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A NEW GENERALIZATION OF HARDY–HILBERT'S INEQUALITY WITH NON–HOMOGENEOUS KERNEL

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Abstract. Let $p>1$, $1/p+1/p^*=1$, and $a=(a_n)_{n=1}^{\infty}$, $b=(b_m)_{m=1}^{\infty}$ be two complex sequences. We exhibit the generalization of Hardy-Hilbert's inequality of the following type:

$$\sum_{n,m\geqslant 1} K(\phi_1(n),\phi_2(m))|a_n||b_m| < C\left(\sum_{n=1}^{\infty}|\frac{a_n}{f_1(\phi_1(n))}|^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}|\frac{b_m}{f_2(\phi_2(m))}|^{p^*}\right)^{\frac{1}{p^*}},$$

where $K:(0,\infty)\times(0,\infty)\to(0,\infty)$, $f_1,f_2,\phi_1,\phi_2:(0,\infty)\to(0,\infty)$ and C is a suitable constant. We also get several interesting inequalities which generalize many recent results.

1. Introduction

The famous Hardy-Hilbert's inequality states that if $p>1$, $1/p+1/p^*=1$, $(a_n)_{n=1}^{\infty}$ and $(b_m)_{m=1}^{\infty}$ are two complex sequences with $0<\sum_{n=1}^{\infty}|a_n|^p<\infty$ and $0<\sum_{m=1}^{\infty}|b_m|^{p^*}<\infty$, then

$$\sum_{n,m\geqslant 1}\frac{|a_n||b_m|}{n+m} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}|b_m|^{p^*}\right)^{\frac{1}{p^*}}, \tag{1.1}$$

where the constant $\pi/\sin\left(\frac{\pi}{p}\right)$ is the best possible (see [3, Theorem 315]). That is, this number in (1.1) can not be replaced by any smaller positive number. This inequality and its varieties provide useful tools and play important roles in analysis and applications. Recently, this topic is still popular and many generalizations of such inequalities are obtained. In [8], B. Yang and L. Debnath exhibited the following results: if $p>1$, $1/p+1/p^*=1$, $2-\min\{p,p^*\}<\lambda\leqslant 2$, $\alpha,\beta>0$, $0<\sum_{n=1}^{\infty}n^{1-\lambda}|a_n|^p<\infty$ and $0<\sum_{m=1}^{\infty}m^{1-\lambda}|b_m|^{p^*}<\infty$, then

$$\sum_{n,m\geqslant 1}\frac{|a_n||b_m|}{(\alpha n+\beta m)^{\lambda}} < C\left(\sum_{n=1}^{\infty}n^{1-\lambda}|a_n|^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}m^{1-\lambda}|b_m|^{p^*}\right)^{\frac{1}{p^*}}, \tag{1.2}$$

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where $C=\alpha^{\frac{2-p-\lambda}{p}}\beta^{\frac{2-p^*-\lambda}{p^*}}B(\frac{p+\lambda-2}{p},\frac{p^*+\lambda-2}{p^*})$ is the best possible and $B(\cdot,\cdot)$ is the beta function (see [5, Theorem 8.20]). It is obvious that (1.2) reduces to (1.1) if $\alpha=\beta=\lambda=1$. Later, more inequalities are examined and the corresponding non-discrete cases are given. For details, we refer the readers to [1,2,4,7,8].

In this paper, we introduce inequalities concluding (1.2) as a special case. The following inequality is under consideration in section 2:

$$\sum_{n,m\geqslant 1} K(\phi_1(n),\phi_2(m))|a_n||b_m| < C\left(\sum_{n=1}^{\infty}|\frac{a_n}{f_1(\phi_1(n))}|^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}|\frac{b_m}{f_2(\phi_2(m))}|^{p^*}\right)^{\frac{1}{p^*}},$$

