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BOUNDEDNESS OF GENERALIZED RIESZ POTENTIALS ON SPACES OF HOMOGENEOUS TYPE

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Abstract. The authors introduce a class of generalized Riesz potentials with kernels having weak regularity on spaces of homogeneous type in the sense of Coifman and Weiss and establish their boundedness on Lebesgue spaces and Hardy spaces. As applications, the authors obtain the boundedness on Lebesgue spaces and Hardy spaces of commutators generated by Lipschitz functions and generalized Riesz potentials or Calderón-Zygmund operators with kernels having weak regularity on spaces of homogeneous type.

1. Introduction

Let Δ be the Laplacian on $\mathbb{R}^n$. It is well known that the Riesz potential, $(-\Delta)^{-\alpha/2}$ with $\alpha \in (0,n)$, is a useful tool in a variety of problems in analysis such as Partial Differential Equations and Harmonic Analysis; see [24, 25, 5, 9, 27, 26, 8]. Riesz potentials on metric measure spaces also attract a lot of attention; see, for example, [10, 11, 21, 22, 29].

In this paper, motivated by Kurtz [16], we introduce a class of generalized Riesz potentials with kernels having weak regularity on spaces of homogeneous type in the sense of Coifman and Weiss [3, 4], and obtain their boundedness on Lebesgue spaces and Hardy spaces. As applications, we also obtain the boundedness on Lebesgue spaces and Hardy spaces of commutators generated by Lipschitz functions and generalized Riesz potentials or Calderón-Zygmund operators with kernels having weak regularity on spaces of homogeneous type.

We first recall some basic facts on spaces of homogeneous type. Let $\mathscr{X}$ be a set. Endow $\mathscr{X}$ with a positive Borel regular measure μ and a quasi-metric d satisfying that there exists $C_1 \geqslant 1$ such that for all $x, y, z \in \mathscr{X}$,

$$d(x,y) \leqslant C_1(d(x,z)+d(y,z)). \tag{1.1}$$

The triple $(\mathscr{X}, d, \mu)$ is called a space of homogeneous type in the sense of Coifman and Weiss ([3, 4]) if μ is doubling, namely, there exists $C_2 \geqslant 1$ such that for all $x \in \mathscr{X}$ and $r > 0$,

$$\mu(B_d(x,2r)) \leqslant C_2\mu(B_d(x,r)), \tag{1.2}$$

where $B_d(x,r) = \{y \in \mathscr{X} : d(x,y) < r\}$.

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We remark that although all balls defined by d satisfy the axioms of complete system of neighborhoods in $\mathscr{X}$, and therefore induce a (separated) topology in $\mathscr{X}$, the balls $B_d(x,r)$ for $x \in \mathscr{X}$ and $r > 0$ need not to be open with respect to this topology. However, by Theorem 2 in [19], we know that there exists a quasi-metric $\widetilde{d}$ such that $\widetilde{d}$ is equivalent to d and the balls corresponding to $\widetilde{d}$ are open in the topology induced by $\widetilde{d}$. Based on this, in what follows, we always assume that the balls corresponding to d are open in the topology induced by d. Otherwise, we replace d by $\widetilde{d}$, since all results in this paper are invariant for equivalent quasi-metrics. Throughout this paper, we also assume that $\mu(\mathscr{X}) = \infty$ and $\mu(\{x\}) = 0$ for all $x \in \mathscr{X}$.

Recall that the measure distance ρ, induced by the quasi-metric d and the measure μ, is defined by that for all $x,\ y \in \mathscr{X}$,

$$\rho(x,y) = \inf\{\mu(B_d): \quad B_d \text{ is a ball containing } x \text{ and } y\};$$

see [4, 19]. Macías and Segovia [19] proved that if the balls corresponding to d are open in the topology induced by d, then ρ is a quasi-metric where we denote by C_3 the corresponding constant in (1.1), the topologies on $\mathscr{X}$ induced by d and ρ coincide; moreover, there exists $C_4 \geqslant 1$ such that for all $x \in \mathscr{X}$ and $r > 0$,

$$C_4^{-1} r \leqslant \mu(B_\rho(x,r)) \leqslant C_4 r; \tag{1.3}$$

see Theorem 3 in [19]. We conveniently mention that if μ and ρ satisfy (1.3), then the triple $(\mathscr{X},\rho,\mu)$ is called to be normal; see [19, p. 258]. In general, ρ is not equivalent to d. We recall that the quasi-metric ρ is said to be equivalent to the quasi-metric d if there exists $C > 0$ such that for all $x,\ y \in \mathscr{X}$, $C^{-1}d(x,y) \leqslant \rho(x,y) \leqslant Cd(x,y)$. Macías and Segovia in [19, Theorem 2] proved that there exists a quasi-metric $\widetilde{\rho}$ on $\mathscr{X}$ which is equivalent to ρ and satisfies that there exist constants $\theta \in (0,1)$ and $C > 0$ such that for all $x,\ x',\ y \in \mathscr{X}$,

$$|\widetilde{\rho}(x,y) - \widetilde{\rho}(x',y)| \leqslant C[\widetilde{\rho}(x,x')]^\theta [\widetilde{\rho}(x,y) + \widetilde{\rho}(x',y)]^{1-\theta}. \tag{1.4}$$

Noticing again that all the conclusions in this paper are invariant for equivalent quasi-metrics, thus, if it is necessary, we may also assume that ρ itself satisfies (1.4). In the sequel, θ is always taken to be the same as in (1.4). Moreover, by the proof of Theorem 2 in [19], we know that θ in (1.4) can be taken to be $1/\log_2[C_3(2C_3+1)]$.

Motivated by [16] on $\mathbb{R}^n$ (see also [13, 23]), we introduce the following classes of kernels with weak regularity of generalized Riesz potentials on $(\mathscr{X},\rho,\mu)$.

DEFINITION 1.1. Let $\kappa \in [1,\infty)$ and K be a locally integrable function on $\mathscr{X} \times \mathscr{X} \setminus \{(x,x):\ x \in \mathscr{X}\}$.

(i) The function K is said to be in $D_\rho(\kappa,\gamma)$ with $\gamma \in [1,\infty)$ if there exist constants $C_K \geqslant 2C_3$ and $C > 0$ such that for all $x,\ y \in \mathscr{X}$,

$$\left\{\int_{\rho(x,z)>C_K\rho(x,y)} |K(z,x) - K(z,y)|^\gamma \, d\mu(z)\right\}^{1/\gamma} \leqslant C[\rho(x,y)]^{1/\gamma - 1/\kappa}.$$

(ii) Let $\eta = \{\eta_j\}_{j\in\mathbb{N}} \subset [0,\infty)$. The function K is said to be in $D_\rho(\kappa,\gamma,\eta)$ with $\gamma \in [1,\infty]$ if there exists a constant $C_K \geqslant 2C_3$ such that for all $x,\ y \in \mathscr{X}$ and $j \in \mathbb{N}$,

$$\left\{\int_{R_j(B_\rho(x,C_K\rho(x,y)))} |K(z,x)-K(z,y)|^\gamma \, d\mu(z)\right\}^{1/\gamma} \leqslant \eta_j [2^j C_K \rho(x,y)]^{1/\gamma - 1/\kappa},$$

where and in what follows, $R_j(B_\rho(x,r)) = B_\rho(x,2^{j+1}r) \setminus B_\rho(x,2^j r)$ for all $x \in \mathscr{X}$ and $r > 0$, and the usual modification is made when $\gamma = \infty$.

We now give some examples of kernels satisfying Definition 1.1 on $\mathbb{R}^n$. Let Ω be homogeneous of zero on $\mathbb{R}^n$ and ω_q be its $L^q(S^{n-1})$ integral modulus of continuity. If $\kappa \in [1,n)$, $q \in (\kappa,\infty]$ and

$$K(x,y) = |x-y|^{-n/\kappa}\Omega(x-y)$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x):\ x \in \mathbb{R}^n\}$ with Ω satisfies the L^q-Dini condition, namely, $\int_0^1 \omega_q(s)s^{-1}\,ds < \infty$, then $K \in D_\rho(\kappa,\kappa)$; if $\kappa \in [1,n/(n-1))$, $q \in (\kappa,\infty]$ and $\int_0^1 \omega_q(s)s^{-[1+n(1-1/\kappa)]}\,ds < \infty$, then $K \in D_\rho(\kappa,\gamma)$ for any $\gamma \in [1,\kappa)$; and if $\kappa \in [1,2n/(2n-1))$, $\widetilde{\kappa} \in (\kappa,n/(n-1)]$, $q \in (1/[1+2n(1/\kappa-1)],\infty]$ and $\int_0^1 \omega_q(s)s^{-[1+n(1-1/\widetilde{\kappa})]}\,ds < \infty$, then $K \in D_\rho(\kappa,\gamma,\eta)$ with $\eta_j \lesssim 2^{-j\varepsilon_\gamma}$ for any $\gamma \in [1,\kappa)$, certain $\varepsilon_\gamma \in (0,1/\kappa - 1/\widetilde{\kappa})$ and all $j \in \mathbb{N}$; see [25, 23, 6].

