^{\frac{q'}{p}-1}\ \text{ a.e. on } X \text{ and } g=\beta\psi^{-q'}G^{\frac{p'}{q}-1}\ \text{ a.e. on } Y, \tag{13}$$

and

$$K=\gamma F^{q'}G^{p'}\ \text{ a.e. on } X\times Y, \tag{14}$$

where α,β and γ are some positive real constants.

The same authors also considered reversed inequalities in (11) and (12), these are the inequalities with reversed sign of inequality. They obtained that reverse inequalities in (11) and (12) hold if

$$p<0,\ q\in\langle 0,1\rangle,\ \frac{1}{p}+\frac{1}{q}\leqslant 1, \tag{15}$$

or

$$p\in\langle 0,1\rangle,\ q<0,\ \frac{1}{p}+\frac{1}{q}\leqslant 1, \tag{16}$$

since the crucial step in proving Theorem 1 was in applying the Hölder inequality, and parameters given by (15) or (16) provide the so-called reversed Hölder inequality (for details, see e.g. [18, Chapter V]). Note that in conjugate case, reverse inequalities in (11) and (12) are achieved if $0<p<1$ and $q<0$.

We proceed with an application of Theorem 1 to homogeneous kernels of degree 0 and certain power weights. All results will be given in two equivalent forms in non-conjugate setting, analogous to (11) and (12). However, the best possible constants will be obtained only in the conjugate case, since described problem in non-conjugate case seems very difficult and remains open.

3. Two kinds of the Hilbert-type operators

In this section we provide an unified treatment of the Hilbert and Hardy-Hilbert type inequalities with homogeneous kernels of degree 0 and power weights. We consider two types of kernels deduced from mentioned homogeneous functions. As an application, we define the corresponding Hilbert-type operators. Before we state and prove our results, we have to establish some definitions.

Let $\langle a,b\rangle \subseteq \mathbb{R}$ be an interval and let $u,v:\langle a,b\rangle \to \mathbb{R}$ be non-negative, measurable functions satisfying following conditions:

(i) u and v are differentiable on $\langle a,b\rangle$;

(ii) u and v are strictly increasing on $\langle a,b\rangle$;

(iii) $\lim_{x\to a^+} u(t) = v(t) = 0$ and $\lim_{x\to b^-} u(t) = v(t) = \infty$.

Further, we suppose that $k_0 : \mathbb{R}_+^2 \mapsto \mathbb{R}$ is non-negative, measurable homogeneous function of degree 0. Also, we consider the integral

$$c_0(\alpha) = \int_0^\infty k_0(1,t)t^{-\alpha}dt, \tag{17}$$

defined in terms of the function k_0. Clearly, the convergence of integral (17) depends on the function k_0, so we consider only the parameters α such that (17) converges.

Now, we define the kernel by means of the function k_0. More precisely, let $K_0 : \langle a,b\rangle \times \langle a,b\rangle \mapsto \mathbb{R}$ be non-negative measurable function defined by

$$K_0(x,y) = k_0\left(u(x), v(y)\right), \tag{18}$$

where functions u and v fulfill conditions (i)-(iii).

Above definitions and notations will be valid through the whole paper. Our first result, in non-conjugate form, reads:

THEOREM 2. *Let real parameters p, q, and λ be as in* (7) *and* (8) *and let u and v be non-negative measurable functions on $\langle a,b\rangle$, fulfilling conditions* (i), (ii) *and* (iii). *Suppose $K_0 : \langle a,b\rangle \times \langle a,b\rangle \mapsto \mathbb{R}$ is non-negative measurable function defined by* (18) *and A_1, A_2 are real parameters such that $c_0(2-p'A_1) < \infty$ and $c_0(q'A_2) < \infty$. Then the inequalities*

$$\int_a^b \int_a^b K_0^\lambda(x,y) f(x) g(y)\, dx dy \leqslant c_0^{\frac{1}{p'}}(2-p'A_1) c_0^{\frac{1}{q'}}(q'A_2) \times$$
$$\times \left[\int_a^b \frac{u^{(A_1-A_2)p+\frac{p}{q'}}(x)}{(u')^{p-1}(x)} f^p(x)\, dx\right]^{\frac{1}{p}} \left[\int_a^b \frac{v^{(A_2-A_1)q+\frac{q}{p'}}(y)}{(v')^{q-1}(y)} g^q(y)\, dy\right]^{\frac{1}{q}} \tag{19}$$

and

$$\left\{\int_a^b \frac{v'(y)}{v^{(A_2-A_1)q'+\frac{q'}{p'}}(y)}\left[\int_a^b K_0^{\lambda}(x,y)f(x)\,dx\right]^{q'}dy\right\}^{\frac{1}{q'}}$$
$$\leqslant c_0^{\frac{1}{p'}}(2-p'A_1)c_0^{\frac{1}{q'}}(q'A_2)\left[\int_a^b \frac{u^{(A_1-A_2)p+\frac{p}{q'}}(x)}{(u')^{p-1}(x)}f^p(x)\,dx\right]^{\frac{1}{p}} \tag{20}$$

hold for all non-negative measurable functions f *and* g *on* $\langle a,b\rangle$ *and are equivalent. Equalities in* (19) *and* (20) *hold if and only if* $f=0$ *or* $g=0$ *a.e. on* $\langle a,b\rangle$.

Proof. It is easy to see that this theorem is a consequence of Theorem 1. We substitute the kernel K_0, defined by (18), in inequalities (11) and (12) and

$$\varphi(x)=\frac{u^{A_1}(x)}{(u')^{\frac{1}{p'}}(x)},\qquad \psi(y)=\frac{v^{A_2}(y)}{(v')^{\frac{1}{q'}}(y)},\qquad x\in\langle a,b\rangle, \tag{21}$$

instead of weight functions φ and ψ as well. Then considering $v(y)=tu(x)$, using the variable substitution theorem, and homogeneity of the function k_0, we get

$$F(x)=\left[\int_a^b k_0(u(x),v(y))v^{-q'A_2}(y)v'(y)dy\right]^{\frac{1}{q'}}$$
$$=u^{\frac{1}{q'}-A_2}(x)\left[\int_0^{\infty}k_0(1,t)t^{-q'A_2}dt\right]^{\frac{1}{q'}}=c_0^{\frac{1}{q'}}(q'A_2)u^{\frac{1}{q'}-A_2}(x),\quad x\in\langle a,b\rangle.$$

Using the same argument, we also obtain

$$G(y)=c_0^{\frac{1}{p'}}(2-p'A_1)v^{\frac{1}{p'}-A_1}(x),\quad y\in\langle a,b\rangle,$$

which gives relations (19) and (20).

It remains to investigate the case of equality in obtained inequalities. According to conditions (13) and (14), necessary condition for equality in (19) and (20) is that the function f has the form

$$f(x)=\alpha c_0^{1-\lambda}(q'A_2)u^{-A_1p'+(1-\lambda)(1-A_2q')}(x)u'(x),\quad \langle a,b\rangle, \tag{22}$$

for some positive real constant α. Now, by inserting (22) in (20), the right-hand side of (20) reads

$$\alpha^p c_0(q'A_2)\int_0^{\infty}t^{1-p'A_1-q'A_2}dt,$$

where we used the substitution $u(x)=t$. Hence, we came to a contradiction since obtained integral diverges. Thus, the equality in (19) and (20) holds only if f or g are zero-functions a.e. on $\langle a,b\rangle$. □

We continue with some interesting examples of functions u and v, fulfilling conditions (i)-(iii). Obviously, functions $u,v:\mathbb{R}_+\mapsto\mathbb{R}$, defined by $u(x)=Ax^{\mu}$, $v(y)=By^{\nu}$, where $A,B,\mu,\nu > 0$, satisfy mentioned conditions. So, as a direct consequence of Theorem 2, we obtain the following result.