where $K:(0,\infty)\times(0,\infty)\to(0,\infty)$ and $f_1,f_2,\phi_1,\phi_2:(0,\infty)\to(0,\infty)$. The readers can verify (1.2) is the corresponding case for $K(x,y)=1/(\alpha x+\beta y)^{\lambda}$, $\phi_1(t)=\phi_2(t)=t$, $f_1(t)=t^{\frac{\lambda-1}{p}}$ and $f_2(t)=t^{\frac{\lambda-1}{p^*}}$. In addition, we also give the sufficient conditions under which the constant C is the best possible. Next, in section 3, some special kernels K are examined to get several interesting inequalities, which extend many well-known results (*cf.* [1,7,8]) and improve some related constants (*cf.* [2,4]). Finally, we derive the corresponding integral analogues in the last section.

2. Main result

Now we present our main result in this section. To reach this aim, we introduce two special classes of nonnegative functions. Let $\mathscr{F}(\alpha)$ be the collection of all differentiable functions $f:(0,\infty)\to(0,\infty)$ with $\inf_{x>0}f'(x)\geqslant\alpha>0$. Note that by [6, Theorem 7.18 & Lemma 7.25], functions in $\mathscr{F}(\alpha)$ must be absolutely continuous. Moreover, for $p>1$, denote by $\mathscr{H}_p$ the set of all positive functions K defined on $(0,\infty)\times(0,\infty)$ having the following properties:

(1) K is homogeneous of degree -1, that is, $K(tx,ty)=t^{-1}K(x,y)$ for all $t>0$,

(2) $K(x,1)x^{-\frac{1}{p}}$ is a strictly decreasing function of x and $K(1,y)y^{-\frac{1}{p^*}}$ is a strictly decreasing function of y, where $1/p+1/p^*=1$, and

(3) $k_p(K)=\int_0^{\infty}K(x,1)x^{-\frac{1}{p}}dx=\int_0^{\infty}K(1,y)y^{-\frac{1}{p^*}}dy<\infty$.

We mention that under the condition (1), these integrals in (3) must be equal, and hence, the condition (3) can be replaced by

$$k_p(K)=\int_0^{\infty}K(1,y)y^{-\frac{1}{p^*}}dy<\infty.$$

Moreover, for convenience, we set ℓ_p the Banach space consisting of all complex sequences $x=(x_n)_{n=1}^{\infty}$ with the norm $\|x\|_p:=(\sum_{n=1}^{\infty}|x_n|^p)^{\frac{1}{p}}<\infty$. In addition, throughout this paper, we suppose K is a positive function defined on $(0,\infty)\times(0,\infty)$ and f_1, f_2 are two positive functions defined on $(0,\infty)$. The main theorem is stated below.

THEOREM 2.1. *Let* $p > 1$, $1/p + 1/p^* = 1$, $\phi_i \in \mathscr{F}(\alpha_i)$ *for* $i = 1,2$. *If* $K(x,y)f_1(x)f_2(y) \in \mathscr{H}_p$, *then*

$$\sum_{n,m\geqslant 1} K(\phi_1(n),\phi_2(m))|a_n||b_m| < \alpha_1^{-\frac{1}{p^*}}\alpha_2^{-\frac{1}{p}} k_p \|\frac{a}{f_1(\phi_1)}\|_p \|\frac{b}{f_2(\phi_2)}\|_{p^*}, \tag{2.1}$$

$$\left(\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} K(\phi_1(n),\phi_2(m))f_2(\phi_2(m))|a_n|\right)^p\right)^{\frac{1}{p}} < \alpha_1^{-\frac{1}{p^*}}\alpha_2^{-\frac{1}{p}} k_p \|\frac{a}{f_1(\phi_1)}\|_p, \tag{2.2}$$

$$\left(\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} K(\phi_1(n),\phi_2(m))f_1(\phi_1(n))|b_m|\right)^{p^*}\right)^{\frac{1}{p^*}} < \alpha_1^{-\frac{1}{p^*}}\alpha_2^{-\frac{1}{p}} k_p \|\frac{b}{f_2(\phi_2)}\|_{p^*}, \tag{2.3}$$