Let $\kappa_0 \in [1,\infty)$, $p_0 \in (1,\infty)$ and $1/q_0 = 1/p_0 + 1/\kappa_0 - 1$. A linear operator T is called a generalized Riesz potential if T is bounded from $L^{p_0}(\mathscr{X})$ to $L^{q_0}(\mathscr{X})$ with kernel K as in Definition 1.1; moreover, T satisfies that for any $f \in L^{p_0}(\mathscr{X})$ with bounded support and $x \notin \operatorname{supp} f$,

$$Tf(x) = \int_{\mathscr{X}} K(x,y)f(y)\,d\mu(y). \tag{1.5}$$

We first point out that on $\mathbb{R}^n$ with the Euclidean metric $|\cdot|$ and the n-dimensional Lebesgue measure μ, if T is bounded from $L^{p_0}(\mathbb{R}^n)$ to $L^{q_0}(\mathbb{R}^n)$ with kernel $K \in D_\rho(\kappa_0,\kappa_0)$, where $\rho(x,y) = |x-y|^n$ for all $x,\ y \in \mathbb{R}^n$, then Hörmander [13] proved that T is bounded from $L^1(\mathbb{R}^n)$ to weak-$L^{\kappa_0}(\mathbb{R}^n)$ with $\kappa_0 \in [1,\infty)$; if $K \in D_\rho(\kappa_0,\gamma,\eta)$ with $\kappa_0 \in (1,\infty)$, $\gamma \in [\kappa_0,\infty]$ and $\{\eta_j\}_{j\in\mathbb{N}}$ being increasing such that $\sum_{j=1}^\infty \eta_j < \infty$, then Kurtz [16] obtained the boundedness of T on weighted Lebesgue spaces when $1 < p < \kappa_0/(\kappa_0-1)$; if $K(x,y) = |x-y|^{-n/\kappa_0}\Omega(x-y)$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x):\ x \in \mathbb{R}^n\}$ with $\kappa_0 \in (1,\infty)$, and Ω being homogeneous of zero and satisfying certain type of Dini condition, then Ding and Lu [6, 7] established the boundedness of T on Hardy spaces. On $(\mathscr{X},\rho,\mu)$, Gatto and Vagi [10] established the boundedness on Lebesgue spaces and Hardy spaces of Riesz potentials with kernel $K(x,y) = \rho(x,y)^{1-\beta}$ and $\beta \in (0,1)$; Coifman and Weiss [3] proved the boundedness from $L^1(\mathscr{X})$ to weak-$L^1(\mathscr{X})$ and from $H^1(\mathscr{X})$ to $L^1(\mathscr{X})$ of T with $K \in D_\rho(1,1)$; and the boundedness on Hardy spaces of T with kernel $K \in D_\rho(1,\gamma,\eta)$ for certain $\gamma \in [1,\infty]$ and η is also considered in [14] via certain molecular characterization of Hardy spaces related to η.

The main results of this paper are the following boundedness conclusions of T with kernel $K \in D_\rho(\kappa_0,\gamma)$ or $K \in D_\rho(\kappa_0,\gamma,\eta)$ and $\kappa_0 > 1$.

THEOREM 1.1. *Let $\kappa_0 \in (1,\infty)$, $p_1 \in [\kappa_0/(2\kappa_0-1),1] \cap [1/(1+\theta),1]$, $1/q_1 = 1/p_1 + 1/\kappa_0 - 1$, T be a linear bounded operator from $L^{p_0}(\mathscr{X})$ to $L^{q_0}(\mathscr{X})$ with certain $p_0 \in (1,\infty)$ and $1/q_0 = 1/p_0 + 1/\kappa_0 - 1$ and T have a kernel $K \in D_\rho(\kappa_0,q_1)$ as in (1.5). For $p \in [p_1,p_0)$, let $1/q = 1/p + 1/\kappa_0 - 1$. Then T is bounded from $H^p(\mathscr{X})$ to $L^q(\mathscr{X})$ for $p \in [p_1,1]$ and from $L^p(\mathscr{X})$ to $L^q(\mathscr{X})$ for $p \in (1,p_0)$; moreover, if $p_1 = 1$, then T is also bounded from $L^1(\mathscr{X})$ to weak-$L^{\kappa_0}(\mathscr{X})$.*

Let T^* denote the dual operator of T. Then the boundedness of T from $L^{p_0}(\mathscr{X})$ to $L^{q_0}(\mathscr{X})$ implies the boundedness of T^* from $L^{q_0'}(\mathscr{X})$ to $L^{p_0'}(\mathscr{X})$.

From this and Theorem 1.1, it also follows that T^* is bounded from $L^{q'}(\mathscr{X})$ to $\mathrm{Lip}(1/p-1)$ for $p \in [p_1,1)$ and from $L^{\kappa_0'}(\mathscr{X})$ to $\mathrm{BMO}(\mathscr{X})$.

It is easy to see that $1/(1+\theta) = \kappa_0/(2\kappa_0-1)$ if and only if $\kappa_0 = 1/(1-\theta)$. In Theorem 1.1, if $\kappa_0 < 1/(1-\theta)$, then $p_1 \in [\kappa_0/(2\kappa_0-1),1]$, and if $\kappa_0 \geqslant 1/(1-\theta)$, then $p_1 \in [1/(1+\theta),1]$.

When $\kappa_0 < 1/(1-\theta)$, we have the following results. In what follows, $T^*(1) = 0$ means that for any $a \in L^1(\mathscr{X})$ with bounded support and $\int_{\mathscr{X}} a(x)\,d\mu(x) = 0$, $\int_{\mathscr{X}} Ta(x)\,d\mu(x) = 0$.

THEOREM 1.2. *Let $1 < \kappa_0 < 1/(1-\theta)$, $p_1 \in [1/(1+\theta),\ \kappa_0/(2\kappa_0-1)]$, $1/q_1 = 1/p_1 + 1/\kappa_0 - 1$, $\gamma \in [1,\infty]$ and $\eta = \{\eta_j\}_{j\in\mathbb{N}} \subset [0,\infty)$ satisfying that $\sum_{j\in\mathbb{N}} 2^{j(1-q_1)}(\eta_j)^{q_1} < \infty$ when $p_1 < \kappa_0/(2\kappa_0-1)$, or $\gamma \in (1,\infty]$ and $\eta = \{\eta_j\}_{j\in\mathbb{N}} \subset [0,\infty)$ satisfying that $\sum_{j\in\mathbb{N}} j\eta_j < \infty$ when $p_1 = \kappa_0/(2\kappa_0-1)$. Let T be a linear bounded operator from $L^{p_0}(\mathscr{X})$ to $L^{q_0}(\mathscr{X})$ with certain $p_0 \in (1,\infty)$ and $1/q_0 = 1/p_0 + 1/\kappa_0 - 1$ and T have a kernel $K \in D_\rho(\kappa_0,\gamma,\eta)$ as in (1.5). For $p \in [p_1,p_0)$, let $1/q = 1/p + 1/\kappa_0 - 1$. Then T is bounded from $H^p(\mathscr{X})$ to $L^q(\mathscr{X})$ for $p \in [p_1,1]$, from $L^p(\mathscr{X})$ to $L^q(\mathscr{X})$ for $p \in (1,p_0)$; if further assume that $T^*(1) = 0$, then T is also bounded from $H^p(\mathscr{X})$ to $H^q(\mathscr{X})$ for $p \in [p_1,\kappa_0/(2\kappa_0-1)]$.*

We remark that Theorem 1.1 and Theorem 1.2 on $\mathbb{R}^n$ imply Theorem 1, Theorem 2 and Theorem 3 in [6].

If $T^*(1) = 0$, from Theorem 1.2, it follows that T^* is bounded from $\mathrm{BMO}(\mathscr{X})$ to $\mathrm{Lip}(1-1/\kappa_0)$ and from $\mathrm{Lip}(1/q-1)$ to $\mathrm{Lip}(1/p-1)$ when $p \in [p_1,\kappa_0/(2\kappa_0-1))$.

By the definition of the Hardy space $H^q(\mathscr{X})$ with $q \in (0,1]$, it is easy to see that $T^*(1) = 0$ is also necessary for T to be bounded from $H^p(\mathscr{X})$ to $H^q(\mathscr{X})$.

Notice that if $\eta = 2^{-j\varepsilon}$, then it is easy to see that $\sum_{j\in\mathbb{N}}(\eta_j)^{q_1}2^{j(1-q_1)} < \infty$ if and only if $q_1 > 1/(1+\varepsilon)$. Let $\kappa_0 \in [1,1/(1-\theta))$, $\varepsilon \in (0,\theta+1/\kappa_0-1]$, $q_1 = 1/(1+\varepsilon)$ and $\eta_j \leqslant C2^{-j\varepsilon}$ for all $j \in \mathbb{N}$ and certain constant $C > 0$. When $\kappa_0 = 1$, if $K \in D_\rho(\kappa_0,\infty,\eta)$, then the boundedness from $H^{1/(1+\varepsilon)}(\mathscr{X})$ to weak-$L^{1/(1+\varepsilon)}(\mathscr{X})$ of T was established in [14]. It is still unclear if there is any similar result when $\kappa_0 \in (1,1/(1-\theta))$.

The proofs of Theorem 1.1 and Theorem 1.2 are given in Section 3 via some general criteria for boundedness of linear operators on Hardy spaces, which were established in [14] via the molecular characterization of Hardy spaces closely related to the kernel of the considered operator T, and are stated in Section 2 for the reader's convenience. In Section 2, we also recall some basic facts about atomic Hardy spaces of Coifman and Weiss on spaces of homogeneous type; moreover, we establish an interpolation theorem on boundedness of operators on $L^p(\mathscr{X})$ when $p \in (1,\infty)$ or $H^p(\mathscr{X})$ when $p \in (1/(1+\theta),1]$, which may have independent interest; see Theorem 2.1 below.

Applying Theorem 1.1 and Theorem 1.2, we in Section 4 establish the boundedness on Lebesgue spaces and Hardy spaces of commutators generated by Lipschitz functions and singular integrals or generalized Riesz potentials with kernels having the weak regularity as in Definition 1.1 (ii) on spaces of homogeneous type; see Proposition 4.2 below. We should point out that Janson [15] first discussed the boundedness on Lebesgue spaces of commutators generated by Calderón-Zygmund operators and Lipschitz functions on $\mathbb{R}^n$. More references on this topic can be found in [18]. To be interesting, if $b \in \mathrm{Lip}(\beta)$ for certain $\beta \in (0,1/\kappa_0)$ with $\kappa_0 \in [1,\infty)$ and T is a bounded linear operator from $L^{p_0}(\mathscr{X})$ to $L^{q_0}(\mathscr{X})$ with kernel $K \in D_\rho(\kappa_0,\gamma,\eta)$ for certain p_0, $q_0 \in (1,\infty)$, $\gamma\kappa_0 \in [1,\infty)$, and sequence η, we then in Section 4 prove that the commutator $[b,T]$ is also a bounded linear operator from $L^{p_1}(\mathscr{X})$ to $L^{q_1}(\mathscr{X})$ with kernel $\widetilde{K} \in D_\rho(\kappa_1,\gamma,\widetilde{\eta})$ for certain κ_1, p_1, $q_1 \in (1,\infty)$ and sequence $\widetilde{\eta}$, namely, $[b,T]$ is also a generalized Riesz potential considered as above; see Proposition 4.1 below. This approach is different from the known approach used for such commutators on Euclidean spaces; see [18, 15].