COROLLARY 1. *Suppose p, q, and λ are as in* (7) *and* (8) *and $k_0 : \mathbb{R}_+^2 \mapsto \mathbb{R}$ is non-negative measurable homogeneous function of degree 0. Let A_1, A_2 be real parameters such that $c_0(2-p'A_1) < \infty$ and $c_0(q'A_2) < \infty$, and let $A,B,\mu,\nu > 0$. Then the inequalities*

$$\int_0^\infty \int_0^\infty k_0^\lambda(Ax^\mu, By^\nu) f(x)g(y)\,dxdy \leqslant c_0^{\frac{1}{p'}}(2-p'A_1)c_0^{\frac{1}{q'}}(q'A_2)C_0 \left[\int_0^\infty x^{\left(A_1-A_2+\frac{1}{q'}\right)p\mu+(p-1)(1-\mu)} f^p(x)\,dx\right]^{\frac{1}{p}} \times \left[\int_0^\infty y^{\left(A_2-A_1+\frac{1}{p'}\right)q\nu+(q-1)(1-\nu)} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{23}$$

and

$$\left\{\int_0^\infty y^{\left(A_1-A_2-\frac{1}{p'}\right)q'\nu+\nu-1} \left[\int_0^\infty k_0^\lambda(Ax^\mu, By^\nu) f(x)\,dx\right]^{q'} dy\right\}^{\frac{1}{q'}} \leqslant c_0^{\frac{1}{p'}}(2-p'A_1)c_0^{\frac{1}{q'}}(q'A_2)C_0 \left[\int_0^\infty x^{\left(A_1-A_2+\frac{1}{q'}\right)p\mu+(p-1)(1-\mu)} f^p(x)\,dx\right]^{\frac{1}{p}}, \tag{24}$$

where

$$C_0 = \mu^{-\frac{1}{p'}}\nu^{-\frac{1}{q'}} A^{A_1-A_2+\frac{1}{q'}-\frac{1}{p'}} B^{A_2-A_1+\frac{1}{p'}-\frac{1}{q'}},$$

hold for all non-negative measurable functions f and g on $\mathbb{R}_+$ and are equivalent. Equalities in (23) *and* (24) *hold if and only if $f = 0$ or $g = 0$ a.e. on $\mathbb{R}_+$.*

REMARK 1. Corollary 1 is obtained in paper [9] (see relations (47) and (48)). So, we can regard Theorem 2 as a generalization of mentioned results from [9]. □

Another interesting application of Theorem 2 appears if we consider the functions $u,v : \mathbb{R} \mapsto \mathbb{R}$, defined by $u(x) = e^x$ and $v(y) = e^y$.

COROLLARY 2. *Let real parameters p, q, and λ be as in* (7) *and* (8) *and let $k_0 : \mathbb{R}_+^2 \mapsto \mathbb{R}$ be non-negative measurable homogeneous function of degree 0. Suppose A_1 and A_2 are real parameters such that $c_0(2-p'A_1) < \infty$ and $c_0(q'A_2) < \infty$. Then the inequalities*

$$\int_{-\infty}^\infty \int_{-\infty}^\infty k_0^\lambda(e^x, e^y) f(x)g(y)\,dxdy \leqslant c_0^{\frac{1}{p'}}(2-p'A_1)c_0^{\frac{1}{q'}}(q'A_2) \times \left[\int_{-\infty}^\infty e^{\left[(A_1-A_2)p+1-\frac{p}{q}\right]x} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^\infty e^{\left[(A_2-A_1)q+1-\frac{q}{p}\right]y} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{25}$$

and

$$\left\{\int_{-\infty}^{\infty} e^{\left[(A_1-A_2)q'+1-\frac{q'}{p'}\right]y}\left[\int_{-\infty}^{\infty} k_0^{\lambda}(e^x,e^y)f(x)\,dx\right]^{q'} dy\right\}^{\frac{1}{q'}}$$
$$\leqslant c_0^{\frac{1}{p'}}(2-p'A_1)c_0^{\frac{1}{q'}}(q'A_2)\left[\int_{-\infty}^{\infty} e^{\left[(A_1-A_2)p+1-\frac{p}{q}\right]x} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{26}$$

hold for all non-negative measurable functions f and g on $\mathbb{R}$ and are equivalent. Equalities in (25) *and* (26) *hold if and only if $f=0$ or $g=0$ a.e. on $\mathbb{R}$.*

Now, we consider another interesting kernel depending on homogeneous function $k_0:\mathbb{R}_+^2\mapsto\mathbb{R}$ of degree 0. Namely, let $\widetilde{K}_0:\langle a,b\rangle\times\langle a,b\rangle\mapsto\mathbb{R}$ be non-negative measurable function defined by

$$\widetilde{K}_0(x,y)=k_0\left(1,u(x)v(y)\right), \tag{27}$$

where the functions u and v fulfill conditions (i)-(iii). The following result is an analogue of Theorem 2.

THEOREM 3. *Let real parameters p, q, and λ be as in* (7) *and* (8) *and let u and v be non-negative measurable functions on $\langle a,b\rangle$, fulfilling conditions* (i), (ii) *and* (iii). *Suppose $\widetilde{K}_0:\langle a,b\rangle\times\langle a,b\rangle\mapsto\mathbb{R}$ is non-negative measurable function defined by* (27) *and A_1, A_2 are real parameters such that $c_0(p'A_1)<\infty$ and $c_0(q'A_2)<\infty$. Then the inequalities*

$$\int_a^b\int_a^b \widetilde{K}_0^{\lambda}(x,y)f(x)g(y)\,dxdy \leqslant c_0^{\frac{1}{p'}}(p'A_1)c_0^{\frac{1}{q'}}(q'A_2)\times$$
$$\times\left[\int_a^b \frac{u^{(A_1+A_2)p-\frac{p}{q'}}(x)}{(u')^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_a^b \frac{v^{(A_1+A_2)q-\frac{q}{p'}}(y)}{(v')^{q-1}(y)} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{28}$$

and

$$\left\{\int_a^b \frac{v'(y)}{v^{(A_1+A_2)q'-\frac{q'}{p'}}(y)}\left[\int_a^b \widetilde{K}_0^{\lambda}(x,y)f(x)\,dx\right]^{q'} dy\right\}^{\frac{1}{q'}}$$
$$\leqslant c_0^{\frac{1}{p'}}(p'A_1)c_0^{\frac{1}{q'}}(q'A_2)\left[\int_a^b \frac{u^{(A_1+A_2)p-\frac{p}{q'}}(x)}{(u')^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{29}$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$ and are equivalent. Equalities in (28) *and* (29) *hold if and only if $f=0$ or $g=0$ a. e. on $\langle a,b\rangle$.*

Proof. We use Theorem 1 and follow the same lines as in the proof of Theorem 2. Considering the weight functions defined by (21), we get

$$F(x)=c_0^{\frac{1}{q'}}(q'A_2)u^{A_2-\frac{1}{q'}}(x),\quad\text{and}\quad G(y)=c_0^{\frac{1}{p'}}(p'A_1)v^{A_1-\frac{1}{p'}}(x),\ x,y\in\langle a,b\rangle,$$

which yields inequalities (28) and (29). The equality in obtained inequalities is established in the same way as in proof of Theorem 2. □

Of course, it is easy to obtain analogues of Corollaries 1 and 2, with the kernel $\widetilde{K}_0$ instead of K_0. Corresponding results are given in the sequel.