where $k_p = k_p(K(x,y)f_1(x)f_2(y))$, $\frac{a}{f_1(\phi_1)} = (\frac{a_n}{f_1(\phi_1(n))})_{n=1}^{\infty}$ *and* $\frac{b}{f_2(\phi_2)} = (\frac{b_m}{f_2(\phi_2(m))})_{m=1}^{\infty}$ *are two complex sequences such that* $0 < \|\frac{a}{f_1(\phi_1)}\|_p, \|\frac{b}{f_2(\phi_2)}\|_{p^*} < \infty$. *Moreover, if (2.4) is true, then the constant* $\alpha_1^{-\frac{1}{p^*}}\alpha_2^{-\frac{1}{p}} k_p$ *is the best possible, where*

$$\lim_{x\to\infty} \phi_i'(x) = \alpha_i \qquad \text{for} \quad i = 1,2. \tag{2.4}$$

Proof. Rewrite $\tilde{a} = \frac{a}{f_1(\phi_1)}$, $\tilde{b} = \frac{b}{f_2(\phi_2)}$ and $\tilde{K}(x,y) = K(x,y)f_1(x)f_2(y)$. By the hypotheses, we get that $0 < \|\tilde{a}\|_p, \|\tilde{b}\|_{p^*} < \infty$, $\tilde{K} \in \mathscr{H}_p$ and (2.1) becomes

$$\sum_{n,m\geqslant 1} \tilde{K}(\phi_1(n),\phi_2(m))|\tilde{a}_n||\tilde{b}_m| < \alpha_1^{-\frac{1}{p^*}}\alpha_2^{-\frac{1}{p}} k_p(\tilde{K})\|\tilde{a}\|_p\|\tilde{b}\|_{p^*}.$$

Hence, it suffices to prove the case that $f_1 = f_2 = 1$. From [3, Theorems 286 & 287], the inequalities (2.1), (2.2) and (2.3) are equivalent, and we only need to show (2.1) holds. Applying Hölder's inequality, we obtain

$$\begin{aligned}\sum_{n,m\geqslant 1} K(\phi_1(n),\phi_2(m))|a_n||b_m| &= \sum_{n,m\geqslant 1} |a_n|K^{\frac{1}{p}}\left(\frac{\phi_1(n)}{\phi_2(m)}\right)^{\frac{1}{pp^*}} |b_m|K^{\frac{1}{p^*}}\left(\frac{\phi_2(m)}{\phi_1(n)}\right)^{\frac{1}{pp^*}} \\ &\leqslant P^{\frac{1}{p}}Q^{\frac{1}{p^*}},\end{aligned}$$

where

$$\begin{aligned} P &= \sum_{n=1}^{\infty}|a_n|^p\sum_{m=1}^{\infty}K(\phi_1(n),\phi_2(m))\left(\frac{\phi_1(n)}{\phi_2(m)}\right)^{\frac{1}{p^*}} \\ &= \sum_{n=1}^{\infty}|a_n|^p\sum_{m=1}^{\infty}K(1,\frac{\phi_2(m)}{\phi_1(n)})\left(\frac{\phi_1(n)}{\phi_2(m)}\right)^{\frac{1}{p^*}}(\phi_1(n))^{-1} \\ &< \sum_{n=1}^{\infty}|a_n|^p\int_0^{\infty}K(1,\frac{\phi_2(x)}{\phi_1(n)})\left(\frac{\phi_2(x)}{\phi_1(n)}\right)^{-\frac{1}{p^*}}(\phi_1(n))^{-1}dx.\end{aligned}$$

Putting $u = \frac{\phi_2(x)}{\phi_1(n)}$, we have

$$P < \sum_{n=1}^{\infty} |a_n|^p \alpha_2^{-1} \int_{\frac{\phi_2(0^+)}{\phi_1(n)}}^{\infty} K(1,u) u^{-\frac{1}{p^*}} du$$
$$\leqslant \sum_{n=1}^{\infty} |a_n|^p \alpha_2^{-1} \int_0^{\infty} K(1,u) u^{-\frac{1}{p^*}} du. \tag{2.5}$$

