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RICCI CURVATURE OF LAGRANGIAN SUBMANIFOLDS IN COMPLEX SPACE FORMS

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Abstract. As we showed in [5] and [6], the basic inequalities, involving Riemannian invariants of a Lagrangian submanifold immersed in a complex space form, can be improved using optimization methods. Also in [1] is showed that the improved Chen's inequality from [5] is optimal. In this paper we find another proof for a Chen's inequality, regarding the Ricci curvature [2] and we improve this inequality in the Lagrangian case.

1. Optimizations on Riemannian submanifolds

Let $(N,\widetilde{g})$ be a Riemannian manifold, (M,g) a Riemannian submanifold of it, and

$$f : N \to R$$

a differentiable function. To these ingredients we attach the optimum problem

$$\min_{x\in M} f(x). \tag{1.1}$$

Let's remember the result obtained in [4].

THEOREM 1.1. *If $x_0 \in M$ is a optimal solution of the problem (1), then*

i) $(\operatorname{grad} f)(x_0) \in T^{\perp}_{x_0}M$,

ii) *the bilinear form*

$$\alpha \ : \ T_{x_0}M \times T_{x_0}M \ \to \ R,$$
$$\alpha(X,Y) = \operatorname{Hess}_f(X,Y) + \widetilde{g}\,(h(X,Y),(\operatorname{grad} f)(x_0))$$

is positive semidefinite, where h is the second fundamental form of the submanifold M in N.

REMARK 1.1. The bilinear form α is nothing else but $\operatorname{Hess}_{f|M}(x_0)$.

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A very nice application of this result is the partial solution of the next problem: *find the 2-plane in the tangent space at the given point x of a Riemannian manifold (M,g) which minimize the sectional curvature.* A equivalent conditioned extremum problem is

$$\min R(X,Y,X,Y), \tag{1.2}$$

$$\text{subject to } \parallel X \parallel = 1, \parallel Y \parallel = 1,\ g(X,Y) = 0,$$

where X and Y are two vectors from T_xM.

A 2-plane $\pi \subset T_xM$, $\pi = \mathrm{Sp}\{X,Y\}$ which verifies the first condition from Theorem 1 is called critical plane for the sectional curvature at the point x. Using Theorem 1, in [4] we showed that *a 2-plane π is a critical plane for the sectional curvature at the point x if and only if for every tangent vectors U, V, $W \in \pi$ the vector $R(U,V)W$ lies in π, where R is the curvature tensor of the Riemannian manifold (M,g).*

2. The Ricci curvature of a submanifold in a real space form

In this section we give a new proof of the next inequality

THEOREM 2.1. (Chen [2]). *Let M be a n-dimensional Riemannian submanifold of a real space form $(\widetilde{M}(c),g)$ and x a point in M. Then, for each unit vector $X \in T_xM$, we have*

$$\mathrm{Ric}(X) \leqslant (n-1)c + \frac{n^2}{4}\|H\|^2,$$

where H is the mean curvature vector of M in $\widetilde{M}(c)$ and Ric(X) *the Ricci curvature of M at x.*

Proof. We fix the point x in M, the vector $X \in T_xM$, with $\|X\| = 1$, the orthonormal frame $\{e_1, e_2, \dots, e_n\}$ in T_xM such that $e_1 = X$ and $\{e_{n+1}, e_{n+2}, \dots, e_m\}$ a orthonormal frame in $T_x^{\perp}M$.

From Gauss equation we have

$$\begin{aligned}\widetilde{R}(e_1,e_j,e_1,e_j) &= R(e_1,e_j,e_1,e_j) - \widetilde{g}(h(e_1,e_1),h(e_j,e_j)) + \widetilde{g}(h(e_1,e_j),h(e_1,e_j)) \\ &= R(e_1,e_j,e_1,e_j) - \sum_{r=n+1}^{m}(h_{11}^r h_{jj}^r - (h_{1j}^r)^2),\ j \in \overline{2,n}.\end{aligned} \tag{2.1}$$

Using the fact that the sectional curvature of $\widetilde{M}(c)$ is constant, we obtain

$$(n-1)c = \mathrm{Ric}(X) - \sum_{r=n+1}^{m}\sum_{j=2}^{n}(h_{11}^r h_{jj}^r - (h_{1j}^r)^2), \tag{2.2}$$

therefore

$$\mathrm{Ric}(X) - (n-1)c = \sum_{r=n+1}^{m}\sum_{j=2}^{n}(h_{11}^r h_{jj}^r - (h_{1j}^r)^2) \leqslant \sum_{r=n+1}^{m}\sum_{j=2}^{n} h_{11}^r h_{jj}^r. \tag{2.3}$$

For $r \in \overline{n+1,m}$, let us consider the quadratic form

$$f_r : R^n \to R,$$
$$f_r(h^r_{11}, h^r_{22}, ..., h^r_{nn}) = \sum_{j=2}^{n} h^r_{11} h^r_{jj}$$

and the constrained extremum problem

$$\max f_r,$$

$$\text{subject to } P : h^r_{11} + h^r_{22} + ... + h^r_{nn} = k^r,$$

where k^r is a real constant.