COROLLARY 3. *Let p, q, and λ be as in* (7) *and* (8) *and let $k_0 : \mathbb{R}_+^2 \mapsto \mathbb{R}$ be non-negative measurable homogeneous function of degree* 0. *Let A_1, A_2 be real parameters such that $c_0(p'A_1) < \infty$ and $c_0(q'A_2) < \infty$, and let $A, \mu, \nu > 0$. Then the inequalities*

$$\int_0^\infty \int_0^\infty k_0^\lambda(1, Ax^\mu y^\nu) f(x) g(y)\, dx dy \\ \leqslant c_0^{\frac{1}{p'}}(p'A_1) c_0^{\frac{1}{q'}}(q'A_2) \widetilde{C}_0 \left[\int_0^\infty x^{\left(A_1+A_2-\frac{1}{q'}\right)p\mu+(p-1)(1-\mu)} f^p(x)\, dx\right]^{\frac{1}{p}} \times \\ \times \left[\int_0^\infty y^{\left(A_1+A_2-\frac{1}{p'}\right)q\nu+(q-1)(1-\nu)} g^q(y)\, dy\right]^{\frac{1}{q}} \tag{30}$$

and

$$\left\{\int_0^\infty y^{\left(-A_1-A_2+\frac{1}{p'}\right)q'\nu+\nu-1} \left[\int_0^\infty k_0^\lambda(1, Ax^\mu y^\nu) f(x)\, dx\right]^{q'} dy\right\}^{\frac{1}{q'}} \\ \leqslant c_0^{\frac{1}{p'}}(p'A_1) c_0^{\frac{1}{q'}}(q'A_2) \widetilde{C}_0 \left[\int_0^\infty x^{\left(A_1+A_2-\frac{1}{q'}\right)p\mu+(p-1)(1-\mu)} f^p(x)\, dx\right]^{\frac{1}{p}}, \tag{31}$$

where

$$\widetilde{C}_0 = \mu^{-\frac{1}{p'}} \nu^{-\frac{1}{q'}} A^{A_1+A_2-\lambda},$$

hold for all non-negative measurable functions f and g on $\mathbb{R}_+$ and are equivalent. Equalities in (30) *and* (31) *hold if and only if $f = 0$ or $g = 0$ a. e. on $\mathbb{R}_+$.*

COROLLARY 4. *Suppose p, q, and λ are as in* (7) *and* (8) *and $k_0 : \mathbb{R}_+^2 \mapsto \mathbb{R}$ is non-negative measurable homogeneous function of degree* 0. *Further, suppose A_1 and A_2 are real parameters such that $c_0(p'A_1) < \infty$ and $c_0(q'A_2) < \infty$. Then the inequalities*

$$\int_{-\infty}^\infty \int_{-\infty}^\infty k_0^\lambda(1, e^{x+y}) f(x) g(y)\, dx dy \leqslant c_0^{\frac{1}{p'}}(p'A_1) c_0^{\frac{1}{q'}}(q'A_2) \times \\ \times \left[\int_{-\infty}^\infty e^{\left[(A_1+A_2)p+1-2p+\frac{p}{q}\right]x} f^p(x)\, dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^\infty e^{\left[(A_1+A_2)q+1-2q+\frac{q}{p}\right]y} g^q(y)\, dy\right]^{\frac{1}{q}} \tag{32}$$

and

$$\left\{\int_{-\infty}^\infty e^{\left[1-(A_1+A_2)q'+\frac{q'}{p'}\right]y} \left[\int_{-\infty}^\infty k_0^\lambda(1, e^{x+y}) f(x)\, dx\right]^{q'} dy\right\}^{\frac{1}{q'}} \\ \leqslant c_0^{\frac{1}{p'}}(p'A_1) c_0^{\frac{1}{q'}}(q'A_2) \left[\int_{-\infty}^\infty e^{\left[(A_1+A_2)p+1-2p+\frac{p}{q}\right]x} f^p(x)\, dx\right]^{\frac{1}{p}} \tag{33}$$

hold for all non-negative measurable functions f *and* g *on* $\mathbb{R}$ *and are equivalent. Equalities in* (32) *and* (33) *hold if and only if* $f=0$ *or* $g=0$ *a.e on* $\mathbb{R}$.

REMARK 2. According to the discussion in Section 2, if non-conjugate exponents p and q satisfy conditions (15) or (16), then the inequality sign in all relations from Theorems 2, 3 and Corollaries 1, 2, 3, 4 is reversed.

The Hardy-Hilbert type inequalities (20) and (29) allow us precise definition of the Hilbert-type operators and some conclusions about their norms as well. We observe the measure space

$$L^r_\Omega\langle a,b\rangle=\left\{f:\langle a,b\rangle\mapsto\mathbb{R};\ \|f\|_{L^r_\Omega\langle a,b\rangle}<\infty\right\},$$

where the norm $\|\cdot\|_{L^r_\Omega\langle a,b\rangle}$ is defined by (5).

Hence, if we denote

$$\Phi(x)=\frac{u^{(A_1-A_2)p+\frac{p}{q'}}(x)}{(u')^{p-1}(x)}\ \text{ and }\ \Psi(y)=\frac{v^{(A_2-A_1)q+\frac{q}{p'}}(y)}{(v')^{q-1}(y)},\quad x,y\in\langle a,b\rangle,$$

we can define the operator $\mathscr{T}_0:L^p_\Phi\langle a,b\rangle\mapsto L^{q'}_{\Psi^{1-q'}}\langle a,b\rangle$ as

$$(\mathscr{T}_0f)(y)=\int_a^b K_0^\lambda(x,y)f(x)\,dx,\quad y\in\langle a,b\rangle. \tag{34}$$

Of course, we suppose p, q and λ are as in (7) and (8). Clearly, the operator $\mathscr{T}_0$ is well defined since from inequality (20) we conclude that $\mathscr{T}_0f\in L^{q'}_{\Psi^{1-q'}}\langle a,b\rangle$. Further, we define the norm of the operator $\mathscr{T}_0$ as

$$\|\mathscr{T}_0\|=\sup_{f\in L^p_\Phi\langle a,b\rangle,f\neq 0}\frac{\|\mathscr{T}_0f\|_{L^{q'}_{\Psi^{1-q'}}\langle a,b\rangle}}{\|f\|_{L^p_\Phi\langle a,b\rangle}}. \tag{35}$$

We conclude, from inequality (20), that the operator $\mathscr{T}_0$ is bounded. More precisely, (20) yields upper bound for the norm of the operator:

$$\|\mathscr{T}_0\|\leqslant c_0^{\frac{1}{p'}}(2-p'A_1)c_0^{\frac{1}{q'}}(q'A_2).$$

The same type of discussion can also be applied in obtaining the Hilbert-type operator related to kernel $\widetilde{K}_0$, defined by (27). More precisely, if we denote

$$\widetilde{\Phi}(x)=\frac{u^{(A_1+A_2)p-\frac{p}{q'}}(x)}{(u')^{p-1}(x)}\ \text{ and }\ \widetilde{\Psi}(y)=\frac{v^{(A_1+A_2)q-\frac{q}{p'}}(y)}{(v')^{q-1}(y)},\qquad x,y\in\langle a,b\rangle,$$

we can define the operator $\widetilde{\mathscr{T}_0}:L^p_{\widetilde{\Phi}}\langle a,b\rangle\mapsto L^{q'}_{\widetilde{\Psi}^{1-q'}}\langle a,b\rangle$ as

$$\left(\widetilde{\mathscr{T}_0}f\right)(y)=\int_a^b \widetilde{K}_0^\lambda(x,y)f(x)\,dx,\quad y\in\langle a,b\rangle. \tag{36}$$

The operator $\widetilde{\mathscr{T}_0}$ is well defined since inequality (29) implies $\widetilde{\mathscr{T}_0}f \in L^{q'}_{\widetilde{\Psi}^{1-q'}}\langle a,b\rangle$. Since the norm of the operator $\widetilde{\mathscr{T}_0}$ is given by

$$\|\widetilde{\mathscr{T}_0}\| = \sup_{f\in L^p_{\widetilde{\Phi}}\langle a,b\rangle, f\neq 0} \frac{\|\widetilde{\mathscr{T}_0}f\|_{L^{q'}_{\widetilde{\Psi}^{1-q'}}\langle a,b\rangle}}{\|f\|_{L^p_{\widetilde{\Phi}}\langle a,b\rangle}}, \tag{37}$$

inequality (29) yields upper bound for the norm of the operator $\widetilde{\mathscr{T}_0}$:

$$\|\widetilde{\mathscr{T}_0}\| \leqslant c_0^{\frac{1}{p'}}(p'A_1)c_0^{\frac{1}{q'}}(q'A_2).$$

In the next section we shall consider some general cases in which we are able to find the norm of operators $\widetilde{T}$ and $\widetilde{T_0}$. Obviously, that problem is equivalent to the problem of the best possible constant factors in inequalities (20) and (29).