Here $\phi_2(0^+)$ is the right-hand limit of ϕ_2 at 0. Similarly, we can get

$$Q < \sum_{m=1}^{\infty} |b_m|^{p^*} \alpha_1^{-1} \int_0^{\infty} K(u,1) u^{-\frac{1}{p}} du. \tag{2.6}$$

In conjunction with (2.5) and (2.6), this proves the inequality (2.1). Finally, we show the number $\alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p$ is the best possible constant if (2.4) holds. Note that ϕ_i is strictly increasing, absolutely continuous and

$$\lim_{x \to \infty} \frac{\phi_i(x)}{\alpha_i x} = 1 \tag{2.7}$$

for $i = 1,2$. Let $(\varepsilon_k)_{k=1}^{\infty}$ be a sequence with $\varepsilon_k \to 0^+$ as $k \to \infty$. We define $a(k) = (a_n(k))_{n=1}^{\infty}$ and $b(k) = (b_m(k))_{m=1}^{\infty}$, where $a_n(k) = (\phi_1(n))^{-\frac{(1+\varepsilon_k)}{p}}$ and $b_m(k) = (\phi_2(m))^{-\frac{(1+\varepsilon_k)}{p^*}}$ for all $n,m \geqslant 1$. (2.7) leads us to the fact $0 < \|a(k)\|_p, \|b(k)\|_{p^*} < \infty$ for all k. Let $N > 0$. We have

$$\sum_{n,m \geqslant 1} K(\phi_1(n), \phi_2(m)) |a_n(k)| |b_m(k)|$$
$$\geqslant \int_{N^2}^{\infty} \int_N^{\infty} K(\phi_1(x), \phi_2(y)) (\phi_1(x))^{-\frac{(1+\varepsilon_k)}{p}} (\phi_2(y))^{-\frac{(1+\varepsilon_k)}{p^*}} dx dy$$

since the integrand is both decreasing in x and in y by the conditions (1)–(2) in the definition of $\mathscr{H}_p$. Furthermore,

$$\int_{N^2}^{\infty} \int_N^{\infty} K(\phi_1(x), \phi_2(y)) (\phi_1(x))^{-\frac{(1+\varepsilon_k)}{p}} (\phi_2(y))^{-\frac{(1+\varepsilon_k)}{p^*}} dx dy$$
$$= \int_{N^2}^{\infty} \int_N^{\infty} K\Big(\frac{\phi_1(x)}{\phi_2(y)}, 1\Big) \left(\frac{\phi_1(x)}{\phi_2(y)}\right)^{-\frac{(1+\varepsilon_k)}{p}} (\phi_2(y))^{-(2+\varepsilon_k)} dx dy$$
$$\geqslant \int_{N^2}^{\infty} \left(\int_{\frac{\phi_1(N)}{\phi_2(N^2)}}^{\infty} K(u,1) u^{-\frac{(1+\varepsilon_k)}{p}} \inf_{x \geqslant N} (\phi_1'(x))^{-1} du \right) (\phi_2(y))^{-(1+\varepsilon_k)} dy. \tag{2.8}$$

From Fatou's lemma, we get

$$\liminf_{k \to \infty} \int_{\frac{\phi_1(N)}{\phi_2(N^2)}}^{\infty} K(u,1) u^{-\frac{(1+\varepsilon_k)}{p}} du \geqslant \int_{\frac{\phi_1(N)}{\phi_2(N^2)}}^{\infty} K(u,1) u^{-\frac{1}{p}} du. \tag{2.9}$$

On the other hand, with the help of (2.7), we can verify that

$$\alpha_2 \int_{N^2}^{\infty} (\phi_2(y))^{-(1+\varepsilon_k)} dy \Big/ \alpha_1^{\frac{1}{p}} \alpha_2^{\frac{1}{p^*}} \|a(k)\|_p \|b(k)\|_{p^*} \longrightarrow 1 \text{ as } k \to \infty. \tag{2.10}$$