We finally make some conventions. Throughout this paper, for any $p \in [1,\infty]$, let $1/p' + 1/p = 1$. We always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line, and $f \lesssim g$ means $f \leqslant Cg$. If $f \lesssim g \lesssim f$, we then write $f \sim g$. Constants with subscripts, such as C_1, do not change in different occurrences. For any given quasi-normed linear spaces $\mathscr{Y}$ and $\mathscr{Z}$ and a linear operator T which maps $\mathscr{Y}$ into $\mathscr{Z}$, T is said to be bounded from $\mathscr{Y}$ to $\mathscr{Z}$, if there exists a positive constant C such that for all $f \in \mathscr{Y}$, $Tf \in \mathscr{Z}$ and $\|Tf\|_{\mathscr{Z}} \leqslant C\|f\|_{\mathscr{Y}}$.

2. Preliminaries

We begin with the definition of atomic Hardy spaces on $(\mathscr{X},d,\mu)$ in [4]. To this end, we first recall the definitions of Lipschitz spaces, the space of functions with bounded mean oscillation and atoms; see [4].

DEFINITION 2.1. Let $\alpha > 0$. A function f is said to be in $\mathrm{Lip}_d(\alpha)$ if there exists $C \geqslant 0$ such that for all x, $y \in \mathscr{X}$ and all balls B_d containing x and y,

$$|f(x) - f(y)| \leqslant C[\mu(B_d)]^\alpha. \tag{2.1}$$

The minimal constant C in (2.1) is defined to be the $\mathrm{Lip}_d(\alpha)$ norm of f and denoted by $\|f\|_{\mathrm{Lip}_d(\alpha)}$.

DEFINITION 2.2. Let $1 \leqslant q < \infty$. A function f is said to be in $\mathrm{BMO}^q(\mathscr{X}, d, \mu)$ if there exists $C \geqslant 0$ such that for all balls $B_d \subset \mathscr{X}$,

$$\left\{ \frac{1}{\mu(B_d)} \int_{B_d} |f(x) - f_{B_d}|^q \, d\mu(x) \right\}^{1/q} \leqslant C, \tag{2.2}$$

where and in the sequel, $f_{B_d} = \frac{1}{\mu(B_d)} \int_{B_d} f(y)\, d\mu(y)$. The minimal constant C in (2.2) is defined to be the $\mathrm{BMO}^q(\mathscr{X}, d, \mu)$ norm of f and denoted by $\|f\|_{\mathrm{BMO}^q(\mathscr{X}, d, \mu)}$.

Denote $\mathrm{BMO}^1(\mathscr{X}, d, \mu)$ simply by $\mathrm{BMO}(\mathscr{X}, d, \mu)$. It is well-known that for $1 < q < \infty$, $\mathrm{BMO}(\mathscr{X}, d, \mu) = \mathrm{BMO}^q(\mathscr{X}, d, \mu)$ with equivalent norms; see [4].

DEFINITION 2.3. Let $0 < p < q$ and $p \leqslant 1 \leqslant q \leqslant \infty$. A function a is called a $(p,q)_d$-atom if

(A1) $\operatorname{supp} a \subset B_d = B_d(x, r)$ for certain $x \in \mathscr{X}$ and $r > 0$;
(A2) $\|a\|_{L^q(\mathscr{X})} \leqslant [\mu(B_d)]^{1/q - 1/p}$;
(A3) $\int_{\mathscr{X}} a(x)\, d\mu(x) = 0$.

Now we state the definition of atomic Hardy spaces. For $\alpha > 0$, let $(\mathrm{Lip}_d(\alpha))^*$ be the dual space of $\mathrm{Lip}_d(\alpha)$.

DEFINITION 2.4. Let $0 < p < q$ and $p \leqslant 1 \leqslant q \leqslant \infty$. A function $f \in L^1(\mathscr{X})$ when $p = 1$ or a linear functional $f \in (\mathrm{Lip}_d(1/p - 1))^*$ when $p < 1$ is said to be in $H^{1,q}(\mathscr{X}, d, \mu)$ when $p = 1$ or in $H^{p,q}(\mathscr{X}, d, \mu)$ when $p < 1$ if there exist $(p,q)_d$-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$, which converges in $L^1(\mathscr{X})$ when $p = 1$ or in $(\mathrm{Lip}_d(1/p - 1))^*$ when $p < 1$, and $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. Moreover, the norm of f in $H^{p,q}(\mathscr{X}, d, \mu)$ is defined by $\|f\|_{H^{p,q}(\mathscr{X}, d, \mu)} = \inf\{(\sum_{j=1}^\infty |\lambda_j|^p)^{1/p}\}$, where the infimum is taken over all the above decompositions of f.

Coifman and Weiss proved that $H^{p,q}(\mathscr{X}, d, \mu) = H^{p,\infty}(\mathscr{X}, d, \mu)$ for $0 < p < q$ and $p \leqslant 1 \leqslant q \leqslant \infty$, $(H^{1,q}(\mathscr{X}, d, \mu))^* = \mathrm{BMO}(\mathscr{X}, d, \mu)$ for $1 < q \leqslant \infty$, and $(H^{p,q}(\mathscr{X}, d, \mu))^* = \mathrm{Lip}_d(1/p - 1)$ for $0 < p < 1 \leqslant q \leqslant \infty$; see Theorem A and Theorem B in [4]. Therefore, in what follows, we denote $H^{p,q}(\mathscr{X}, d, \mu)$ simply by $H^p(\mathscr{X}, d, \mu)$.

If we replace d by ρ in Definition 2.1 through Definition 2.4, we then obtain $\mathrm{Lip}_\rho(\alpha)$, $\mathrm{BMO}^q(\mathscr{X}, \rho, \mu)$, $(p,q)_\rho$-atoms and atomic Hardy spaces $H^{p,q}(\mathscr{X}, \rho, \mu)$. All the conclusions stated above still hold for $H^{p,q}(\mathscr{X}, \rho, \mu)$, $\mathrm{BMO}^q(\mathscr{X}, \rho, \mu)$ and $\mathrm{Lip}_\rho(1/p - 1)$. Thus, in what follows, we denote $H^{p,q}(\mathscr{X}, \rho, \mu)$ simply by $H^p(\mathscr{X}, \rho, \mu)$.

Generally speaking, for two topologically equivalent spaces of homogeneous type, the corresponding Hardy spaces are not necessary to be equivalent; see, for example, [1, Theorem 10.5]. We recall that two quasi-Banach spaces $\mathscr{B}_1$ and $\mathscr{B}_2$ are said to be equivalent if they are equal as a set and their norms are equivalent. However, for all $\alpha > 0$, Macías and Segovia [19] proved that $\mathrm{Lip}_\rho(\alpha)$ and $\mathrm{Lip}_d(\alpha)$ are equivalent. For $p \in (0, 1]$, it was proved in [14] that $H^p(\mathscr{X}, \rho, \mu)$ and $H^p(\mathscr{X}, d, \mu)$ are equivalent, which was also mentioned in [4, p. 594] and [20, p. 271]. By the dual theory, we also have $\mathrm{BMO}(\mathscr{X}, d, \mu) = \mathrm{BMO}(\mathscr{X}, \rho, \mu)$ with equivalent norms. Thus in what follows, we denote them simply by $\mathrm{Lip}(\alpha)$, $H^p(\mathscr{X})$ and $\mathrm{BMO}(\mathscr{X})$; respectively.

The following kind of molecules in [14] is closely related to the classes of kernels in Definition 1.1 (ii).

DEFINITION 2.5. Let $0 < p < q$, $p \leqslant 1 \leqslant q \leqslant \infty$ and $\eta = \{\eta_k\}_{k\in\mathbb{N}} \subset [0,\infty)$ satisfying

$$\sum_{k=1}^{\infty} k\eta_k < \infty, \tag{2.3}$$

or when $p < 1$,

$$\sum_{k=1}^{\infty} (\eta_k)^p 2^{k(1-p)} < \infty. \tag{2.4}$$

A function $M \in L^q(\mathscr{X})$ is called a $(p,q,\eta)_\rho$-molecule centered at a ball B_ρ if
(M1) $\|M\|_{L^q(\mathscr{X})} \leqslant [\mu(B_\rho)]^{1/q-1/p}$;
(M2) for all $k \in \mathbb{N}$, $\|M\chi_{R_k(B_\rho)}\|_{L^q(\mathscr{X})} \leqslant \eta_k 2^{k(1/q-1)}[\mu(B_\rho)]^{1/q-1/p}$;
(M3) $\int_{\mathscr{X}} M(x)\,d\mu(x) = 0$.

REMARK 2.1. (a) From Definition 2.4 and Definition 2.5, it is easy to see that if a is a $(p,q)_\rho$-atom supported in a ball B_ρ, then a is a $(p,q,\eta)_\rho$-molecule centered at the same ball B_ρ. Conversely, if $\eta_k = 0$ for all $k \in \mathbb{N}$, then a $(p,q,\eta)_\rho$-molecule is just a $(p,q)_\rho$-atom.

(b) By Definition 2.5, it is easy to see that if $q_1 < q_2$ and M is a $(p,q_2,\eta)_\rho$-molecule, then there exists a constant $C > 0$ independent of M such that $\frac{1}{C}M$ is a $(p,q_1,\eta)_\rho$-molecule.

(c) Theorem 2.2 in [14] characterizes $H^p(\mathscr{X})$ with $p \in (0,1]$ by molecules in Definition 2.5; moreover, this characterization is sharp when $p < 1$.

To establish the boundedness of operators on Hardy spaces on $\mathscr{X}$, we need the following bounded criteria established in Theorem 3.2 and Corollary 3.1 of [14] (see also [28, 12] for some more general results on this topic).