The gradient vector of f_r is given by

$$\operatorname{grad} f_r = (\sum_{j=2}^{n} h^r_{jj}, h^r_{11}, ..., h^r_{11}).$$

Let us denote with $p = (h^r_{11}, h^r_{22}, ..., h^r_{nn})$ a optimal solution of the problem in question.

As $\operatorname{grad} f_r$ is normal to P at the point p, we obtain

$$h^r_{11} = \sum_{j=2}^{n} h^r_{jj} = \frac{k^r}{2}. \tag{2.4}$$

Let $q \in P$ be an arbitrary point.

The bilinear form $\alpha : T_qP \times T_qP \to R$ has the expression

$$\alpha(X,Y) = \operatorname{Hess}(f_r)(X,Y) + \langle h^{'}(X,Y), \operatorname{grad} f_r(q) \rangle,$$

where $h^{'}$ is the second fundamental form of P in R^n, and $\langle \ , \ \rangle$ is the standard inner-product on R^n.

In the standard frame of R^n, the Hessian of f_r has the matrix

$$\begin{pmatrix} 0 & 1 & . & 1 \\ 1 & 0 & . & 0 \\ . & . & . & . \\ 1 & 0 & . & 0 \end{pmatrix}.$$

A vector $X \in T_qP$ satisfies $\sum_{i=1}^{n} X^i = 0$.

As P is totally geodesic in R^n, we have $\alpha(X,X) = 2\sum_{j=2}^{n} X^1 X^j = -2(X^1)^2$.

So $f_r|P$ is a concave function, therefore the points which satisfies the relation (2.4) are global maximum points for $f_r|P$.

One gets

$$f_r \leqslant \frac{(k^r)^2}{4} = \frac{1}{4}(\sum_{i=1} h^r_{ii})^2 = \frac{n^2}{4}(H^r)^2. \tag{2.5}$$

Using (2.3) and (2.5) we find

$$\mathrm{Ric}\,(X)-(n-1)c\leqslant\sum_{r=n+1}^{m}\frac{n^2}{4}(H^r)^2=\frac{n^2}{4}\,\|H\|^2. \tag{2.6}$$

3. The Ricci curvature of a Lagrangian submanifold in a complex space form

Let $(\widetilde{M},\widetilde{g},J)$ be a Kähler manifold of real dimension $2m$. A submanifold M of dimension n of $(\widetilde{M},\widetilde{g},J)$ is called a totally real submanifold if for any point x in M the relation $J(T_xM)\subset T_x^{\perp}M$ holds.

If, in addition, $n=m$, then M is called Lagrangian submanifold. For a Lagrangian submanifold, the relation $J(T_xM)=T_x^{\perp}M$ occurs.

A Kähler manifold with constant holomorphic sectional curvature is called a complex space form and is denoted by $\widetilde{M}(c)$. The Riemann curvature tensor $\widetilde{R}$ of $\widetilde{M}(c)$ satisfies the relation

$$\widetilde{R}(X,Y)Z=\frac{c}{4}\{\widetilde{g}(Y,Z)X-\widetilde{g}(X,Z)Y+\widetilde{g}(JY,Z)JX-\widetilde{g}(JX,Z)JY+2\widetilde{g}(X,JY)JZ\}.$$

A totally real submanifold of real dimension n in a complex space form $\widetilde{M}(c)$ of real dimension $2m$ satisfies a Chen's inequality

THEOREM 3.1. (Chen). *Let M be a n-dimensional Riemannian submanifold of a complex space form $(\widetilde{M}(c),g)$ and x a point in M. Then, for each unit vector $X\in T_xM$, we have*

$$\mathrm{Ric}\,(X)\leqslant(n-1)\frac{c}{4}+\frac{n^2}{4}\,\|H\|^2,$$

where H is the mean curvature vector of M in $\widetilde{M}(c)$ and Ric (X) *the Ricci curvature of M at x.*

REMARK 3.1. i) If M is a totally real submanifold of real dimension n in a complex space form $\widetilde{M}(c)$ of real dimension $2m$, then

$$A_{JY}X=-Jh(X,Y)=A_{JX}Y,$$

where X and Y are two arbitrary vector fields.

ii) Let $m=n$ (M is Lagrangian in $\widetilde{M}(c)$). If we consider the point $x\in M$, the orthonormal frames $\{e_1,...,e_n\}$ in T_xM and $\{Je_1,...,Je_n\}$ in $T_x^{\perp}M$, then

$$h^i_{jk}=h^j_{ik},\forall\; i,j,k\in\overline{1,n},$$

where h^i_{jk} is the component after Je_i of the vector $h(e_j,e_k)$.