It follows from (2.8)–(2.10) that

$$\begin{aligned} \liminf_{k\to\infty} \sum_{n,m\geqslant 1} K(\phi_1(n), \phi_2(m)) \frac{|a_n(k)|}{\|a(k)\|_p} \frac{|b_m(k)|}{\|b(k)\|_{p^*}} \\ \geqslant \alpha_1^{\frac{1}{p}} \alpha_2^{-\frac{1}{p}} \int_{\frac{\phi_1(N)}{\phi_2(N^2)}}^{\infty} K(u,1) u^{-\frac{1}{p}} \inf_{x\geqslant N} (\phi_1'(x))^{-1} du. \end{aligned} \tag{2.11}$$

By (2.7), the last number in (2.11) converges to $\alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p$ as $N \to \infty$. This finishes the proof. □

Set $f_1 = f_2 = 1$ and Theorem 2.1 can be reduced to the following corollary. It is a generalization of Hardy-Hilbert's result, which corresponds to the case $\phi_1(t) = \phi_2(t) = t$ (*cf.* [3, Theorem 318]).

COROLLARY 2.2. *Let* $p > 1$, $1/p + 1/p^* = 1$, *and* $\phi_i \in \mathscr{F}(\alpha_i)$ *for* $i = 1,2$. *If* $K(x,y) \in \mathscr{H}_p$, *then*

$$\sum_{n,m\geqslant 1} K(\phi_1(n), \phi_2(m)) |a_n||b_m| < \alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p \|a\|_p \|b\|_{p^*},$$

$$\left(\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} K(\phi_1(n), \phi_2(m)) |a_n| \right)^p \right)^{\frac{1}{p}} < \alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p \|a\|_p,$$

$$\left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} K(\phi_1(n), \phi_2(m)) |b_m| \right)^{p^*} \right)^{\frac{1}{p^*}} < \alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p \|b\|_{p^*},$$

where $k_p = k_p(K)$, $a = (a_n)_{n=1}^{\infty}$ *and* $b = (b_m)_{m=1}^{\infty}$ *are two complex sequences such that* $0 < \|a\|_p, \|b\|_{p^*} < \infty$. *Moreover, if (2.4) is true, then the constant* $\alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p$ *is the best possible.*

We mention that Corollary 2.2 also provides a technique to evaluate the norms of some special Hilbert's type operators on ℓ_p, where $p > 1$. Suppose $T : \ell_p \to \ell_p$ is a linear operator defined by $T(a) = (T_m(a))_{m=1}^{\infty}$ for each $a = (a_n)_{n=1}^{\infty} \in \ell_p$, where $T_m(a) = \sum_{n=1}^{\infty} K(\phi_1(n), \phi_2(m)) a_n$ for each $m \geqslant 1$. We conclude that T is bounded for such a kernel $K(\phi_1, \phi_2)$ involved here, and furthermore, the corresponding operator norm is exactly the constant $\alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p(K)$.

3. Some applications

Now it is time to investigate varieties of Hilbert's type kernel and obtain some interesting results. Let $p>1$ and $1/p+1/p^*=1$. From elementary calculation, we can verify that $k_p(K)=\pi/\sin(\frac{\pi}{p})$ if $K(x,y)=1/(x+y)$ and $k_p(K)=pp^*$ if $K(x,y)=1/\max\{x,y\}$. Select appropriate ϕ_1, ϕ_2 in Corollary 2.2 and we get following inequalities:

$$\sum_{n,m\geqslant 1}\frac{|a_n||b_m|}{An+Bm+C}<\frac{\pi}{A^{\frac{1}{p^*}}B^{\frac{1}{p}}\sin\left(\frac{\pi}{p}\right)}\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}|b_m|^{p^*}\right)^{\frac{1}{p^*}};$$

