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CERTAIN INEQUALITIES AND THEIR APPLICATIONS TO MULTIVALENTLY ANALYTIC FUNCTIONS–II

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This paper is decided
to the memory of Prof. Dr. Doğan Çoker

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Abstract. In the present investigation, some results concerning multivalently analytic functions and fractional calculus which were recently obtained in [4, *Math. Inequal. Appl.* **8** (3) (2005), 451-458] are restated by using a different technique proposed by M. Nunokawa [10, *Proc. Japan Acad. Ser. A Math. Sci.*, **68**, 6 (1992), 152–153].

1. Introduction and Definitions

Let $\mathscr{A}(p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots \tag{1.1}$$

$$(p \in \mathscr{N} := \{1,\ 2,\ 3,\ \cdots\} =: \mathscr{N}_0 \backslash \{0\}),$$

which are *analytic* and *multivalent* in the *open unit disk* $\mathbf{U} = \{z\ :\ z \in \mathbf{C} \text{ and } |z| < 1\}$, where $\mathbf{C}$ is the set of complex numbers.

We denote by $\mathscr{S}_p(\alpha)$, $\mathscr{C}_p(\alpha)$ and $\mathscr{K}_p(\alpha)$ the subclasses of the class $\mathscr{A}(p)$ consisting of all functions which are, respectively, *multivalently starlike* (or, *starlike* when $p = 1$) *of order* α in $\mathbf{U}$, *multivalently convex* (or, *convex* when $p = 1$) *of order* α in $\mathbf{U}$ and *multivalently close-to-convex* (or, *close-to-convex* when $p = 1$) *of order* α in $\mathbf{U}$. (For the details of the above special functions classes, one may refree to [1], [3], and also [13].)

In this section and also throughout this paper, the symbol D_z^{μ} denotes an operator of fractional calculus (that is that fractional derivative(s)), which is defined as follows (cf., [11], and see also [2] and [5]):

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DEFINITION 1. Let a function $\kappa(z)$ be analytic in a simply-connected region of the z-plane containing the origin. Then, the fractional derivative of order μ is defined by

$$D_z^{\mu}\{\kappa(z)\} = \frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\int_0^z \frac{\kappa(\xi)}{(z-\xi)^{\mu}}d\xi \quad (0 \leqslant \mu < 1), \tag{1.2}$$

where the multiplicity of $(z-\xi)^{-\mu}$ involved in (1.2) is removed by requiring $\log(z-\xi)$ to be real when $z-\xi>0$.

DEFINITION 2. Under the hypotheses of the Definition 1, the fractional derivative of order $m+\mu$ is defined by

$$D_z^{m+\mu}\{\kappa(z)\} = \frac{d^m}{dz^m}D_z^{\mu}\{\kappa(z)\} \quad (m \in \mathcal{N}_0; 0 \leqslant \mu < 1). \tag{1.3}$$

It follows from (1.1), and (1.2) together with (1.3) that

$$D_z^{m+\mu}\{f(z)\} = \frac{\Gamma(p+1)}{\Gamma(p-m-\mu+1)}z^{p-m-\mu} + \sum_{k=p+1}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k-m-\mu+1)}a_k z^{k-m-\mu}, \tag{1.4}$$

where

$$f(z) \in \mathscr{A}(p),\ 0 \leqslant \mu < 1,\ p \in \mathcal{N},\ \text{ and } m \in \mathcal{N}_0\ (m < p-\mu+1).$$

As we said in the abstract of this paper, this investigation includes some results concerning certain derivatives of multivalently analytic function in the class $\mathscr{A}(p)$ and fractional calculus which were similarly results given in the paper in [4] are reorganized by using a different technique obtained by M. Nunokawa [10]. In addition, special and very special consequences of these main results can be compared with several results which were earlier obtained by certain researches (see, [5], [6], [7], [8], [9], and [12]). We note that this present investigation is not only a generalization of the results which cited above papers.

In order to prove our main results, we shall require the following assertion due to M. Nunokawa.

LEMMA. ([10]) *Let $p(z)$ be an analytic function in $\mathbf{U}$ with $p(0)=1$. If there exists a point $z_0 \in \mathbf{U}$ such that*

$$\Re e\{p(z)\} > 0\ (|z| < |z_0|),\ \Re e\{p(z_0)\} = 0 \ \textit{and}\ p(z_0) \neq 0,$$

then

$$p(z_0) = ia \ \textit{and}\ \left.\frac{zp'(z)}{p(z)}\right|_{z=z_0} = i\frac{c}{2}\left(a+\frac{1}{a}\right),$$

where $a \neq 0$ and $c \geqslant 1$.