With these ingredients we prove the next result which improve Chen's inequality in the Lagrangian case.

THEOREM 3.2. *Let M be a Lagrangian submanifold in a complex space form $\widetilde{M}(c)$ of real dimension $2n$, $n \geqslant 2$, x a point in M and X a unit tangent vector in T_xM. Then we have*

$$\mathrm{Ric}\,(X) \leqslant \frac{n-1}{4}(c+n\|H\|^2).$$

Proof. We fix the point x in M, the tangent vector $X \in T_xM$, with $\|X\| = 1$, the orthonormal frame $\{e_1, e_2, ..., e_n\}$ in T_xM such that $e_1 = X$ and $\{Je_1, Je_2, ..., Je_n\}$ a orthonormal frame in $T_x^{\perp}M$.

From the Gauss equation we get

$$\begin{aligned}\widetilde{R}(e_1,e_j,e_1,e_j) &= R(e_1,e_j,e_1,e_j) - \widetilde{g}(h(e_1,e_1),h(e_j,e_j)) + \widetilde{g}(h(e_1,e_j),h(e_1,e_j)) \\ &= R(e_1,e_j,e_1,e_j) - \sum_{r=1}^{n}(h^r_{11}h^r_{jj} - (h^r_{1j})^2), \quad \forall j \in \overline{2,n}.\end{aligned} \tag{4.1}$$

Therefore

$$(n-1)\frac{c}{4} = \mathrm{Ric}\,(X) - \sum_{r=1}^{n}\sum_{j=2}^{n}(h^r_{11}h^r_{jj} - (h^r_{1j})^2) \tag{4.2}$$

which implies

$$\begin{aligned}\mathrm{Ric}\,(X) - (n-1)\frac{c}{4} &= \sum_{r=1}^{n}\sum_{j=2}^{n}(h^r_{11}h^r_{jj} - (h^r_{1j})^2) \\ &\leqslant (\sum_{r=1}^{n}\sum_{j=2}^{n} h^r_{11}h^r_{jj}) - \sum_{j=2}^{n}(h^1_{1j})^2 - \sum_{j=2}^{n}(h^j_{1j})^2 \\ &= (\sum_{r=1}^{n}\sum_{j=2}^{n} h^r_{11}h^r_{jj}) - \sum_{j=2}^{n}(h^j_{11})^2 - \sum_{j=2}^{n}(h^1_{jj})^2.\end{aligned} \tag{4.3}$$

Let us consider the quadratic forms

$$\begin{gathered} f_1,\ f_r : R^n \to R, \\ f_1(h^1_{11}, h^1_{22}, ..., h^1_{nn}) = \sum_{j=2}^{n} h^1_{11}h^1_{jj} - \sum_{j=2}^{n}(h^1_{jj})^2, \\ f_r(h^r_{11}, h^r_{22}, ..., h^r_{nn}) = \sum_{j=2}^{n} h^r_{11}h^r_{jj} - (h^r_{11})^2, \end{gathered}$$

where $r \in \overline{2,n}$.

We need the maximum of f_1 and f_2. For f_r, $r \in \overline{3,n}$, we can solve similar problems.

We start with the problem

$$\max f_1,$$

$$\text{subject to } P : h^1_{11} + h^1_{22} + ... + h^1_{nn} = k^1,$$

where k^1 is a real constant.

The partial derivatives of the function f_1 are

$$\frac{\partial f_1}{\partial h_{11}^1} = \sum_{j=2}^{n} h_{jj}^1, \tag{4.4}$$

$$\frac{\partial f_1}{\partial h_{jj}^1} = h_{11}^1 - 2h_{jj}^1, \quad \forall j \in \overline{2,n}. \tag{4.5}$$