4. On the best possible constant factors and norms

As already mentioned, the problem of the best possible constant factors in the Hilbert and Hardy-Hilbert type inequalities seems to be very difficult in the case of non-conjugate exponents and remains still open.

In spite of that, we can solve mentioned problem in some general settings with conjugate exponents. Hence, in this section parameters p and q are assumed to be conjugate, that is, when $p=q'$, $q=p'$ and $\lambda=1$. We consider constant factors involved in the right-hand sides of inequalities from Theorems 2 and 3. Of course, it is enough to investigate the problem for the Hilbert-type inequalities since we have pairs of equivalent inequalities.

The main idea in obtaining the best possible constant factors is a reduction of the constant in the form without exponents, by appropriate choice of real parameters A_1 and A_2. More precisely, in the conjugate case, the constant involved in the right-hand sides of inequalities (19) and (20) reads $c_0^{\frac{1}{q}}(2-qA_1)c_0^{\frac{1}{p}}(pA_2)$, hence it is natural to set the condition

$$pA_2+qA_1=2. \tag{38}$$

Thus, if parameters A_1 and A_2 satisfy (38), the previous constant factor reduce to $c_0(pA_2)$. Moreover, if the constraint (38) is satisfied, inequalities (19) and (20) from Theorem 2 become respectively (in conjugate case)

$$\int_a^b\int_a^b K_0(x,y)f(x)g(y)\,dxdy \leqslant c_0(pA_2)\left[\int_a^b \frac{u^{pqA_1-1}(x)}{(u')^{p-1}(x)}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_a^b \frac{v^{pqA_2-1}}{(v')^{q-1}(y)}g^q(y)\,dy\right]^{\frac{1}{q}} \tag{39}$$

and

$$\left\{\int_a^b \frac{v'(y)}{v^{-pqA_1+p+1}(y)}\left[\int_a^b K_0(x,y)f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}} \leqslant c_0(pA_2)\left[\int_a^b \frac{u^{pqA_1-1}(x)}{(u')^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}}. \tag{40}$$

Our next step is to show that the constant $c_0(pA_2)$ is the best possible in both inequalities (39) and (40). For that sake, we need to establish the following lemma.

LEMMA 1. *Suppose that p and q are conjugate parameters. Let functions K_0,u,v be as in the statement of Theorem 2 and let $u(c)=1, v(d)=1,\ c,d\in\langle a,b\rangle$. Let parameters A_1,A_2 fulfill conditions as in the statement of Theorem 2 and let $qA_1+pA_2=2$. Then the relation*

$$\varepsilon\int_c^b \frac{u'(x)}{u^{qA_1+\frac{\varepsilon}{p}}(x)}\left[\int_d^b \frac{K_0(x,y)v'(y)}{v^{pA_2+\frac{\varepsilon}{q}}(y)}\,dy\right]dx = c_0(pA_2)+o(1) \tag{41}$$

hold for $\varepsilon\searrow 0$.

Proof. Let us denote the left-hand side of relation (41) with I_ε. By using substitution $\widetilde{x}=u(x)$ and $\widetilde{y}=v(y)$ we have

$$I_\varepsilon=\varepsilon\int_1^\infty \widetilde{x}^{-qA_1-\frac{\varepsilon}{p}}\left[\int_1^\infty k_0(\widetilde{x},\widetilde{y})\widetilde{y}^{-pA_2-\frac{\varepsilon}{q}}d\widetilde{y}\right]d\widetilde{x}.$$

Further, considering substitution $\widetilde{y}=\widetilde{x}t$, homogeneity of the function k_0 and the condition $qA_1+pA_2=2$, we obtain the expression

$$I_\varepsilon=\varepsilon\int_1^\infty \widetilde{x}^{-1-\varepsilon}\left[\int_{\frac{1}{\widetilde{x}}}^\infty k_0(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\right]d\widetilde{x}.$$

If we separate the inner integral in I_ε into two integrals and apply the Fubini theorem, we have

$$\begin{aligned}
I_\varepsilon &= \varepsilon\int_1^\infty \widetilde{x}^{-1-\varepsilon}\left[\int_{\frac{1}{\widetilde{x}}}^1 k_0(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt+\int_1^\infty k_0(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\right]d\widetilde{x}\\
&= \varepsilon\int_1^\infty \widetilde{x}^{-1-\varepsilon}\left[\int_{\frac{1}{\widetilde{x}}}^1 k_0(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\right]d\widetilde{x}+\int_1^\infty k_0(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\\
&= \varepsilon\int_0^1 k_0(1,t)t^{-pA_2-\frac{\varepsilon}{q}}\left[\int_{\frac{1}{t}}^\infty \widetilde{x}^{-1-\varepsilon}d\widetilde{x}\right]dt+\int_0^1 k_0(t,1)t^{-qA_1+\frac{\varepsilon}{q}}dt\\
&= \int_0^1 k_0(1,t)t^{-pA_2+\frac{\varepsilon}{p}}dt+\int_0^1 k_0(t,1)t^{-qA_1+\frac{\varepsilon}{q}}dt.
\end{aligned} \tag{42}$$

Now, we have to distinguish two cases, depending on signs of conjugate parameters p and q. If $p>1$, then $q>1$, so $\frac{\varepsilon}{p}>0$ and $\frac{\varepsilon}{q}>0$. Hence, relation (42) yields

$$I_\varepsilon \leqslant \int_0^1 k_0(1,t)t^{-pA_2}dt + \int_0^1 k_0(t,1)t^{-qA_1}dt = \int_0^\infty k_0(1,t)t^{-pA_2}dt = c_0(pA_2).$$

Finally, by the Lebesgue control convergent theorem (see [16]) we obtain (41).

It remains to consider the case when $0<p<1$ and $q<0$. Then for $\varepsilon>0$, there exist $\sigma>0$ such that $\varepsilon \leqslant -q\sigma$. Namely, we can choose $\sigma=\frac{2\varepsilon}{-q}$. Thus, we can estimate I_ε in the following way:

$$\begin{aligned} I_\varepsilon &\leqslant \int_0^1 k_0(1,t)t^{-pA_2+\frac{\varepsilon}{q}}dt + \int_0^1 k_0(t,1)t^{-qA_1+\frac{\varepsilon}{q}}dt \\ &\leqslant \int_0^1 k_0(1,t)t^{-pA_2-\sigma}dt + \int_0^1 k_0(t,1)t^{-qA_1-\sigma}dt. \end{aligned}$$

Clearly, $\varepsilon \searrow 0$ implies $\sigma \searrow 0$ and again, by the Lebesgue control convergent theorem we obtain (41). □

Now, we are ready to state and prove the main result, concerning the best possible constant factors in inequalities (39) and (40).