LEMMA 2.1. *Let $p_0,\ q_0 \in [1,\infty)$, $p \in [1/(1+\theta),1]$, $q \in [1/(1+\theta),\infty)$ and T be a linear operator bounded from $L^{p_0}(\mathscr{X})$ to $L^{q_0}(\mathscr{X})$.*

(i) If $q \in [1,\infty)$ and there exists a positive constant C such that for all (p,∞)-atoms a, $\|Ta\|_{L^q(\mathscr{X})} \leqslant C$, then T is bounded from $H^p(\mathscr{X})$ to $L^q(\mathscr{X})$.

(ii) If $q \in [p,1]$, and there exists a positive constant C such that for all (p,∞)-atoms a, $\|Ta\|_{H^q(\mathscr{X})} \leqslant C$, then T is bounded from $H^p(\mathscr{X})$ to $H^q(\mathscr{X})$. Especially, if there exists $\widetilde{q} \in [1,\infty)$ and η satisfying $\sum_{j\in\mathbb{N}}(\eta_j)^q 2^{j(1-q)} < \infty$ when $q < 1$, or $\widetilde{q} \in (1,\infty)$ and η satisfying $\sum_{j\in\mathbb{N}} j\eta_j < \infty$ when $q = 1$, such that for all $(p,\infty)_\rho$-atoms, $\frac{1}{C}Ta$ is a $(q,\widetilde{q},\eta)_\rho$-molecule, then T is bounded from $H^p(\mathscr{X})$ to $H^q(\mathscr{X})$.

(iii) If $q \in [p,1)$, and there exist positive constants C and η satisfying $\sum_{j\in\mathbb{N}}(\eta_j)^q 2^{j(1-q)} < \infty$ such that for any (p,∞)-atom a, $\frac{1}{C}Ta$ satisfies conditions (M1) and (M2) of $(q,1,\eta)_\rho$-molecules in Definition 2.5, then T is bounded from $H^p(\mathscr{X})$ to $L^q(\mathscr{X})$.

REMARK 2.2. We point out that Lemma 2.1 (i) still holds for $p \in (1/(1+\theta), 1]$ and $q \in (1/(1+\theta), 1]$. In fact, by Theorem 5.6 of [12], if $\|Ta\|_{L^q(\mathscr{X})} \leqslant C$ for all continuous $(p,\infty)_\rho$-atoms, then T extends a bounded linear operator from $H^p(\mathscr{X})$ to $L^q(\mathscr{X})$. By the boundedness of T from $L^{p_0}(\mathscr{X})$ to $L^{q_0}(\mathscr{X})$, we know that this extension coincides with T on all $(p,\infty)_\rho$ atoms. By the same reason, the results of Lemma 2.1 can also be deduced from Theorem 5.6 of [12] when $p \in (1/(1+\theta), 1]$, but not when $p = 1/(1+\theta)$.

Finally, in this section, we establish an interpolation theorem for linear operators, which is used in the next section and may have independent interesting.

THEOREM 2.1. *Let* $1/(1+\theta) \leqslant p_1 \leqslant 1 < p_2 < \infty$, $q_1 \in [p_1,\infty)$ *and* $q_2 \in [p_2,\infty)$ *satisfying* $1/q_1 - 1/p_1 = 1/q_2 - 1/p_2$. *Let* T *be a bounded linear operator from* $H^{p_1}(\mathscr{X})$ *to weak-*$L^{q_1}(\mathscr{X})$ *and from* $L^{p_2}(\mathscr{X})$ *to weak-*$L^{q_2}(\mathscr{X})$.

(i) *If* $p \in (1,p_2)$ *and* $q \in [p,\infty)$ *satisfy* $1/q_1 - 1/p_1 = 1/q - 1/p$, *then* T *is bounded from* $L^p(\mathscr{X})$ *to* $L^q(\mathscr{X})$.

(ii) *If* $p \in (p_1,1]$ *and* $q \in [p,q_2)$ *satisfy* $1/q_1 - 1/p_1 = 1/q - 1/p$, *then* T *is bounded from* $H^p(\mathscr{X})$ *to* $L^q(\mathscr{X})$.

Proof. To show (i), let $p \in (1,p_2)$, and $f \in L^\infty(\mathscr{X})$ with bounded support and $\|f\|_{L^p(\mathscr{X})} \leqslant 1$. Let $\widetilde{p} = (1+p)/2$, $\alpha > 0$ and $\Omega^\alpha = \{x \in \mathscr{X} : \mathscr{M}_{\widetilde{p}}(f)(x) > \alpha\}$, where $\mathscr{M}_{\widetilde{p}}(f) = [\mathscr{M}(|f|^{\widetilde{p}})]^{1/\widetilde{p}}$ and $\mathscr{M}(f)$ denotes the Hardy-Littlewood maximal function of f. From the $L^{p/\widetilde{p}}(\mathscr{X})$-boundedness of $\mathscr{M}$ (see [3]), we deduce that

$$\mu(\Omega^\alpha) \lesssim \alpha^{-p} \||f|^{\widetilde{p}}\|^{p/\widetilde{p}}_{L^{p/\widetilde{p}}(\mathscr{X})} \lesssim \alpha^{-p}\|f\|^p_{L^p(\mathscr{X})} < \infty.$$

From this, it is easy to see that Ω^α is an open and bounded set by the definition of $\mathscr{M}$ and Lemma 3.9 in [4], respectively. Applying the Whitney type covering lemma (Theorem 3.2 in [4]), we obtain a collection of balls, $\{B_j^\alpha = B_j^\alpha(x_j^\alpha, r_j^\alpha)\}_j$ such that $\Omega^\alpha = \cup_j B_j^\alpha$, $(3C_1B_j^\alpha) \cap (\mathscr{X} \setminus \Omega^\alpha) \neq \emptyset$ and each $x \in \mathscr{X}$ is contained in at most N balls for certain $N \in \mathbb{N}$ independent of α and f, where $3C_1B_j^\alpha = B_j^\alpha(x_j^\alpha, 3C_1r_j^\alpha)$.

Set $\chi_j^\alpha = \chi_{B_j^\alpha}$, $\eta_j^\alpha = \chi_j^\alpha(\sum_j \chi_j^\alpha)^{-1}$, $g^\alpha = \sum_j (\eta_j^\alpha f)_{B_j^\alpha}\chi_j^\alpha + f\chi_{\mathscr{X}\setminus\Omega^\alpha}$, $b_j^\alpha = f\eta_j^\alpha - (\eta_j^\alpha f)_{B_j^\alpha}\chi_j^\alpha$, where $(\eta_j^\alpha f)_{B_j^\alpha} = [\mu(B_j^\alpha)]^{-1}\int_{B_j^\alpha} \eta_j^\alpha(x) f(x)\, d\mu(x)$, and $b = \sum_j b_j^\alpha$. Then $f = g^\alpha + b^\alpha$. By an argument similar to that used in [3, 4], there exists a constant $C > 0$, independent of α and f, such that

(I) $|g^\alpha(x)| \leqslant C\alpha$ for all $x \in \mathscr{X}$;

(II) $\sum_j \mu(B_j^\alpha) \leqslant C(\|f\|_{L^p(\mathscr{X})}/\alpha)^p$;

(III) $\|b_j^\alpha\|_{L^{\widetilde{p}}(\mathscr{X})} \leqslant C\alpha[\mu(B_j^\alpha)]^{1/\widetilde{p}}$;

(IV) $\int_{B_j^\alpha} b_j^\alpha(x)\, d\mu(x) = 0$ and $\sum_j \|b_j^\alpha\|^p_{L^p(\mathscr{X})} \leqslant C\|f\|^p_{L^p(\mathscr{X})}$.

Set $a_j^\alpha = [C\alpha\mu(B_j^\alpha)^{1/p_1}]^{-1} b_j^\alpha$. Notice that $\operatorname{supp} b_j^\alpha \subset B_j^\alpha$. Thus a_j^α is a $(p_1, \widetilde{p})$-atom.

Hence, by (III) and Definition 2.4, we have $b^\alpha = C\alpha\sum_j \mu(B_j^\alpha)^{1/p_1}a_j^\alpha \in H^{p_1}(\mathscr{X})$ and

$$\|b^\alpha\|_{H^{p_1}(\mathscr{X})} \lesssim \alpha\left\{\sum_j \mu(B_j^\alpha)\right\}^{1/p_1} \lesssim \alpha[\mu(\Omega^\alpha)]^{1/p_1}. \tag{2.5}$$

Applying the Minkowski inequality, we have

$$\begin{aligned}
\|Tf\|_{L^q(\mathscr{X})}^q &= \int_0^\infty q\lambda^{q-1}\mu(\{x\in\mathscr{X} : |Tf(x)|>\lambda\})\,d\lambda \\
&= \int_0^\infty p\alpha^{p-1}\mu(\{x\in\mathscr{X} : |Tf(x)|>\alpha^{p/q}\})\,d\alpha \\
&\lesssim \int_0^\infty \alpha^{p-1}\mu(\{x\in\mathscr{X} : |Tg^\alpha(x)|>2^{-1}\alpha^{p/q}\})\,d\alpha \\
&\quad + \int_0^\infty \alpha^{p-1}\mu(\{x\in\mathscr{X} : |Tb^\alpha(x)|>2^{-1}\alpha^{p/q}\})\,d\alpha \\
&\lesssim \int_0^\infty \alpha^{p(q-q_2)/q-1}\|g^\alpha\|_{L^{p_2}(\mathscr{X})}^{q_2}\,d\alpha + \int_0^\infty \alpha^{p(q-q_1)/q-1}\|b^\alpha\|_{H^{p_1}(\mathscr{X})}^{q_1}\,d\alpha \\
&\equiv J_1 + J_2.
\end{aligned}$$