$$(A,B>0;\ C\geqslant 0)$$

$$\sum_{n,m\geqslant 1}\frac{|a_n||b_m|}{A\max\{Bn+C,Dm+E\}+F}<\frac{pp^*}{AB^{\frac{1}{p^*}}D^{\frac{1}{p}}}\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}|b_m|^{p^*}\right)^{\frac{1}{p^*}},$$

$$(A,B,D>0;\ C,E\geqslant -A^{-1}F)$$

where $0<\sum_{n=1}^{\infty}|a_n|^p<\infty$ and $0<\sum_{m=1}^{\infty}|b_m|^{p^*}<\infty$. We emphasize that these estimations can not be improved anymore, that is, the corresponding constants are the best possible. On the other hand, as a consequence, Theorem 2.1 generalizes many well-known results. The first is the following corollary which extends [7, Theorem 1].

COROLLARY 3.1. *Let* $p>1$, $1/p+1/p^*=1$, $\phi_i\in\mathscr{F}(\alpha_i)$ *for* $i=1,2$, $0<\lambda\leqslant 2$, $\alpha,\beta,\gamma\geqslant 0$ *with* $\alpha+\beta\gamma>0$, $0<\sum_{n=1}^{\infty}\phi_1(n)^{p(1-\frac{\lambda}{2})-1}|a_n|^p<\infty$ *and* $0<\sum_{m=1}^{\infty}\phi_2(m)^{p^*(1-\frac{\lambda}{2})-1}|b_m|^{p^*}<\infty$. *We have*

$$\sum_{n,m\geqslant 1}\frac{|a_n||b_m|}{\alpha\max\{\phi_1(n)^{\lambda},\phi_2(m)^{\lambda}\}+\beta\phi_1(n)^{\lambda}+\gamma\phi_2(m)^{\lambda}}$$
$$<\alpha_1^{-\frac{1}{p^*}}\alpha_2^{-\frac{1}{p}}C_{\lambda}(\alpha,\beta,\gamma)\left(\sum_{n=1}^{\infty}\phi_1(n)^{p(1-\frac{\lambda}{2})-1}|a_n|^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}\phi_2(m)^{p^*(1-\frac{\lambda}{2})-1}|b_m|^{p^*}\right)^{\frac{1}{p^*}}, \tag{3.1}$$

where

$$C_{\lambda}(\alpha,\beta,\gamma):=\begin{cases}\frac{2}{\lambda\sqrt{\gamma(\alpha+\beta)}}\arctan\sqrt{\frac{\gamma}{\alpha+\beta}}+\frac{2}{\lambda\sqrt{\beta(\alpha+\gamma)}}\arctan\sqrt{\frac{\beta}{(\alpha+\gamma)}} & \text{for } \alpha,\beta,\gamma>0;\\ \frac{2}{\lambda\sqrt{\alpha\gamma}}\arctan\sqrt{\frac{\gamma}{\alpha}}+\frac{2}{\lambda(\alpha+\gamma)} & \text{for } \beta=0,\alpha>0,\gamma>0;\\ \frac{2}{\lambda\sqrt{\alpha\beta}}\arctan\sqrt{\frac{\beta}{\alpha}}+\frac{2}{\lambda(\alpha+\beta)} & \text{for } \gamma=0,\alpha>0,\beta>0;\\ \frac{4}{\lambda\alpha} & \text{for } \beta=\gamma=0,\alpha>0;\\ \frac{\pi}{\lambda\sqrt{\beta\gamma}} & \text{for } \alpha=0,\beta>0,\gamma>0.\end{cases}$$

Moreover, if (2.4) is true, then the constant is the best possible.