2. The Main Results

By using the above assertion, we first begin by proving the following result:

THEOREM 1. *If a function $f(z)$ in $\mathscr{A}(p)$ satisfies any one of the cases of the following inequalities:*

$$\Re e\left(\frac{zD_z^{1+\mu}\{f(z)\}}{D_z^{\mu}\{f(z)\}}\right)\begin{cases} > p-\mu+\frac{1}{2\delta}\left(1-\frac{1}{1-\beta}\right) & \text{if } \delta>0,\ 0\leqslant\beta\leqslant\frac{1}{2} \\ > p-\mu+\frac{1}{2\delta}\left(1-\frac{1}{\beta}\right) & \text{if } \delta>0,\ \frac{1}{2}\leqslant\beta<1 \\ < p-\mu+\frac{1}{2\delta}\left(1-\frac{1}{1-\beta}\right) & \text{if } \delta<0,\ 0\leqslant\beta\leqslant\frac{1}{2} \\ < p-\mu+\frac{1}{2\delta}\left(1-\frac{1}{\beta}\right) & \text{if } \delta<0,\ \frac{1}{2}\leqslant\beta<1 \end{cases}, \tag{2.1}$$

then

$$\Re e\left[\left(\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)}\cdot\frac{D_z^{\mu}\{f(z)\}}{z^{p-\mu}}\right)^{\delta}\right]>\beta, \tag{2.2}$$

where $z\in\mathbf{U},\ p\in\mathscr{N},\ 0\leqslant\mu<1,\ \delta\in\mathbf{R}^*=\mathbf{R}-\{0\}$, and $0\leqslant\beta<1$, and the value of the above complex power is considered to be as its principal value.

Proof. From (1.1) in conjunction with the representation (1.4), it easily follows that

$$\frac{D_z^{\mu}\{f(z)\}}{z^{p-\mu}}=\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}+\sum_{k=p+1}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k-\mu+1)}a_k z^{k-p}, \tag{2.3}$$

where $f(z)\in\mathscr{A}(p),\ 0\leqslant\mu<1$, and $p\in\mathscr{N}$ with $m<p-\mu+1$.

In view of (2.3), we now define a function $p(z)$ by

$$\left(\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)}\cdot\frac{D_z^{\mu}\{f(z)\}}{z^{p-\mu}}\right)^{\delta}=\beta+(1-\beta)p(z). \tag{2.4}$$

Then, we observe that for $z=0,\ p(0)=1$, and further $p(z)$ given by (2.4) is analytic for every z in $\mathbf{U}$. We also obtain from the logarithmic differentiation of (2.4) that

$$\delta\cdot z\cdot\left(\frac{D_z^{1+\mu}\{f(z)\}}{D_z^{\mu}\{f(z)\}}-\frac{p-\mu}{z}\right)=\frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}.$$

We also define a function $\mathscr{H}(z)$ as

$$\mathscr{H}(z):=\frac{zD_z^{1+\mu}\{f(z)\}}{D_z^{\mu}\{f(z)\}}-(p-\mu)=\frac{1}{\delta}\cdot\frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}. \tag{2.5}$$

Suppose now there exists a point $z_0\in\mathbf{U}$ such that

$$\Re e\{p(z)\}>0\ \ (|z|<|z_0|),\ \ \Re e\{p(z_0)\}=0,\ \text{ and }\ p(z_0)\neq 0\ \ (z\in\mathbf{U}).$$

Then, by using Lemma, we have

$$p(z_0) = ia \text{ and } \frac{z_0 p'(z_0)}{p(z_0)} = i\frac{c}{2}\left(a + \frac{1}{a}\right), \tag{2.6}$$

where $a \neq 0,\ c \geqslant 1,$ and $z_0 \in \mathbf{U}$.