By the Minkowski inequality and $q<q_2$, we obtain

$$\begin{aligned}
J_1 &\sim \int_0^\infty \alpha^{p(q-q_2)/q-1}\left(\int_{\mathscr{X}} |g^\alpha(x)|^{p_2}\chi_{\{y\in\mathscr{X}:\,|g^\alpha(y)|\leqslant C\alpha\}}(x)\,d\mu(x)\right)^{q_2/p_2} d\alpha \\
&\lesssim \left\{\int_{\mathscr{X}} |g^\alpha(x)|^{p_2}\left\{\int_{|g^\alpha(x)|/C}^\infty \alpha^{p(q-q_2)/q-1}\,d\alpha\right\}^{p_2/q_2} d\mu(x)\right\}^{q_2/p_2} \\
&\lesssim \left\{\int_{\mathscr{X}} |g^\alpha(x)|^{p_2+p(q-q_2)p_2/(qq_2)}\,d\mu(x)\right\}^{q_2/p_2},
\end{aligned}$$

which together with (I) & (IV) as above and $1/q_2-1/p_2=1/q-1/p$, implies that

$$J_1 \lesssim \|g^\alpha\|_{L^p(\mathscr{X})}^{pq_2/p_2} \lesssim \|f-b^\alpha\|_{L^p(\mathscr{X})}^{pq_2/p_2} \lesssim \|f\|_{L^p(\mathscr{X})}^{pq_2/p_2} + \|b^\alpha\|_{L^p(\mathscr{X})}^{pq_2/p_2} \lesssim 1.$$

By (2.5), the Minkowski inequality and the $L^{p/\widetilde{p}}$-boundedness of $\mathscr{M}$, we have

$$\begin{aligned}
J_2 &\lesssim \int_0^\infty \alpha^{p(q-q_1)/q-1}\alpha^{q_1}[\mu(\Omega^\alpha)]^{q_1/p_1}\,d\alpha \\
&\lesssim \int_0^\infty \alpha^{p+q_1-pq_1/q-1}\left(\int_{\mathscr{X}}\chi_{\Omega^\alpha}(x)\,d\mu(x)\right)^{q_1/p_1} d\alpha \\
&\lesssim \left\{\int_{\mathscr{X}}\left(\int_0^{\mathscr{M}_{\widetilde{p}}(f)(x)} \alpha^{p+q_1-pq_1/q-1}\,d\alpha\right)^{p_1/q_1} d\mu(x)\right\}^{q_1/p_1} \\
&\lesssim \left\{\int_{\mathscr{X}}\left[\mathscr{M}_{\widetilde{p}}(f)(x)\right]^{(p+q_1-pq_1/q)p_1/q_1}\,d\mu(x)\right\}^{q_1/p_1}.
\end{aligned}$$

Since $1/q_1 - 1/p_1 = 1/q - 1/p$ implies $(p + q_1 - pq_1/q)p_1/q_1 = p$, we then obtain

$$J_2 \lesssim \|\mathscr{M}_{\widetilde{p}}(f)\|_{L^p(\mathscr{X})}^{pq_1/p_1} \lesssim \|f\|_{L^p(\mathscr{X})}^{pq_1/p_1} \lesssim 1.$$

This combined with the estimate for J_1 and a density argument yields (i).

To prove (ii), let $p \in (p_1, 1]$. Then it suffices to prove that $\|Ta\|_{L^q(\mathscr{X})} \lesssim 1$ for all $(p,\infty)_\rho$-atoms a. Assume this for the moment. Let $p_0 \in (1, p_2)$ and $1/q_0 - 1/p_0 = 1/q_1 - 1/p_1$; then by (i), T is bounded from $L^{p_0}(\mathscr{X})$ to $L^{q_0}(\mathscr{X})$. Thus by Lemma 2.1 (i) and Remark 2.2, we obtain the boundedness of T from $H^p(\mathscr{X})$ to $L^q(\mathscr{X})$. Assume that a is a $(p,\infty)_\rho$-atom supported in certain ball B_ρ. Set $b = [\mu(B)]^{1/p-1/p_1}a$. Obviously, b is a $(p_1,\infty)_\rho$-atom and hence $\|a\|_{H^{p_1}(\mathscr{X})} \leqslant [\mu(B)]^{1/p_1-1/p}$. By the fact $-(q-q_1)/q + q_1/p_1 - q_1/p = 0$ and $-(q-q_2)/q + q_2/p_2 - q_2/p = 0$, we have

$$\begin{aligned}
\|Ta\|_{L^q(\mathscr{X})}^q &= \int_0^\infty q\lambda^{q-1}\mu(\{x \in \mathscr{X} : |Ta(x)| > \lambda\})\,d\lambda \\
&\lesssim \int_0^{[\mu(B)]^{-1/q}} \lambda^{q-1}\mu(\{x \in \mathscr{X} : |Ta(x)| > \lambda\})\,d\lambda \\
&\quad + \int_{[\mu(B)]^{-1/q}}^\infty \lambda^{q-1}\mu(\{x \in \mathscr{X} : |Ta(x)| > \lambda\})\,d\lambda \\
&\lesssim \int_0^{[\mu(B)]^{-1/q}} \lambda^{q-q_1-1}\|a\|_{H^{p_1}(\mathscr{X})}^{q_1}\,d\lambda + \int_{[\mu(B)]^{-1/q}}^\infty \lambda^{q-q_2-1}\|a\|_{L^{p_2}(\mathscr{X})}^{q_2}\,d\lambda \\
&\lesssim [\mu(B)]^{-(q-q_1)/q+q_1/p_1-q_1/p} + [\mu(B)]^{-(q-q_2)/q+q_2/p_2-q_2/p} \lesssim 1,
\end{aligned}$$

which completes the proof of Theorem 2.1.

REMARK 2.3. It is easy to see that Theorem 2.1 (i) still holds when T is a sublinear operator. We should also point out that Theorem 2.1 (i) when $p_1 = q_1$ and Theorem 2.1 (ii) when $p_1 = q_1$ and $p = 1$ are included in Theorem D in [4].

3. Proofs of Theorem 1.1 and Theorem 1.2

We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. We first prove the boundedness of T from $L^1(\mathscr{X})$ to weak-$L^{\kappa_0}(\mathscr{X})$ when $p_1 = 1$. By a density argument, it suffices to verify that for any $\lambda > 0$ and $f \in L^\infty(\mathscr{X})$ with bounded support and $\|f\|_{L^1(\mathscr{X})} \leqslant 1$, we have $\mu(\{x \in \mathscr{X} : |Tf(x)| > 2\lambda\}) \lesssim \lambda^{-\kappa_0}$.

To this end, let $\lambda > 0$, $\alpha = \lambda^{\kappa_0}$ and $f \in L^\infty(\mathscr{X})$ with bounded support and $\|f\|_{L^1(\mathscr{X})} \leqslant 1$. We recall the Calderón-Zygmund decomposition on spaces of homogeneous type; see [3, 4]. There exists $N \in \mathbb{N}$, independent of f and α, such that f has a decomposition $f = g + b = g + \sum_j b_j$ satisfying

(I) $|g(x)| \lesssim \alpha$ for all $x \in \mathscr{X}$;
(II) $\operatorname{supp} b_j \subset B_j = B_\rho(x_j, r_j)$ for certain $x_j \in \mathscr{X}$ and $r_j > 0$ and $\sum_j \mu(B_j) \lesssim \|f\|_{L^1(\mathscr{X})}/\alpha$;
(III) $[\mu(B_j)]^{-1} \int_{B_j} |b_j(x)|\, d\mu(x) \lesssim \alpha$;
(IV) $\int_{B_j} b_j(x)\, d\mu(x) = 0$ and $\sum_{j\in\mathbb{N}} \|b_j\|_{L^1(\mathscr{X})} \lesssim \|f\|_{L^1(\mathscr{X})}$;
(V) For any $x \in \mathscr{X}$, x belongs to at most N balls B_j.

We then write

$$\begin{aligned}\mu(\{x \in \mathscr{X} : |Tf(x)| > 2\lambda\}) &\leqslant \mu(\{x \in \mathscr{X},\ |Tg(x)| > \lambda\}) + \mu(\{x \in \mathscr{X},\ |Tb(x)| > \lambda\}) \\ &\equiv J_1 + J_2.\end{aligned}$$

By (II) & (III), we have $\|b\|_{L^1(\mathscr{X})} \lesssim \sum_i \|b_i\|_{L^1(\mathscr{X})} \lesssim \sum_i \alpha\mu(B_i) \lesssim 1$, which together with $g = f - b$ further implies that $\|g\|_{L^1(\mathscr{X})} \lesssim 1$. From this, the boundedness of T from $L^{p_0}(\mathscr{X})$ to $L^{q_0}(\mathscr{X})$, $\alpha = \lambda^{\kappa_0}$ and $1/q_0 = 1/p_0 + 1/\kappa_0 - 1$, we deduce that

$$J_1 \leqslant \left(\lambda^{-1}\|Tg\|_{L^{q_0}(\mathscr{X})}\right)^{q_0} \lesssim \lambda^{-q_0}\|g\|_{L^{p_0}(\mathscr{X})}^{q_0} \lesssim \lambda^{-q_0}\alpha^{(1-1/p_0)q_0}\|g\|_{L^1(\mathscr{X})}^{q_0/p_0} \leqslant \lambda^{-\kappa_0}.$$

On the other hand, by (II) through (V) above, we have

$$\begin{aligned}\|b\|_{L^{p_0}(\mathscr{X})}^{p_0} &\lesssim \sum_i \|b_i\|_{L^{p_0}(\mathscr{X})}^{p_0} \lesssim \sum_i \int_{B_i} [\alpha^{p_0} + |f(x)|^{p_0}]\, d\mu(x) \\ &\lesssim \sum_i \alpha^{p_0}\mu(B_i) + \sum_i \int_{B_i} |f(x)|^{p_0}\, d\mu(x) \lesssim \alpha^{p_0-1}\|f\|_{L^1(\mathscr{X})} + \|f\|_{L^{p_0}(\mathscr{X})}^{p_0},\end{aligned}$$

which implies that $b = \sum_j b_j$ in $L^{p_0}(\mathscr{X})$. Therefore, from this and the boundedness of T from $L^{p_0}(\mathscr{X})$ to $L^{q_0}(\mathscr{X})$ again, it follows that for almost all $x \in \mathscr{X}$, $Tb(x) = \sum_j Tb_j(x)$ and $|Tb(x)| \leqslant \sum_j |Tb_j(x)|$.