Proof. Set

$$K(x,y) = \frac{1}{\alpha \max\{x^\lambda, y^\lambda\} + \beta x^\lambda + \gamma y^\lambda}, \ \ f_1(x) = x^{\frac{\lambda}{2} - \frac{1}{p^*}} \ \text{ and } \ f_2(y) = y^{\frac{\lambda}{2} - \frac{1}{p}}.$$

It is easy to see that $K(x,y)f_1(x)f_2(y)$ satisfies (1)–(2) in the definition of $\mathscr{H}_p$. Furthermore, since

$$k_p = \int_0^\infty K(1,y)f_1(1)f_2(y)y^{-\frac{1}{p^*}}dy = \int_0^\infty \frac{y^{\frac{\lambda}{2}-1}}{\alpha \max\{1, y^\lambda\} + \beta + \gamma y^\lambda}dy,$$

using change of variable $u = y^\lambda$ and we find that the integral above is equal to [7, Eq. (8)]. This confirms that $k_p = C_\lambda(\alpha, \beta, \gamma) < \infty$ and $K(x,y)f_1(x)f_2(y) \in \mathscr{H}_p$. Applying Theorem 2.1 and we find that (2.1) reduces to (3.1). This completes the proof. □

The second application generalizes [8, Theorems 3.3] (that is, (1.2)) and gives a partial extension of [2, Theorem 2] and [4, Theorem 1]. The statement is as follows.

COROLLARY 3.2. *Let* $p > 1$, $1/p + 1/p^* = 1$, $\phi_i \in \mathscr{F}(\alpha_i)$ *for* $i = 1,2$, $\alpha, \beta, \lambda > 0$, $-1 < \frac{\lambda-2}{p} + A_2 - A_1 \leqslant 0$, $-1 < \frac{\lambda-2}{p^*} + A_1 - A_2 \leqslant 0$, $0 < \sum_{n=1}^\infty \phi_1(n)^{1-\lambda+p(A_1-A_2)}|a_n|^p < \infty$ *and* $0 < \sum_{m=1}^\infty \phi_2(m)^{1-\lambda+p^*(A_2-A_1)}|b_m|^{p^*} < \infty$. *We have*

$$\sum_{n,m \geqslant 1} \frac{|a_n||b_m|}{(\alpha\phi_1(n) + \beta\phi_2(m))^\lambda}$$
$$< \alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p \left(\sum_{n=1}^\infty \phi_1(n)^{1-\lambda+p(A_1-A_2)}|a_n|^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^\infty \phi_2(m)^{1-\lambda+p^*(A_2-A_1)}|b_m|^{p^*} \right)^{\frac{1}{p^*}}, \tag{3.2}$$

where $k_p = \alpha^{\frac{2-p-\lambda}{p}+A_1-A_2}\beta^{\frac{2-p^*-\lambda}{p^*}+A_2-A_1}B(\frac{p+\lambda-2}{p}+A_2-A_1, \frac{p^*+\lambda-2}{p^*}+A_1-A_2)$. *Moreover, if (2.4) is true, then the constant is the best possible.*

Proof. Let

$$K(x,y) = \frac{1}{(\alpha x + \beta y)^\lambda}, \ \ f_1(x) = x^{\frac{\lambda-1}{p} - A_1 + A_2} \ \text{ and } \ f_2(y) = y^{\frac{\lambda-1}{p^*} - A_2 + A_1}.$$

One may check that $K(x,y)f_1(x)f_2(y)$ satisfies (1)–(2) in the definition of $\mathscr{H}_p$ and

$$\begin{aligned}
k_p &= \int_0^\infty K(1,y)f_1(1)f_2(y)y^{-\frac{1}{p^*}}dy \\
&= \alpha^{-\lambda}\int_0^\infty y^{\frac{\lambda-2}{p^*}-A_2+A_1}\left(1+\frac{\beta}{\alpha}y\right)^{-\lambda} dy \\
&= \alpha^{\frac{2-p-\lambda}{p}+A_1-A_2}\beta^{\frac{2-p^*-\lambda}{p^*}+A_2-A_1}\int_0^1 u^{\frac{\lambda-2}{p}+A_2-A_1}(1-u)^{\frac{\lambda-2}{p^*}+A_1-A_2}du \\
&= \alpha^{\frac{2-p-\lambda}{p}+A_1-A_2}\beta^{\frac{2-p^*-\lambda}{p^*}+A_2-A_1}B\left(\frac{p+\lambda-2}{p}+A_2-A_1, \frac{p^*+\lambda-2}{p^*}+A_1-A_2\right),
\end{aligned}$$

where $u = (1+\frac{\beta}{\alpha}y)^{-1}$. This guarantees that $K(x,y)f_1(x)f_2(y) \in \mathscr{H}_p$. Hence, (3.2) holds by Theorem 2.1. $\square$