As for a optimal solution $(h_{11}^1, h_{22}^1, ..., h_{nn}^1)$ of the problem in question the vector $\operatorname{grad} f_1$ is normal to P, we obtain

$$h_{22}^1 = h_{33}^1 = ... = h_{nn}^1 = a^1 \tag{4.6}$$

and

$$h_{11}^1 - 2h_{22}^1 = \sum_{j=2}^{n} h_{jj}^1. \tag{4.7}$$

Using (4.6) and (4.7) we find

$$h_{11}^1 = (n+1)a^1. \tag{4.8}$$

From the relation $h_{11}^1 + h_{22}^1 + ... + h_{nn}^1 = k^1$, we get

$$(n+1)a^1 + (n-1)a^1 = k^1, \tag{4.9}$$

therefore

$$a^1 = \frac{k^1}{2n}. \tag{4.10}$$

As f_1 is obtained from the function studied in previous section by subtracting some square terms, $f_1\,|P$ will have the Hessian negative semidefinite. Consequently the point $(h_{11}^1, h_{22}^1, ..., h_{nn}^1)$ given by the relations (4.6), (4.8) and (4.10) is a maximum point, and hence

$$f_1 \leqslant (n+1)a^1(n-1)a^1 - (n-1)(a^1)^2 = \frac{n-1}{4n}(k^1)^2 = \frac{n(n-1)}{4}(H^1)^2. \tag{4.11}$$

Further on, we shall consider the problem

$$\max f_2,$$

$$\text{subject to } P : h_{11}^2 + h_{22}^2 + ... + h_{nn}^2 = k^2,$$

where k^2 is a real constant.

The first two partial derivatives of the function f_2 are

$$\frac{\partial f_2}{\partial h_{11}^2} = \sum_{j=2}^{n} h_{jj}^2 - 2h_{11}^2, \tag{4.12}$$

$$\frac{\partial f_2}{\partial h_{jj}^2} = h_{11}^2, \quad \forall j \in \overline{2,n}. \tag{4.13}$$

As for a optimal solution $(h_{11}^2, h_{22}^2, ..., h_{nn}^2)$ of the problem in question the vector $\operatorname{grad} f_2$ is normal to P, we obtain

$$3h_{11}^2 = \sum_{j=2}^{n} h_{jj}^2 = 3a^2. \tag{4.14}$$

Using the relation $h_{11}^2 + h_{22}^2 + ... + h_{nn}^2 = k^2$, one gets

$$a^2 + 3a^2 = k^2, \tag{4.15}$$

therefore

$$a^2 = \frac{k^2}{4}. \tag{4.16}$$

With an similar argument to those in the previous problem we obtain that the points $(h_{11}^2, h_{22}^2, ..., h_{nn}^2)$ given by the relations (4.14) and (4.16) are global maximum points. Therefore

$$f_2 \leqslant a^2 3a^2 - (a^2)^2 = 2(a^2)^2 = \frac{(k^2)^2}{8} = \frac{n^2}{8}(H^2)^2. \tag{4.17}$$

Similarly one gets

$$f_r \leqslant \frac{n^2}{8}(H^r)^2, \quad \forall r \in \overline{2,n}. \tag{4.18}$$

As $\frac{n(n-1)}{4} \geqslant \frac{n^2}{8}$, $\forall\ n \geqslant 2$, using (4.11) and (4.18) we find

$$f_r \leqslant \frac{n(n-1)}{4}(H^r)^2, \quad \forall r \in \overline{1,n}. \tag{4.19}$$

From (4.3) and (4.19) it follows

$$\operatorname{Ric}(X) - (n-1)\frac{c}{4} \leqslant \frac{n(n-1)}{4} \sum_{r=1}^{n} (H^r)^2 = \frac{n(n-1)}{4} \|H\|^2, \tag{4.20}$$

therefore

$$\operatorname{Ric}(X) \leqslant \frac{n-1}{4}(c + n\|H\|^2). \tag{4.21}$$

REMARK 3.2. i) If the mean curvature vector field H vanish at the point $x \in M$, then in precedent inequality the equality occurs for a tangent vector $X \in T_xM$ if and only if $h(X,Y) = 0, \forall\ Y \in T_xM$.

ii) If $n \geqslant 3$, the mean curvature vector field H don't vanish at the point $x \in M$ and the tangent vector $X \in T_xM$ satisfies

$$\operatorname{Ric}(X) = \frac{n-1}{4}(c + n\|H\|^2),$$

then $X = \frac{\pm JH}{\|H\|}$. Indeed, the above equality implies that $H^2, H^3, \dots, H^n$ are 0. Consequently, the mean curvature vector H is colinear with Je_1. It follows that JH is colinear with X, therefore $X = \frac{\pm JH}{\|H\|}$.

iii) If $n \geqslant 3$, in theorem 4 the equality occurs for any tangent vector $X \in T_xM$ if and only if x is a totally geodesic point.

iv) If $n = 2$, in theorem 4 the equality occurs for any tangent vector $X \in T_xM$ if and only if there is an orthonormal frame $\{e_1, e_2\}$ in T_xM in which the Weingarten operators take the following form

$$A_{Je_1} = \begin{pmatrix} 3a^1 & a^2 \\ a^2 & a^1 \end{pmatrix}, \quad A_{Je_2} = \begin{pmatrix} a^2 & a^1 \\ a^1 & 3a^2 \end{pmatrix}.$$

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