Let $\widetilde{B}_i = B_\rho(x_i, 2C_3C_K r_i)$. Then for any $y \in B_i$ and $x \notin \widetilde{B}_i$, we have $\rho(x_i, y) \leqslant 2C_K\rho(x_i, x)$. By this, $|Tb(x)| \leqslant \sum_j |Tb_j(x)|$ for almost all $x \in \mathscr{X}$, (III) & (IV), the Minkowski inequality and $K \in D_\rho(\kappa_0, \kappa_0)$, we have

$$\begin{aligned}&\left\{\int_{\mathscr{X}\setminus\cup_i\widetilde{B}_i} |Tb(x)|^{\kappa_0}\, d\mu(x)\right\}^{1/\kappa_0} \\ &\quad\lesssim \sum_i \left\{\int_{\mathscr{X}\setminus\widetilde{B}_i} |Tb_i(x)|^{\kappa_0}\, d\mu(x)\right\}^{1/\kappa_0} \\ &\quad\lesssim \sum_i \left\{\int_{\mathscr{X}\setminus\widetilde{B}_i} \left|\int_{\mathscr{X}} [K(x,y) - K(x,x_i)]b_i(y)\, d\mu(y)\right|^{\kappa_0} d\mu(x)\right\}^{1/\kappa_0} \\ &\quad\lesssim \sum_i \int_{B_i} \left\{\int_{\mathscr{X}\setminus\widetilde{B}_i} |K(x,y) - K(x,x_i)|^{\kappa_0}\, d\mu(x)\right\}^{1/\kappa_0} |b_i(y)|\, d\mu(y) \\ &\quad\lesssim \sum_i \int_{B_i} |b_i(y)|\, d\mu(y) \lesssim \sum_i \alpha\mu(B_i) \lesssim 1.\end{aligned}$$

This shows that

$$J_2 \leqslant \mu\left(\bigcup_i \widetilde{B}_i\right) + \mu\left(\left\{x \in \mathscr{X} \setminus \bigcup_i \widetilde{B}_i : \ |Tb(x)| > \lambda\right\}\right)$$
$$\lesssim \lambda^{-\kappa_0} + \lambda^{-\kappa_0} \int_{\mathscr{X} \setminus \bigcup_{i=1}^\infty \widetilde{B}_i} |Tb(x)|^{\kappa_0} \, d\mu(x) \lesssim \lambda^{-\kappa_0},$$

which together with the estimate for J_1 confirms the claim. Thus, T is bounded from $L^1(\mathscr{X})$ to weak-$L^{\kappa_0}(\mathscr{X})$.

To verify the boundedness of T from $H^{p_1}(\mathscr{X})$ to $L^{q_1}(\mathscr{X})$, let a be any (p_1,∞)-atom supported in $B_\rho = B_\rho(x_0,r)$ for certain $x_0 \in \mathscr{X}$ and $r > 0$. Then by the Hölder inequality, we have

$$\left\{\int_{B_\rho(x_0,C_K r)} |Ta(x)|^{q_1} \, d\mu(x)\right\}^{1/q_1} \lesssim [\mu(B_\rho)]^{1/q_1 - 1/q_2} \|Ta\|_{L^{q_2}(\mathscr{X})}$$
$$\lesssim [\mu(B_\rho)]^{1/p_1 - 1/p_2} \|a\|_{L^{p_2}(\mathscr{X})} \lesssim 1.$$

On the other hand,

$$\left\{\int_{\rho(x,x_0)\geqslant C_K r} |Ta(x)|^{q_1} \, d\mu(x)\right\}^{1/q_1}$$
$$= \left\{\int_{\rho(x,x_0)\geqslant C_k r} \left|\int_{B_\rho} [K(x,y) - K(x,x_0)] a(y) \, d\mu(y)\right|^{q_1} d\mu(x)\right\}^{1/q_1}$$
$$\lesssim [\mu(B_\rho)]^{-1/p_1} \int_{B_\rho} \left\{\int_{\rho(x,x_0)\geqslant C_k \rho(x_0,y)} |K(x,y) - K(x,x_0)|^{q_1} \, d\mu(x)\right\}^{1/q_1} d\mu(y) \lesssim 1.$$

Thus $Ta \in L^{q_1}(\mathscr{X})$ and $\|Ta\|_{L^{q_1}(\mathscr{X})} \lesssim 1$, which together with Lemma 2.1 (i) gives the boundedness of T from $H^{p_1}(\mathscr{X})$ to $L^{q_1}(\mathscr{X})$.

From this and Theorem 2.1, we deduce the boundedness of T from $H^p(\mathscr{X})$ to $L^q(\mathscr{X})$ when $p \in [p_1, 1]$ and from $L^p(\mathscr{X})$ to $L^q(\mathscr{X})$ when $p \in (1, p_0)$, which completes the proof of Theorem 1.1. $\square$

REMARK 3.1. We remark that some ideas of the proof for the boundedness of T from $L^1(\mathscr{X})$ to weak-$L^{\kappa_0}(\mathscr{X})$ in Theorem 1.1 come from [13, Theorem 2.2] on $\mathbb{R}^n$.

Proof of Theorem 1.2. Let $p \in [p_1, \kappa_0/(2\kappa_0 - 1)]$. It is easy to see that if $\gamma > \kappa_0$ then $D_\rho(\kappa_0, \gamma, \eta) \subset D_\rho(\kappa_0, \kappa_0, \eta)$. Thus, without loss of generality, we may assume that $1 < \gamma \leqslant \kappa_0$ when $p = \kappa_0/(2\kappa_0 - 1)$ or $\gamma = 1$ when $p < \kappa_0/(2\kappa_0 - 1)$.

Let a be any (p,∞)-atom supported in $B_\rho = B_\rho(x_0, r)$ for some $x_0 \in \mathscr{X}$ and $r > 0$. We now claim that there exists a positive constant C independent of a such that $\frac{1}{C}Ta$ satisfies conditions (M1) and (M2) of $(q,\gamma,\widetilde{\eta})_\rho$-molecules centered at $B_\rho(x_0, C_K r)$ in Definition 2.5, where $\widetilde{\eta}_j = \sum_{k=j+1}^\infty \eta_k 2^{j-k}$; moreover, if $T^*(1) = 0$, $\frac{1}{C}Ta$ is a $(q,\gamma,\widetilde{\eta})_\rho$-molecule centered at $B_\rho(x_0, C_K r)$.

If this is true, then when $p \in [p_1, \kappa_0/(2\kappa_0 - 1)]$, Lemma 2.1 (iii) implies the boundedness of T from $H^p(\mathscr{X})$ to $L^q(\mathscr{X})$; moreover, if $T^*(1) = 0$, Lemma 2.1 (ii) yields the boundedness of T from $H^p(\mathscr{X})$ to $H^q(\mathscr{X})$ for $p \in [p_1, \kappa_0/(2\kappa_0 - 1)]$. Using the boundedness of T from $H^{p_1}(\mathscr{X})$ to $L^{q_1}(\mathscr{X})$ and from $L^{p_0}(\mathscr{X})$ to $L^{q_0}(\mathscr{X})$ together with Theorem 2.1, we then obtain the boundedness of T from $H^p(\mathscr{X})$ to $L^q(\mathscr{X})$ when $p \in [\kappa_0/(2\kappa_0 - 1), 1]$ and from $L^p(\mathscr{X})$ to $L^q(\mathscr{X})$ when $p \in (1, p_0)$.

To prove the claim, notice that $D_\rho(\kappa_0, \gamma, \eta) \subset D_\rho(\kappa_0, \gamma)$ together with Theorem 1.1 implies the boundedness of T from $H^{\widetilde{p}}(\mathscr{X})$ to $L^\gamma(\mathscr{X})$, where $1/\gamma = 1/\widetilde{p} + 1/\kappa_0 - 1$; then from (1.3), it is easy to see that

$$\begin{aligned}\|Ta\|_{L^\gamma(\mathscr{X})} &\lesssim \|a\|_{H^{\widetilde{p}}(\mathscr{X})} \lesssim [\mu(B_\rho)]^{1/\widetilde{p} - 1/p} \\ &\lesssim [\mu(B_\rho(x_0, C_K r))]^{1/\gamma - 1/q},\end{aligned}$$

which gives (M1). Since $T^*(1) = 0$ implies (M3), it remains to verify (M2). For $j \in \mathbb{N}$, by $\int_{\mathscr{X}} a(x)\, d\mu(x) = 0$, the Minkowski inequality, $K \in D_\rho(\kappa_0, \gamma, \eta)$ and (1.3), we obtain

$$\begin{aligned}&\left\{\int_{R_j(B_\rho(x_0, C_K r))} |Ta(x)|^\gamma\, d\mu(x)\right\}^{1/\gamma} \\ &\leqslant [\mu(B_\rho)]^{-1/p} \int_{B_\rho} \left\{\int_{R_j(B_\rho(x_0, C_K r))} |K(x,y) - K(x,x_0)|^\gamma\, d\mu(x)\right\}^{1/\gamma} d\mu(y) \\ &\leqslant [\mu(B_\rho)]^{-1/p} \sum_{k=1}^\infty \int_{2^{-k} r \leqslant \rho(x_0, y) < 2^{-k+1} r} \\ &\quad \times \left\{\int_{2^{j+k-1} C_K \rho(x_0,y) \leqslant \rho(x_0,x) < 2^{j+k+1} C_K \rho(x_0,y)} |K(x,y) - K(x,x_0)|^\gamma\, d\mu(x)\right\}^{1/\gamma} d\mu(y) \\ &\leqslant [\mu(B_\rho)]^{-1/p} \sum_{k=1}^\infty \left[\eta_{j+k} 2^{(j+k)(1/\gamma - 1/\kappa_0)} + \eta_{j+k+1} 2^{(j+k+1)(1/\gamma - 1/\kappa_0)}\right] \\ &\quad \times \int_{2^{-k} r \leqslant \rho(x_0,y) < 2^{-k+1} r} [\rho(x_0, y)]^{1/\gamma - 1/\kappa_0} d\mu(y) \\ &\lesssim [\mu(B_\rho(x_0, C_K r))]^{1/\gamma - 1/q} 2^{j(1/\gamma - 1/\kappa_0)} \sum_{k=j+1}^\infty \eta_k 2^{j-k}.\end{aligned}$$

If $p = \kappa_0/(2\kappa_0 - 1)$, we then have

$$\begin{aligned}\sum_{j=1}^\infty j\widetilde{\eta}_j &= \sum_{j=1}^\infty j \left(\sum_{k=j+1}^\infty \eta_k 2^{j-k}\right) = \sum_{k=1}^\infty \eta_k 2^{-k} \left(\sum_{j=1}^k j 2^j\right) \\ &\lesssim \sum_{k=1}^\infty k\eta_k < \infty.\end{aligned}$$

If $p \in (p_1, \kappa_0/(2\kappa_0 - 1))$, then by $q > q_1$, we have

$$\sum_{j=1}^{\infty} 2^{j(1-q)} (\widetilde{\eta}_j)^q \leqslant \sum_{j=1}^{\infty} 2^j \left(\sum_{k=j+1}^{\infty} (\eta_k)^q 2^{-kq} \right)$$
$$\lesssim \sum_{k=1}^{\infty} (\eta_k)^q 2^{k(1-q)} \lesssim \left\{ \sum_{k=1}^{\infty} 2^{k(1-q_1)} (\eta_k)^{q_1} \right\}^{q/q_1} < \infty.$$

This verifies the claim, and thus finishes the proof of Theorem 1.2. □

4. Some applications

In this section, we apply Theorem 1.1 and Theorem 1.2 to the boundedness of commutators generated by Lipschitz functions and integral operators with kernels having weak regularity as in Definition 1.1 (ii) on spaces of homogeneous type.