Since

$$0 \leqslant \beta \leqslant \frac{1}{2} \quad \Rightarrow \quad \frac{1+a^2}{\left(\frac{\beta}{1-\beta}\right)^2 + a^2} \geqslant 1$$

and

$$\frac{1}{2} \leqslant \beta < 1 \quad \Rightarrow \quad \frac{1+a^2}{1 + \left(\frac{1-\beta}{\beta}\right)^2 a^2} \geqslant 1,$$

and from (2.5) together with (2.6), we immediately get that

$$\begin{aligned} \Re e\{\mathscr{H}(z_0)\} &= \Re e\left(\frac{1}{\delta}\cdot\frac{(1-\beta)zp'(z)}{p(z)}\cdot\left.\frac{p(z)}{\beta+(1-\beta)p(z)}\right|_{z=z_0}\right) \\ &= -\frac{c\beta(1-\beta)(1+a^2)}{2\delta[\beta^2+a^2(1-\beta)^2]} \\ &\left\{\begin{array}{ll} \leqslant \frac{c}{2\delta}\left(1-\frac{1}{1-\beta}\right) & \textit{if } \delta > 0, 0 \leqslant \beta \leqslant \frac{1}{2} \\ \leqslant \frac{c}{2\delta}\left(1-\frac{1}{\beta}\right) & \textit{if } \delta > 0, \frac{1}{2} \leqslant \beta < 1 \\ \geqslant \frac{c}{2\delta}\left(1-\frac{1}{1-\beta}\right) & \textit{if } \delta < 0, 0 \leqslant \beta \leqslant \frac{1}{2} \\ \geqslant \frac{c}{2\delta}\left(1-\frac{1}{\beta}\right) & \textit{if } \delta < 0, \frac{1}{2} \leqslant \beta < 1 \end{array}\right\}. \end{aligned} \tag{2.7}$$

But, for $c = 1,$ the inequalities in (2.7) contradict our assumptions imposed in (2.1), respectively. Hence, $\Re e\{p(z)\} > 0$ for all $z \in \mathbf{U}$. Therefore, (2.4) evidently yields the inequality in (2.2). So, the desired proof of Theorem 1 is completed. □

The other result is contained in

THEOREM 2. *If a function $f(z)$ in $\mathscr{A}(p)$ satisfies any one of the cases of the following inequalities:*

$$1 + \Re e\left[z\left(\frac{D_z^{2+\mu}\{f(z)\}}{D_z^{1+\mu}\{f(z)\}} - \frac{D_z^{1+\mu}\{f(z)\}}{D_z^{\mu}\{f(z)\}}\right)\right]\left\{\begin{array}{ll} > \frac{\beta}{2\delta(\beta-1)} & \textit{if } \delta > 0, 0 \leqslant \beta \leqslant \frac{1}{2} \\ > \frac{\beta-1}{2\delta\beta} & \textit{if } \delta > 0, \frac{1}{2} \leqslant \beta < 1 \\ < \frac{\beta}{2\delta(\beta-1)} & \textit{if } \delta < 0, 0 \leqslant \beta \leqslant \frac{1}{2} \\ < \frac{\beta-1}{2\delta\beta} & \textit{if } \delta < 0, \frac{1}{2} \leqslant \beta < 1 \end{array}\right\}, \tag{2.8}$$

then

$$\Re e\left[\left(\frac{1}{p-\mu}\cdot\frac{zD_z^{1+\mu}\{f(z)\}}{D_z^{\mu}\{f(z)\}}\right)^{\delta}\right] > \beta, \tag{2.9}$$

where $z \in \mathbf{U},\ p \in \mathcal{N},\ 0 \leqslant \mu < 1,\ \delta \in \mathbf{R}^*$, *and* $0 \leqslant \beta < 1$, *and the value of the above complex power is taken to be as its principal value.*

Proof. Under the hypothesis of Theorem 2 and in view of (1.4), we need again to define a function $p(z)$ by

$$\left(\frac{1}{p-\mu}\cdot\frac{zD_z^{1+\mu}\{f(z)\}}{D_z^{\mu}\{f(z)\}}\right)^{\delta} = \beta + (1-\beta)p(z). \tag{2.10}$$

If we take the both sides derivation of (2.10) by logarithmically, then we easily get the following equality:

$$1 + z\left(\frac{D_z^{2+\mu}\{f(z)\}}{D_z^{1+\mu}\{f(z)\}} - \frac{D_z^{1+\mu}\{f(z)\}}{D_z^{\mu}\{f(z)\}}\right) = \frac{1}{\delta}\cdot\frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}, \tag{2.11}$$

where $f(z) \in \mathscr{A}(p),\ z \in \mathbf{U},\ 0 \leqslant \mu < 1,\ \delta \in \mathbf{R}^*$, and $0 \leqslant \beta < 1$.