THEOREM 4. *Let p and q be conjugate exponents, let A_1 and A_2 be real parameters such that $qA_1+pA_2=2$, and let $c_0(pA_2)<\infty$. Then the constant factor $c_0(pA_2)$ is the best possible in both inequalities* (39) *and* (40).

Proof. We consider two cases depending on weather $p>1$ or $0<p<1$.

If $p>1$ then $q>1$. Suppose that the constant factor $c_0(pA_2)$ is not the best possible in inequality (39). That means that there exist the constant factor $k_0<c_0(pA_2)$ such that inequality (39) holds if we replace $c_0(pA_2)$ with k_0. For $\varepsilon>0$, we substitute functions

$$f_\varepsilon(x) = \begin{cases} 0, & x \in \langle a,c\rangle, \\ \frac{u'(x)}{u^{qA_1+\frac{\varepsilon}{p}}(x)}, & x\in [c,b\rangle, \end{cases}$$

and

$$g_\varepsilon(y) = \begin{cases} 0, & x \in \langle a,d\rangle, \\ \frac{v'(y)}{v^{pA_2+\frac{\varepsilon}{q}}(y)}, & x\in [d,b\rangle, \end{cases}$$

where $c,d\in\langle a,b\rangle$, $u(c)=1$, $v(d)=1$, in inequality (39). We get

$$\int_a^b\int_a^b K_0(x,y)f_\varepsilon(x)g_\varepsilon(y)\,dxdy \leqslant k_0\left[\int_c^b u^{-1-\varepsilon}(x)u'(x)dx\right]^{\frac{1}{p}}\left[\int_d^b v^{-1-\varepsilon}(y)v'(y)dy\right]^{\frac{1}{q}},$$

and further

$$\varepsilon\int_a^b\int_a^b K_0(x,y)f_\varepsilon(x)g_\varepsilon(y)\,dxdy \leqslant k_0. \tag{43}$$

The left-hand side of the inequality (43) coincides with the left-hand side of relation (41), so by Lemma 1 we have

$$c_0(pA_2)+o(1)\leqslant k_0.$$

Now by letting $\varepsilon\searrow 0$ we obtain that $c_0(pA_2)\leqslant k_0$ which contradicts with the assumption $k_0<c_0(pA_2)$. Thus, the constant $c_0(pA_2)$ is the best possible.

It remains to consider the case when $0<p<1$, that is, the case of the reverse inequality in (39). Suppose that $c_0(pA_2)$ is not the best possible constant. That means that there exist the constant factor $k_0'>c_0(pA_2)$ such that the reverse inequality in (39) holds if we replace $c_0(pA_2)$ with k_0'. Now, for above choice of functions f_ε and g_ε, and with the use of Lemma 1 we obtain

$$c_0(pA_2)+o(1)\geqslant k_0',$$

and also $c_0(pA_2)\geqslant k_0'$, when $\varepsilon\searrow 0$. Hence, we get a contradiction and $c_0(pA_2)$ is the best possible in the reverse case as well.

Finally, since (39) and (40) is a pair of equivalent inequalities, it follows that $c_0(pA_2)$ is also the best possible constant factor in (40). □

According to discussion in Section 3, the previous result also yields the norm of the operator $\mathscr{T}_0$, defined by (34), in the case of conjugate exponents.

COROLLARY 5. *Let p and q be conjugate exponents, let A_1 and A_2 be real parameters such that $qA_1+pA_2=2$, and let $c_0(pA_2)<\infty$. Then, the norm of the operator $\mathscr{T}_0: L^p_\Phi\langle a,b\rangle\mapsto L^p_{\Psi^{1-p}}\langle a,b\rangle$, defined by $(\mathscr{T}_0 f)(y)=\int_a^b K_0(x,y)f(x)\,dx$, $y\in\langle a,b\rangle$, is $\|\mathscr{T}_0\|=c_0(pA_2)$.*

REMARK 3. It is very interesting that we can deduce inequalities (3) and (4) from (39). First, we show that inequality (39) implies (3). For that sake, we suppose that parameters A_1 and A_2 satisfy condition $pA_2+qA_1=2-l$, $l>0$, as in (3). Note that parameters $A_1+\frac{l}{q}$ and A_2 satisfy condition (38). Further, since $k_0(x,y)=(x+y)^{-l}x^l$ is homogeneous function of degree 0, we have

$$c_0(pA_2)=\int_0^\infty(1+t)^{-l}t^{-pA_2}dt=B(1-A_2p,l-1+A_2p),$$

where $B(\cdot,\cdot)$ is the usual Beta function. Hence, if we substitute $u(x)=x$, $v(y)=y$, $K_0(x,y)=(x+y)^{-l}x^l$ in (39), and replace A_1 and $f(x)$ respectively with $A_1+\frac{l}{q}$ and $x^{-l}f(x)$, we obtain (3).

To obtain (4), we suppose that $k_l(x,y)$ is homogeneous function of degree $-l$ and r,s is a pair of conjugate exponents. Then, $k_0(x,y)=k_l(x,y)x^{\frac{l}{r}}y^{\frac{l}{s}}$ is homogeneous function of degree 0. Now, if we substitute $u(x)=x$, $v(y)=y$, $A_1=\frac{1}{q}$, $A_2=\frac{1}{p}$, $K_0(x,y)=k_l(x,y)x^{\frac{l}{r}}y^{\frac{l}{s}}$ in (39), and replace $f(x)$ and $g(y)$ respectively with $x^{-\frac{l}{r}}f(x)$ and $y^{-\frac{l}{s}}g(y)$, we obtain (4) since

$$c_0(pA_2)=c_0(1)=\int_0^\infty k_l(1,t)t^{\frac{l}{s}-1}dt=\int_0^\infty k_l(u,1)u^{\frac{l}{r}-1}du.$$

The previous discussion about the best possible constant factors can also be extended to Theorem 3. Namely, the constant involved in the right-hand sides of inequalities (28) and (29) becomes $c_0^{\frac{1}{q}}(qA_1)c_0^{\frac{1}{p}}(pA_2)$ in the conjugate case, so it is natural to set the condition

$$pA_2 = qA_1. \tag{44}$$

Hence, in this setting the constant becomes $c_0(pA_2)$ and inequalities (28), (29) respectively read

$$\int_a^b \int_a^b \widetilde{K}_0(x,y)f(x)g(y)\,dxdy \leqslant c_0(pA_2)\left[\int_a^b \frac{u^{pqA_1-1}(x)}{(u')^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_a^b \frac{v^{pqA_2-1}}{(v')^{q-1}(y)} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{45}$$

and

$$\left\{\int_a^b \frac{v'(y)}{v^{pqA_1-p+1}(y)}\left[\int_a^b \widetilde{K}_0(x,y)f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}} \leqslant c_0(pA_2)\left[\int_a^b \frac{u^{pqA_1-1}(x)}{(u')^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{46}$$

The crucial step in proving that the constant $c_0(pA_2)$ in (45) and (46) can not be replaced with the smaller one, is the following lemma.

LEMMA 2. *Let p and q be conjugate parameters, let functions $\widetilde{K}_0, u, v$ be as in the statement of Theorem 3 and let $u(c) = 1, v(d) = 1,\ c,d \in \langle a,b\rangle$. Suppose A_1, A_2 fulfill conditions as in the statement of Theorem 3 and $qA_1 = pA_2$. Then the relation*

$$\varepsilon \int_c^b \frac{u'(x)}{u^{qA_1+\frac{\varepsilon}{p}}(x)}\left[\int_a^d \frac{\widetilde{K}_0(x,y)v'(y)}{v^{pA_2-\frac{\varepsilon}{q}}(y)}\,dy\right] dx = c_0(pA_2) + \widetilde{o}(1) \tag{47}$$

hold for $\varepsilon \searrow 0$.