In what follows of this subsection, we always let $\kappa_0 \in [1,\infty)$, $\gamma \in [1,\infty]$, $\eta = \{\eta_j\}_{j\in\mathbb{N}}$ with $\eta_j > 0$ and $K \in D_\rho(\kappa_0,\gamma,\eta)$ satisfying that $|K(x,y)| \lesssim [\rho(x,y)]^{-1/\kappa_0}$ for all $x \neq y$; let $p_0 \in (1,\infty)$, $1/q_0 = 1/p_0 + 1/\kappa_0 - 1$, T be a linear bounded operator from $L^{p_0}(\mathscr{X})$ to $L^{q_0}(\mathscr{X})$ and have a kernel K as in (1.5), and let $b \in \mathrm{Lip}(\beta)$ with $\beta \in (0, 1/\kappa_0)$. The commutator $[b,T]$ is defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x) \tag{4.1}$$

for all $f \in L^\infty(\mathscr{X})$ with bounded supports and almost all $x \in \mathscr{X}$.

Recall that on $\mathbb{R}^n$, if $b \in \mathrm{Lip}(\beta)$ with $\beta \in (0,1]$ and $K(x,y) = |x-y|^{-n}\Omega((x-y)/|x-y|)$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) :\ x \in \mathbb{R}^n\}$ with $\Omega \in C^\infty(\mathbb{S}^{n-1})$, Janson [15] then proved that $[b,T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $p \in (1, n/\beta)$ and $1/q = 1/p - \beta/n$; and it was also proved in [18] that $[b,T]$ is bounded from $H^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $p \in (n/(n+\beta), 1]$ and $1/q = 1/p - \beta/n$.

For all $(x,y) \in \mathscr{X} \times \mathscr{X} \setminus \{(x,x) :\ x \in \mathscr{X}\}$, let $\widetilde{K}(x,y) = K(x,y)[b(x) - b(y)]$.

PROPOSITION 4.1. *Let $\kappa_0 \in [1,\infty)$, $\beta \in (0,1/\kappa_0)$, $\beta_1 = 1 + \beta - 1/\kappa_0$, $[b,T]$ and $\widetilde{K}$ be the same as above. Then $[b,T]$ is bounded from $L^{p_1}(\mathscr{X})$ to $L^{q_1}(\mathscr{X})$ for any $p_1 \in (1, 1/\beta_1)$ and $1/q_1 = 1/p_1 - \beta_1$, and from $L^1(\mathscr{X})$ to weak-$L^{1/(1-\beta_1)}(\mathscr{X})$. Moreover, $\widetilde{K} \in D_\rho(1/(1-\beta_1),\gamma,\widetilde{\eta})$ with $\gamma \in [1,\infty]$ and $\widetilde{\eta}_j = \eta_j + 2^{-j\beta}$ for all $j \in \mathbb{N}$, and for any $f \in L^\infty(\mathscr{X})$ with bounded support,*

$$[b,T]f(x) = \int_{\mathscr{X}} \widetilde{K}(x,y) f(y)\, d\mu(y) \tag{4.2}$$

holds in both $L^{q_0}(\mathscr{X})$ and almost everywhere.

To prove Proposition 4.1, we need the following dyadic decomposition on $\mathscr{X}$ of Christ in [2].

LEMMA 4.1. *Let $\mathscr{X}$ be a space of homogeneous type. Then there exists a collection*

$$\{Q_\nu^k \subset \mathscr{X} : k \in \mathbb{Z}_+, \nu \in I_k\}$$

of open subsets, where I_k is index set, and constants $\delta \in (0,1)$ and $C_5, C_6 > 0$ such that

(i) $\mu(\mathscr{X} \setminus \cup_\nu Q_\nu^k) = 0$ *for each fixed k and $Q_\nu^k \cap Q_\tau^k = \emptyset$ if $\nu \neq \tau$;*

(ii) for any ν, τ, k, ℓ with $\ell \geqslant k$, either $Q_\tau^\ell \subset Q_\nu^k$ or $Q_\tau^\ell \cap Q_\nu^k = \emptyset$;

(iii) for each (k,ν) and each $\ell < k$, there exists a unique τ such that $Q_\nu^k \subset Q_\tau^\ell$;

(iv) $\sup_{x,y\in Q_\nu^k} \rho(x,y) \leqslant C_5\delta^k$;

(v) each Q_ν^k contains some ball $B(z_\nu^k, C_6\delta^k)$, where $z_\nu^k \in \mathscr{X}$.

Proof of Proposition 4.1. We first prove (4.2) holds in both $L^{q_0}(\mathscr{X})$ and almost everywhere. Let $f \in L^\infty(\mathscr{X})$ with $\operatorname{supp} f \subset Q_{\nu_0}^{k_0}$. From (1.5) for K, it is easy to see that $\widetilde{K}$ is the kernel of $[b,T]$ in the sense of (1.5), namely, (4.2) holds for all $x \notin \operatorname{supp} f$.

On the other hand, by Lemma 4.1, when $k > k_0$, there exists a finite index set $\widetilde{I}_k \subset I_k$ such that $Q_{\nu_0}^{k_0} = \cup_{\nu\in\widetilde{I}_k} Q_\nu^k$. From (1.3) and Lemma 4.1 (iv) & (v), it is easy to deduce that $\sharp\widetilde{I}_k \lesssim \delta^{-k}$, where $\sharp\widetilde{I}_k$ denotes the number of indices in $\widetilde{I}_k$. Let $f_\nu^k = f\chi_{Q_\nu^k}$ for $\nu \in \widetilde{I}_k$. Then $f = \sum_{\nu\in\widetilde{I}_k} f_\nu^k$ and $[b,T]f = \sum_{\nu\in\widetilde{I}_k}[b,T]f_\nu^k$. For each $\nu \in \widetilde{I}_k$, we have

$$\begin{aligned}
&\left\|\left\{[b,T]f_\nu^k - \int_{\mathscr{X}} \widetilde{K}(\cdot,y) f_\nu^k(y)\,d\mu(y)\right\}\chi_{Q_\nu^k}\right\|_{L^{q_0}(\mathscr{X})} \\
&\leqslant \left\|\{[b,T]f_\nu^k\}\chi_{Q_\nu^k}\right\|_{L^{q_0}(\mathscr{X})} + \left\|\left\{\int_{\mathscr{X}} \widetilde{K}(\cdot,y) f_\nu^k(y)\,d\mu(y)\right\}\chi_{Q_\nu^k}\right\|_{L^{q_0}(\mathscr{X})} \\
&\leqslant \left\|\{[b-b(z_\nu^k)]Tf_\nu^k\}\chi_{Q_\nu^k}\right\|_{L^{q_0}(\mathscr{X})} + \left\|T([b-b(z_\nu^k)]f_\nu^k)\right\|_{L^{q_0}(\mathscr{X})} \\
&\quad + \left\{\int_{Q_\nu^k}\left[\int_{Q_\nu^k}[\rho(x,y)]^{\beta-1/\kappa_0}|f_\nu^k(y)|\,d\mu(y)\right]^{q_0} d\mu(x)\right\}^{1/q_0} \\
&\lesssim \delta^{k\beta}\|Tf_\nu^k\|_{L^{q_0}(\mathscr{X})} + \|[b-b(z_\nu^k)]f_\nu^k\|_{L^{p_0}(\mathscr{X})} \\
&\quad + \|f_\nu^k\|_{L^\infty(\mathscr{X})}\left\{\int_{Q_\nu^k}\left(\int_{Q_\nu^k}[\rho(x,y)]^{\beta-1/\kappa_0}\,d\mu(y)\right)^{q_0} d\mu(x)\right\}^{1/q_0} \\
&\lesssim \delta^{k\beta}\|f_\nu^k\|_{L^{p_0}(\mathscr{X})} + \delta^{k(1/q_0+1+\beta-1/\kappa_0)}\|f_\nu^k\|_{L^\infty(\mathscr{X})}.
\end{aligned}$$

Moreover, by (4.2) for all $\nu \in \widetilde{I}_k$ and $x \notin Q_\nu^k$, we have

$$[b,T]f_\nu^k(x) = \int_{\mathscr{X}} \widetilde{K}(x,y) f_\nu^k(y)\,d\mu(y).$$

Thus, from $p_0 \leqslant q_0$, $1+\beta-1/\kappa_0>0$ and $\sharp \widetilde{I}_k \lesssim \delta^{-k}$, we deduce that

$$\begin{aligned}
&\left\|[b,T]f-\int_{\mathscr{X}}\widetilde{K}(\cdot,y)f(y)\,d\mu(y)\right\|_{L^{q_0}(\mathscr{X})}^{q_0}\\
&\quad=\left\|\sum_{\nu\in\widetilde{I}_k}\left\{[b,T]f_\nu^k-\int_{\mathscr{X}}\widetilde{K}(\cdot,y)f_\nu^k(y)\,d\mu(y)\right\}\chi_{Q_\nu^k}\right\|_{L^{q_0}(\mathscr{X})}^{q_0}\\
&\quad=\sum_{\nu\in\widetilde{I}_k}\left\|[b,T]f_\nu^k-\int_{\mathscr{X}}\widetilde{K}(\cdot,y)f_\nu^k(y)\,d\mu(y)\right\|_{L^{q_0}(\mathscr{X})}^{q_0}\\
&\quad\lesssim\delta^{k\beta q_0}\sum_{\nu\in\widetilde{I}_k}\|f_\nu^k\|_{L^{p_0}(\mathscr{X})}^{q_0}+\delta^{k(1+q_0+\beta q_0-q_0/\kappa_0)}\sum_{\nu\in\widetilde{I}_k}\|f_\nu^k\|_{L^\infty(\mathscr{X})}^{q_0}\\
&\quad\lesssim\delta^{k\beta q_0}\|f\|_{L^{p_0}(\mathscr{X})}^{q_0}+\delta^{k(q_0+\beta q_0-q_0/\kappa_0)}\|f\|_{L^\infty(\mathscr{X})}^{q_0}\to 0,
\end{aligned}$$

as $k\to\infty$. Therefore (4.2) holds in both $L^{q_0}(\mathscr{X})$ and almost everywhere.