It is easily seen that the defined function $p(z)$ satisfies the conditions of the Lemma. Then, under the same assumptions, and by appealing the same technique as in the proof of the Theorem 1 to the equality (2.11), we easily arrive at some contradictions to the inequalities in (2.8), respectively. Therefore, the inequality in (2.9) immediately follows from (2.10) which completes the proof of the Theorem 2. So, we need not give the detail.

It is clear that the our main results (Theorems 1 and 2) include several useful consequences which will be important and/or interesting for analytic and geometric function theory. By selecting the suitable values of $\delta,\ \mu,\ \beta$ and/or p in the Theorems 1 and 2, respectively, can be then arrived at the consequences of the main results mentioned above. Although we used another technique for their proofs, these results have some very special consequences of the main results which are comparable with certain results of certain authors in the references in [4], [5], [6], [7], [8], [9], and [12]. Especially, the main two results were again restated by making use of the other method which were recently presented in [10]. Therefore, the two basic and important results (Theorems 1 and 2) are also comparable with the results in [10]. In addition, we want only to emphasize the three results of Theorems 1 and 2 including geometric properties, as examples (Corollaries 1-3 below). □

If we first let $\delta = 1$ and $\mu \to 1-$ in the Theorem 1, then we get the following result.

COROLLARY 1. *If a function* $f(z) \in \mathscr{A}(p)$ *satisfies any one of the cases of the following inequalities:*

$$\Re e\left(\frac{zf''(z)}{f'(z)}\right)\begin{Bmatrix} > p-1-\frac{\beta}{2(1-\beta)} & \text{if } 0 \leqslant \beta \leqslant \frac{1}{2} \\ > p-1-\frac{1-\beta}{2\beta} & \text{if } \frac{1}{2} \leqslant \beta < 1 \end{Bmatrix},$$

then

$$\Re e\left(\frac{f'(z)}{z^{p-1}}\right) > p\beta, \quad i.e., \quad f(z) \in \mathscr{K}_p(p\beta),$$

where $z \in \mathbf{U},\ p \in \mathscr{N}$ *and* $0 \leqslant \beta < 1$.

If we next take ($\delta = 1$ and $\mu = 0$) and ($\delta = 1$ and $\mu \to 1-$) in the Theorem 2, then we also receive the following two results (Corollaries 2 and 3 below), respectively.

COROLLARY 2. *If a function* $f(z) \in \mathscr{A}(p)$ *satisfies any one of the cases of the following inequalities:*

$$\Re e\left[z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right)\right] \begin{Bmatrix} > -1 - \frac{\beta}{2(1-\beta)} & \text{if } 0 \leqslant \beta \leqslant \frac{1}{2} \\ > -1 - \frac{1-\beta}{2\beta} & \text{if } \frac{1}{2} \leqslant \beta < 1 \end{Bmatrix},$$

then

$$\Re e\left(\frac{zf'(z)}{f(z)}\right) > p\beta, \quad i.e., \quad f(z) \in \mathscr{S}_p(p\beta),$$

where $z \in \mathbf{U},\ p \in \mathscr{N}$ *and* $0 \leqslant \beta < 1$.

COROLLARY 3. *If a function* $f(z) \in \mathscr{A}(p)$ *satisfies any one of the cases of the following inequalities:*

$$\Re e\left[z\left(\frac{f'''(z)}{f''(z)} - \frac{f''(z)}{f'(z)}\right)\right] \begin{Bmatrix} > -1 - \frac{\beta}{2(1-\beta)} & \text{if } 0 \leqslant \beta \leqslant \frac{1}{2} \\ > -1 - \frac{1-\beta}{2\beta} & \text{if } \frac{1}{2} \leqslant \beta < 1 \end{Bmatrix},$$

then

$$\Re e\left(\frac{zf''(z)}{f'(z)}\right) > \beta(p-1), \quad i.e., \quad f(z) \in \mathscr{C}_p(1+\beta(p-1)),$$

where $z \in \mathbf{U},\ p \in \mathscr{N}$ *and* $0 \leqslant \beta < 1$.

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