Proof. By using substitution $\widetilde{x} = u(x)$ and $\widetilde{y} = v(y)$ we have

$$\widetilde{I}_\varepsilon = \varepsilon \int_1^\infty \widetilde{x}^{-qA_1-\frac{\varepsilon}{p}}\left[\int_0^1 k_0(1,\widetilde{x}\widetilde{y})\widetilde{y}^{-pA_2+\frac{\varepsilon}{q}} d\widetilde{y}\right] d\widetilde{x},$$

where $\widetilde{I}_\varepsilon$ denotes the left-hand side of relation (47). Further, considering substitution $\widetilde{x}\widetilde{y} = t$, homogeneity of the function k_0 and the condition $qA_1 = pA_2$, we get

$$\widetilde{I}_\varepsilon = \varepsilon \int_1^\infty \widetilde{x}^{-1-\varepsilon}\left[\int_0^{\widetilde{x}} k_0(1,t)t^{-pA_2+\frac{\varepsilon}{q}} dt\right] d\widetilde{x}.$$

Now, if we separate the inner integral in $\widetilde{I}_\varepsilon$, into two integrals and apply the Fubini theorem, we have

$$\begin{aligned}
\widetilde{I}_\varepsilon &= \varepsilon\int_1^\infty \widetilde{x}^{-1-\varepsilon}\left[\int_0^1 k_0(1,t)t^{-pA_2+\frac{\varepsilon}{q}}dt + \int_1^{\widetilde{x}} k_0(1,t)t^{-pA_2+\frac{\varepsilon}{q}}dt\right]d\widetilde{x} \\
&= \int_0^1 k_0(1,t)t^{-pA_2+\frac{\varepsilon}{q}}dt + \varepsilon\int_1^\infty \widetilde{x}^{-1-\varepsilon}\left[\int_1^{\widetilde{x}} k_0(1,t)t^{-pA_2+\frac{\varepsilon}{q}}dt\right]d\widetilde{x} \\
&= \int_0^1 k_0(1,t)t^{-pA_2+\frac{\varepsilon}{q}}dt + \varepsilon\int_1^\infty k_0(1,t)t^{-pA_2+\frac{\varepsilon}{q}}\left[\int_t^\infty \widetilde{x}^{-1-\varepsilon}d\widetilde{x}\right]dt \\
&= \int_0^1 k_0(1,t)t^{-pA_2+\frac{\varepsilon}{q}}dt + \int_0^1 k_0(t,1)t^{-2+pA_2+\frac{\varepsilon}{p}}dt. \qquad (48)
\end{aligned}$$

Similarly as in Lemma 1, we have to distinguish two cases, depending on signs of conjugate parameters p and q. If $p>1$, then $q>1$, so $\frac{\varepsilon}{p}>0$ and $\frac{\varepsilon}{q}>0$. Hence, relation (48) yields

$$\begin{aligned}
\widetilde{I}_\varepsilon &\leqslant \int_0^1 k_0(1,t)t^{-pA_2}dt + \int_0^1 k_0(t,1)t^{-2+pA_2}dt \\
&= \int_0^1 k_0(1,t)t^{-pA_2}dt + \int_1^\infty k_0(1,t)t^{-pA_2}dt = c_0(pA_2).
\end{aligned}$$

Thus, according to the Lebesgue control convergent theorem (see [16]) we obtain (41).

Now we consider the case when $0<p<1$ and $q<0$. Then, for $\varepsilon>0$, there exist $\sigma>0$ such that $\varepsilon\leqslant -q\sigma$. Namely, if we choose $\sigma=\frac{2\varepsilon}{-q}$, we get

$$\begin{aligned}
\widetilde{I}_\varepsilon &\leqslant \int_0^1 k_0(1,t)t^{-pA_2+\frac{\varepsilon}{q}}dt + \int_0^1 k_0(t,1)t^{-2+pA_2+\frac{\varepsilon}{q}}dt \\
&\leqslant \int_0^1 k_0(1,t)t^{-pA_2-\sigma}dt + \int_0^1 k_0(t,1)t^{-2+pA_2-\sigma}dt.
\end{aligned}$$

Clearly, $\varepsilon\searrow 0$ implies $\sigma\searrow 0$ and again, by the Lebesgue control convergent theorem (see [16]) we obtain (47). The proof is now completed. □

Finally, by means of Lemma 2, we obtain the best possible constant factors in inequalities (45) and (46).

THEOREM 5. *Let p and q be conjugate exponents, let A_1 and A_2 be real parameters such that $qA_1=pA_2$, and let $c_0(pA_2)<\infty$. Then the constant factor $c_0(pA_2)$ is the best possible in both inequalities* (45) *and* (46).

Proof. We perform the proof similarly as the proof of Theorem 4, by means of Lemma 2. For $p>1$, we suppose that there exist the constant factor k_0, smaller than $c_0(pA_2)$ such that inequality (45) still holds if we replace $c_0(pA_2)$ with k_0. Now, we consider functions

$$\widetilde{f}_\varepsilon(x)=\begin{cases}0, & x\in\langle a,c\rangle,\\ \frac{u'(x)}{u^{qA_1+\frac{\varepsilon}{p}}(x)}, & x\in[c,b\rangle,\end{cases}$$

and

$$\widetilde{g}_\varepsilon(y) = \begin{cases} \frac{v'(y)}{v^{pA_2+\frac{\varepsilon}{q}}(y)}, & x \in \langle a,d\rangle, \\ 0, & x \in [d,b\rangle, \end{cases}$$

where $\varepsilon > 0$ and $c,d \in \langle a,b\rangle$, $u(c) = v(d) = 1$. Clearly, for defined functions f_ε and g_ε, inequality (45), with smaller constant factor k_0, reads

$$\int_a^b \int_a^b \widetilde{K}_0(x,y) f_\varepsilon(x) g_\varepsilon(y)\, dxdy \leqslant k_0 \left[\int_a^d u^{-1-\varepsilon}(x)u'(x)dx\right]^{\frac{1}{p}} \left[\int_d^b v^{-1+\varepsilon}(y)v'(y)dy\right]^{\frac{1}{q}},$$

that is,

$$\varepsilon \int_a^b \int_a^b \widetilde{K}_0(x,y) f_\varepsilon(x) g_\varepsilon(y)\, dxdy \leqslant k_0. \tag{49}$$

As the left-hand side of inequality (49) represent the left-hand side of relation (47), we have

$$c_0(pA_2) + \widetilde{o}(1) \leqslant k_0.$$

Thus, by letting $\varepsilon \searrow 0$ we get $c_0(pA_2) \leqslant k_0$ which contradicts with the assumption $k_0 < c_0(pA_2)$. Hence $c_0(pA_2)$ is the best possible constant factor in (45).

For the reverse inequality in (45) we suppose that there exist constant factor $k_0' > c_0(pA_2)$ such that reverse inequality in (45) holds if we replace $c_0(pA_2)$ with k_0'. Clearly, for above choice of functions f_ε and g_ε, with the use of Lemma 2, we obtain

$$c_0(pA_2) + \widetilde{o}(1) \geqslant k_0',$$

which implies $c_0(pA_2) \geqslant k_0'$ when $\varepsilon \searrow 0$. Obviously, we came to a contradiction since by assumption $k_0' > c_0(pA_2)$.

Of course, since (45) and (46) is a pair of equivalent inequalities, it follows that $c_0(pA_2)$ is also the best possible constant factor in (46). $\square$

We conclude the previous discussion with the operator analogue of Theorem 5.