By (4.2) and $|\widetilde{K}(x,y)|\lesssim[\rho(x,y)]^{-1/\kappa_0+\beta}$, we have

$$|[b,T]f(x)|\lesssim\int_{\mathscr{X}}[\rho(x,y)]^{-1/\kappa_0+\beta}|f(y)|\,d\mu(y),$$

from which and Theorem 1.1 in [10], we deduce that $[b,T]$ is bounded from $L^{p_1}(\mathscr{X})$ to $L^{q_1}(\mathscr{X})$ and from $L^1(\mathscr{X})$ to weak-$L^{1/(1-\beta_1)}(\mathscr{X})$.

To verify that $\widetilde{K}\in D_\rho(1/(1-\beta_1),\gamma,\widetilde{\eta})$, for any $x\neq y$ and $z\in R_j(B_\rho(x,C_K\rho(x,y)))$ with $j\in\mathbb{N}$, we have

$$\begin{aligned}
\left|\widetilde{K}(z,x)-\widetilde{K}(z,y)\right|&=|K(z,x)[b(z)-b(x)]-K(z,y)[b(z)-b(y)]|\\
&\leqslant|[K(z,y)-K(z,x)][b(z)-b(y)]|+|K(z,x)[b(x)-b(y)]|\\
&\lesssim[\rho(z,y)]^\beta|K(z,y)-K(z,x)|+[\rho(z,x)]^{-1/\kappa_0}[\rho(x,y)]^\beta.
\end{aligned}$$

Thus, by the Minkowski inequality, we obtain

$$\begin{aligned}
&\left\{\int_{R_j(B_\rho(x,C_K\rho(x,y)))}\left|\widetilde{K}(z,x)-\widetilde{K}(z,y)\right|^\gamma d\mu(z)\right\}^{1/\gamma}\\
&\quad\lesssim\left\{\int_{R_j(B_\rho(x,C_K\rho(x,y)))}|K(z,x)-K(z,y)|^\gamma[\rho(z,y)]^{\beta\gamma}\,d\mu(z)\right\}^{1/\gamma}\\
&\qquad+\left\{\int_{R_j(B_\rho(x,C_K\rho(x,y)))}[\rho(z,x)]^{-\gamma/\kappa_0}[\rho(x,y)]^{\beta\gamma}\,d\mu(z)\right\}^{1/\gamma}\\
&\quad\lesssim[2^{j+1}\rho(x,y)]^\beta\left\{\int_{R_j(B_\rho(x,C_K\rho(x,y)))}|K(z,x)-K(z,y)|^\gamma\,d\mu(z)\right\}^{1/\gamma}\\
&\qquad+[\rho(x,y)]^\beta[2^j\rho(x,y)]^{-1/\kappa_0}[2^j\rho(x,y)]^{1/\gamma}\\
&\quad\lesssim(\eta_j+2^{-j\beta})\left[\mu(B_\rho(x,2^jC_K\rho(x,y)))\right]^{1/\gamma+\beta-1/\kappa_0}.
\end{aligned}$$

This shows that $\widetilde{K} \in D_\rho(1/(1-\beta_1),\gamma,\widetilde{\eta})$ with $\widetilde{\eta}_j = C(\eta_j + 2^{-j\beta})$ for all $j \in \mathbb{N}$ and certain positive constant C, and hence, finishes the proof of Proposition 4.1. $\square$

Moreover, applying Proposition 4.1, Theorem 1.1 and Theorem 1.2, we obtain the following conclusions.

PROPOSITION 4.2. *Let $\kappa_0 \in [1,\infty)$, $\beta \in (0,1/\kappa_0)$, $\beta_1 = 1+\beta-1/\kappa_0$, and $[b,T]$ be the same as in (4.1). For any $p \in (1/(1+\beta_1),1]$, let $1/q = 1/p - \beta_1$.*

(i) If $p_2 \in [1/(1+\beta_1),1] \cap [1/(1+\theta),1]$, $1/q_2 = 1/p_2 - \beta_1$, $K \in D_\rho(\kappa_0,q_2,\eta)$ with $\eta \in \ell^1$, then $[b,T]$ is bounded from $H^p(\mathscr{X})$ to $L^q(\mathscr{X})$ for $p \in [p_2,1]$.

(ii) If $\beta \in (1-1/\kappa_0, \theta+1/\kappa_0-1]$, $p_2 \in [1/(1+\theta),1/(1+\beta_1)] \cap (1/(1+2\beta),1/(1+\beta_1)]$, $1/q_2 = 1/p_2-\beta_1$, $\gamma \in (1,\infty]$ and η satisfying $\sum_{j\in\mathbb{N}} j\eta_j < \infty$ when $p_2 = 1/(1+\beta_1)$, or $\gamma \in [1,\infty]$ and η satisfying $\sum_{j\in\mathbb{N}} 2^{j(1-q_2)}(\eta_j)^{q_2} < \infty$ when $p_2 < 1/(1+\beta_1)$, and $K \in D_\rho(\kappa_0,\gamma,\eta)$, then $[b,T]$ is bounded from $H^p(\mathscr{X})$ to $L^q(\mathscr{X})$ for $p \in [p_2,1]$; if further assume that $([b,T])^*(1) = 0$, then $[b,T]$ is bounded from $H^p(\mathscr{X})$ to $H^q(\mathscr{X})$ with $p \in [1/(1+\theta),1/(1+\beta_1)]$.

Proof. Notice that $\sum_{j\in\mathbb{N}} \eta_j < \infty$ implies that $\sum_{j\in\mathbb{N}} \widetilde{\eta}_j = \sum_{j\in\mathbb{N}} (\eta_j + 2^{-j\beta}) < \infty$. Thus, if $K \in D_\rho(\kappa_0,q_2,\eta)$ with $\sum_{j\in\mathbb{N}} \eta_j < \infty$, then by Proposition 4.1, $\widetilde{K} \in D_\rho(1/(1-\beta_1),q_2)$ which together with Theorem 1.1 gives Proposition 4.2 (i).

To verify (ii), notice that $K \in D_\rho(\kappa_0,q_2,\eta)$ implies $\widetilde{K} \in D_\rho(1/(1-\beta_1),q_2,\widetilde{\eta})$ by Proposition 4.1, $\sum_{j\in\mathbb{N}} j\eta_j < \infty$ if and only if $\sum_{j\in\mathbb{N}} j\widetilde{\eta}_j < \infty$, $p_2 > 1/(1+2\beta)$ implies that $q_2 > 1/(1+\beta)$, and $\sum_{j\in\mathbb{N}} 2^{j(1-q_2)}(\eta_j)^{q_2} < \infty$ if and only if $\sum_{j\in\mathbb{N}} 2^{j(1-q_2)}(\widetilde{\eta}_j)^{q_2} < \infty$. This together with Theorem 1.2 gives Proposition 4.2 (ii), and hence, finishes the proof of Proposition 4.2.

REMARK 4.1. (a) We remark that $([b,T])^* = [\widetilde{b},T^*]$, where $\widetilde{b}(x) = b(-x)$ for all $x \in \mathscr{X}$ and T^* is the dual of T; and if $T^*(1) = 0$ and $T^*(b) = 0$, namely, $\int_{\mathscr{X}} b(x)Tf(x)\,d\mu(x) = 0$ for all $f \in L^{p_1}(\mathscr{X})$ with bounded support and $\int_{\mathscr{X}} f(x)\,d\mu(x) = 0$, then $([b,T])^*(1) = 0$.

(b) Notice $[b,T^*] = ([\widetilde{b},T])^*$. From this and Proposition 4.2, it is easy to deduce the boundedness of $[b,T^*]$ from Lebesgue spaces to $\mathrm{BMO}(\mathscr{X})$ or Lipschitz spaces, and from $\mathrm{BMO}(\mathscr{X})$ or Lipschitz spaces to Lipschitz spaces.

(c) We point out that the regularity of the kernel of the operator T in Proposition 4.2 is weak than the corresponding result in [18]. In fact, in [18],

$$K(x,y) = |x-y|^{-n}\Omega((x-y)/|x-y|)$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $x \neq y$; thus $\kappa_0 = 1$. It was assumed that $\Omega \in \mathrm{Lip}_1(\mathbb{S}^{n-1})$ in [18], which implies that $K \in D_\rho(1,\infty,\eta)$ with $\eta_j = C2^{-j/n}$ for all $j \in \mathbb{N}$ and certain positive constant C. However, in Proposition 4.2, we only assume that $K \in D_\rho(1,\gamma,\eta)$ with γ and η as in Proposition 4.2. It is easy to see that $D_\rho(1,\infty,\eta) \subsetneq D_\rho(1,\gamma,\eta)$.

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