COROLLARY 6. *Let p and q be conjugate exponents, let A_1 and A_2 be real parameters such that $pA_2 = qA_1$, and let $c_0(pA_2) < \infty$. Then the norm of the operator $\widetilde{\mathscr{T}_0} : L^p_{\widetilde{\Phi}}\langle a,b\rangle \mapsto L^p_{\widetilde{\Psi}^{1-p}}\langle a,b\rangle$, defined by $\big(\widetilde{\mathscr{T}_0} f\big)(y) = \int_a^b \widetilde{K}_0(x,y) f(x)\, dx$, $y \in \langle a,b\rangle$, is $\|\widetilde{\mathscr{T}_0}\| = c_0(pA_2)$.*

5. The Hardy-type operators

In Section 3 we have defined two kinds of the Hilbert-type operators. We can also generate some interesting Hardy-type operators from the Hilbert-type operators. More precisely, let $k_0 : \mathbb{R}^2_+ \mapsto \mathbb{R}$ be non-negative measurable homogeneous function of degree 0. Then the function $\overline{k}_0 : \mathbb{R}^2_+ \mapsto \mathbb{R}$, defined by

$$\overline{k}_0(x,y) = k_0(x,y)\chi_{x\geqslant y}(x,y) = \begin{cases} 0, & x < y, \\ k_0(x,y), & x \geqslant y, \end{cases} \tag{50}$$

is also homogeneous of degree 0. Now, we define two Hardy-type operators as the Hilbert-type operators $\mathscr{T}_0$ and $\widetilde{\mathscr{T}_0}$ with respect to the kernel $\overline{k}_0 : \mathbb{R}_+^2 \mapsto \mathbb{R}$. More precisely, we define the operator $\overline{\mathscr{T}}_0 : L^p_\Phi\langle a,b\rangle \mapsto L^{q'}_{\Psi^{1-q'}}\langle a,b\rangle$ as

$$\left(\overline{\mathscr{T}}_0 f\right)(y) = \int_{u^{(-1)}(v(y))}^{b} K_0^\lambda(x,y) f(x)\,dx, \quad y \in \langle a,b\rangle, \tag{51}$$

and $\overline{\widetilde{\mathscr{T}}}_0 : L^p_{\widetilde{\Phi}}\langle a,b\rangle \mapsto L^{q'}_{\widetilde{\Psi}^{1-q'}}\langle a,b\rangle$ as

$$\left(\overline{\widetilde{\mathscr{T}}}_0 f\right)(y) = \int_a^{u^{(-1)}\left(\frac{1}{v(y)}\right)} \widetilde{K}_0^\lambda(x,y) f(x)\,dx, \quad y \in \langle a,b\rangle, \tag{52}$$

where $u^{(-1)}$ is the inverse of the function u. Further, since

$$c_0(\alpha) = \int_0^\infty \overline{k}_0(1,t)t^{-\alpha}dt = \int_0^1 k_0(1,t)t^{-\alpha}dt,$$

we define

$$\overline{c}_0(\alpha) = \int_0^1 k_0(1,t)t^{-\alpha}dt. \tag{53}$$

Now, we easily obtain corresponding analogues of Theorems 2 and 3 for described Hardy-type kernels of degree 0.

COROLLARY 7. *Let p, q, and λ be as in* (7) *and* (8) *and let u, v be non-negative measurable functions on $\langle a,b\rangle$, fulfilling conditions* (i), (ii) *and* (iii). *Suppose $K_0 : \langle a,b\rangle \times \langle a,b\rangle \mapsto \mathbb{R}$ is non-negative measurable function defined by* (18) *and A_1, A_2 are real parameters such that $\overline{c}_0(2-p'A_1) < \infty$ and $\overline{c}_0(q'A_2) < \infty$. Then the inequalities*

$$\int_a^b \int_{u^{(-1)}(v(y))}^{b} K_0^\lambda(x,y) f(x) g(y)\,dxdy \leqslant \overline{c}_0^{\frac{1}{p'}}(2-p'A_1)\overline{c}_0^{\frac{1}{q'}}(q'A_2) \times$$
$$\times \left[\int_a^b \frac{u^{(A_1-A_2)p+\frac{p}{q'}}(x)}{(u')^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_a^b \frac{v^{(A_2-A_1)q+\frac{q}{p'}}(y)}{(v')^{q-1}(y)} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{54}$$

and

$$\left\{\int_a^b \frac{v'(y)}{v^{(A_2-A_1)q'+\frac{q'}{p'}}(y)} \left[\int_{u^{(-1)}(v(y))}^{b} K_0^\lambda(x,y) f(x)\,dx\right]^{q'} dy\right\}^{\frac{1}{q'}}$$
$$\leqslant \overline{c}_0^{\frac{1}{p'}}(2-p'A_1)\overline{c}_0^{\frac{1}{q'}}(q'A_2) \left[\int_a^b \frac{u^{(A_1-A_2)p+\frac{p}{q'}}(x)}{(u')^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{55}$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$ and are equivalent. Equalities in (54) *and* (55) *hold if and only if $f=0$ or $g=0$ a.e. on $\langle a,b\rangle$.*

COROLLARY 8. *Let p, q, and λ be as in* (7) *and* (8) *and let u, v be non-negative measurable functions on $\langle a,b\rangle$, fulfilling conditions* (i), (ii) *and* (iii). *Suppose $\widetilde{K}_0 : \langle a,b\rangle \times \langle a,b\rangle \mapsto \mathbb{R}$ is non-negative measurable function defined by* (27) *and A_1, A_2 are real parameters such that $\overline{c}_0(p'A_1) < \infty$ and $\overline{c}_0(q'A_2) < \infty$. Then the inequalities*

$$\int_a^b \int_a^{u^{(-1)}\left(\frac{1}{v(y)}\right)} \widetilde{K}_0^{\lambda}(x,y)f(x)g(y)\,dxdy \leqslant \overline{c}_0^{\frac{1}{p'}}(p'A_1)\overline{c}_0^{\frac{1}{q'}}(q'A_2)\times \times \left[\int_a^b \frac{u^{(A_1+A_2)p-\frac{p}{q'}}(x)}{(u')^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_a^b \frac{v^{(A_1+A_2)q-\frac{q}{p'}}(y)}{(v')^{q-1}(y)} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{56}$$

and

$$\left\{\int_a^b \frac{v'(y)}{v^{(A_1+A_2)q'-\frac{q'}{p'}}(y)} \left[\int_a^{u^{(-1)}\left(\frac{1}{v(y)}\right)} \widetilde{K}_0^{\lambda}(x,y)f(x)\,dx\right]^{q'} dy\right\}^{\frac{1}{q'}} \leqslant \overline{c}_0^{\frac{1}{p'}}(p'A_1)\overline{c}_0^{\frac{1}{q'}}(q'A_2)\left[\int_a^b \frac{u^{(A_1+A_2)p-\frac{p}{q'}}(x)}{(u')^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{57}$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$ and are equivalent. Equalities in (56) *and* (57) *hold if and only if $f=0$ or $g=0$ a.e. on $\langle a,b\rangle$.*

REMARK 4. According to Corollaries 7 and 8, we can easily obtain analogues of Corollaries 1, 2, 3 and 4. We just need to replace the constant factor $c_0(\alpha)$ with $\overline{c}_0(\alpha)$ and change the integration sets according to definitions (51) and (52). For example, if $u(x)=e^x$ and $v(y)=e^y$, then $u^{(-1)}(v(y))=y$ and $u^{(-1)}\left(\frac{1}{v(y)}\right)=-y$.

REMARK 5. The discussion about the best possible constant factors, carried out in Section 4, also holds for the Hardy-type operators (51) and (52). More precisely, if p, q are conjugate parameters and A_1, A_2 parameters satisfying condition (38), then $\overline{c}_0(pA_2)$ is the best possible constant factor in inequalities (54) and (55). Further, the same constant is also the best possible in (56) and (57), when parameters A_1 and A_2 fulfill condition (44) in conjugate case. Of course, under the same assumptions, we obtain norms of operators $\overline{\mathscr{T}}_0$ and $\overline{\widetilde{\mathscr{T}}}_0$ in the conjugate case: $\|\overline{\mathscr{T}}_0\| = \|\overline{\widetilde{\mathscr{T}}}_0\| = \overline{c}_0(pA_2)$.

6. Applications

This section is dedicated to some special choices of parameters A_1, A_2 and homogeneous kernels of degree 0 in obtained results. At first, we simplify inequalities in Theorem 2, Theorem 3, Corollary 7 and Corollary 8 by suitable choices of parameters A_1 and A_2. Namely, if $A_1 = \frac{1}{p'}$ and $A_2 = \frac{1}{q'}$, then the inequalities (19) and (20) become respectively

$$\int_a^b \int_a^b k_0^\lambda(u(x),v(y))f(x)g(y)\,dxdy \leqslant c_0^\lambda(1)\left[\int_a^b \left(\frac{u}{u'}\right)^{p-1}(x)f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_a^b \left(\frac{v}{v'}\right)^{p-1}(y)g^q(y)\,dy\right]^{\frac{1}{q}} \tag{58}$$

and

$$\left\{\int_a^b \left(\frac{v'}{v}\right)(y)\left[\int_a^b k_0^\lambda(u(x),v(y))f(x)\,dx\right]^{q'} dy\right\}^{\frac{1}{q'}} \leqslant c_0^\lambda(1)\left[\int_a^b \left(\frac{u}{u'}\right)^{p-1}(x)f^p(x)\,dx\right]^{\frac{1}{p}}. \tag{59}$$

REMARK 6. For the same choice of parameters A_1 and A_2, that is $A_1 = \frac{1}{p'}$ and $A_2 = \frac{1}{q'}$, Theorem 3 yields the same inequalities as (58) and (59), with the kernel $k_0(1,u(x)v(y))$ instead of $k_0(u(x),v(y))$. In other words, inequalities (58) and (59) also hold if we replace the kernel $k_0(u(x),v(y))$ with $k_0(1,u(x)v(y))$.

We obtain similar results for the Hardy-type kernels defined by (50). Namely, if $A_1 = \frac{1}{p'}$ and $A_2 = \frac{1}{q'}$, Corollary 7 yields inequalities

$$\int_a^b \int_{u^{(-1)}(v(y))}^b k_0^\lambda(u(x),v(y))f(x)g(y)\,dxdy \leqslant \overline{c}_0^\lambda(1)\left[\int_a^b \left(\frac{u}{u'}\right)^{p-1}(x)f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_a^b \left(\frac{v}{v'}\right)^{p-1}(y)g^q(y)\,dy\right]^{\frac{1}{q}} \tag{60}$$

and

$$\left\{\int_a^b \left(\frac{v'}{v}\right)(y)\left[\int_{u^{(-1)}(v(y))}^b k_0^\lambda(u(x),v(y))f(x)\,dx\right]^{q'} dy\right\}^{\frac{1}{q'}} \leqslant \overline{c}_0^\lambda(1)\left[\int_a^b \left(\frac{u}{u'}\right)^{p-1}(x)f^p(x)\,dx\right]^{\frac{1}{p}}. \tag{61}$$

REMARK 7. According to Corollary 8, inequalities (60) and (61) also hold if we replace the kernel $k_0(u(x),v(y))$ with $k_0(1,u(x)v(y))$ and integration set $\langle u^{(-1)}(v(y)),b\rangle$ with $\langle a,u^{(-1)}(\frac{1}{v(y)})\rangle$.

REMARK 8.. In the conjugate case, parameters that generate inequalities (58), (59), (60), and (61) are $A_1 = \frac{1}{q}$ and $A_2 = \frac{1}{p}$. Such parameters satisfy condition (38), so we get the best possible constant facctors $c_0(1)$ and $\overline{c}_0(1)$ in this setting.

Finally, to conclude the paper, we shall consider some interesting examples of homogeneous kernels of degree 0 in inequalities (58), (59), (60), and (61). We obtain the constant factors expressed in terms of some special functions, such as Beta and Gamma function. Of course, the constant factors will be the best possible in the conjugate case.

EXAMPLE 1. Let $\alpha>0$, $\beta>-1$ and

$$k_0(x,y)=\left(\frac{\min\{x,y\}}{\max\{x,y\}}\right)^{\alpha}\left|\ln\left(\frac{y}{x}\right)\right|^{\beta}. \tag{62}$$

Then,

$$\int_0^{\infty}\left(\frac{\min\{1,t\}}{\max\{1,t\}}\right)^{\alpha}|\ln t|^{\beta}t^{-1}dt=\int_0^1 t^{\alpha-1}(-\ln t)^{\beta}dt+\int_1^{\infty}t^{-\alpha-1}(\ln t)^{\beta}dt.$$

It is easy to verify that

$$\int_0^1 t^{\alpha-1}(-\ln t)^{\beta}dt=\int_1^{\infty}t^{-\alpha-1}(\ln t)^{\beta}dt=\int_0^{\infty}e^{-\alpha v}v^{\beta}dv=\frac{\Gamma(\beta+1)}{\alpha^{\beta+1}},$$

where Γ denotes the well-known Gamma function. Hence, in this case we obtain constant factors

$$c_0(1)=\frac{2\Gamma(\beta+1)}{\alpha^{\beta+1}}\qquad\text{and}\qquad \overline{c}_0(1)=\frac{\Gamma(\beta+1)}{\alpha^{\beta+1}}.$$

This kernel was also considered in paper [6]. Further, if $\alpha=1$ and $\beta=0$, we have $c_0(1)=2$, that is, we get inequality (6) from the Introduction.

EXAMPLE 2. For homogeneous function defined by

$$k_0(x,y)=\frac{\min\{x,y\}}{\max\{x,y\}}\arctan\left(\frac{y}{x}\right), \tag{63}$$

we have

$$\int_0^{\infty}\frac{\min\{1,t\}}{\max\{1,t\}}\arctan t\cdot t^{-1}dt=\int_0^1\arctan t\,dt+\int_1^{\infty}t^{-2}\arctan t\,dt.$$

Clearly, by means of partial integration, we easily obtain required constant factors

$$c_0(1)=\frac{\pi}{2}\qquad\text{and}\qquad \overline{c}_0(1)=\frac{\pi}{4}-\frac{\ln 2}{2}.$$

EXAMPLE 3. Let $0<\alpha<1$ and

$$k_0(x,y)=\left(\frac{\min\{x,y\}}{|x-y|}\right)^{\alpha}. \tag{64}$$

Then

$$\int_0^{\infty}\left(\frac{\min\{1,t\}}{|1-t|}\right)^{\alpha}t^{-1}dt=\int_0^1 t^{\alpha-1}(1-t)^{-\alpha}dt+\int_1^{\infty}t^{-1}(t-1)^{-\alpha}dt.$$

Further, it is easy to see that

$$\int_0^1 t^{\alpha-1}(1-t)^{-\alpha}dt = \int_1^\infty t^{-1}(t-1)^{-\alpha}dt = B(1-\alpha,\alpha),$$

where B is the usual Beta function. Hence, in this setting we obtain constants

$$c_0(1) = 2B(1-\alpha,\alpha) \qquad \text{and} \qquad \overline{c}_0(1) = B(1-\alpha,\alpha).$$

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