It is easy to see that (3.2) reduces to (1.2) if we set $2-\min\{p,p^*\} < \lambda \leqslant 2$, $\phi_1(t) = \phi_2(t) = t$ and $A_1 = A_2$. Moreover, the readers can check that $-1 < \frac{\lambda-2}{p} + A_2 - A_1 \leqslant 0$ and $-1 < \frac{\lambda-2}{p^*} + A_1 - A_2 \leqslant 0$ hold if $0 < \lambda \leqslant 1$, $A_1 \in (\frac{1-\lambda}{p^*}, \frac{1}{p^*})$ and $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$. Hence, Corollary 3.2 improves the constant given in [2, Theorem 2] and [4, Theorem 1] for the case $0 < \lambda \leqslant 1$. For example, select $p = 2$, $\alpha = \beta = \lambda = 1$, $\phi_1(t) = \phi_2(t) = t$ and $A_1 = A_2 \in (0, \frac{1}{2})$. The constant $B(\frac{1}{2}, \frac{1}{2})$ obtained here is the best possible, which is less than the constant $B(1-2A_1, 2A_1)$ in [2, Theorem 2] and [4, Theorem 1] for $A_1 \neq \frac{1}{4}$ since the function $\log(\Gamma(x))$ is convex on $(0,\infty)$ (*cf.* [5, Theorem 8.18]). The third application is a generalization of [1, Theorem 3.3].

COROLLARY 3.3. *Let* $p > 1$, $1/p + 1/p^* = 1$, $\phi_i \in \mathscr{F}(\alpha_i)$ *for* $i = 1,2$, $0 < \lambda \leqslant 2$, $\alpha \geqslant 0$, $\beta > 0$, $0 < \sum_{n=1}^{\infty} \phi_1(n)^{p(1-\frac{\lambda}{2})-1}|a_n|^p < \infty$ *and* $0 < \sum_{m=1}^{\infty} \phi_2(m)^{p^*(1-\frac{\lambda}{2})-1}|b_m|^{p^*} < \infty$. *We have*

$$\sum_{n,m\geqslant 1} \frac{|a_n||b_m|}{\alpha \min\{\phi_1(n)^\lambda, \phi_2(m)^\lambda\} + \beta \max\{\phi_1(n)^\lambda, \phi_2(m)^\lambda\}}$$
$$< \alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} C_\lambda(\alpha,\beta) \left(\sum_{n=1}^{\infty} \phi_1(n)^{p(1-\frac{\lambda}{2})-1}|a_n|^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} \phi_2(m)^{p^*(1-\frac{\lambda}{2})-1}|b_m|^{p^*} \right)^{\frac{1}{p^*}}, \tag{3.3}$$

where

$$C_\lambda(\alpha,\beta) := \begin{cases} \frac{4}{\lambda\sqrt{\alpha\beta}} \arctan \sqrt{\frac{\alpha}{\beta}} \text{ for } \alpha > 0, \beta > 0; \\ \frac{4}{\lambda\beta} \text{ for } \alpha = 0, \beta > 0. \end{cases}$$

Moreover, if (2.4) is true, then the constant is the best possible.

Proof. Set

$$K(x,y) = \frac{1}{\alpha \min\{x^\lambda, y^\lambda\} + \beta \max\{x^\lambda, y^\lambda\}}, \quad f_1(x) = x^{\frac{\lambda}{2} - \frac{1}{p^*}} \quad \text{and} \quad f_2(y) = y^{\frac{\lambda}{2} - \frac{1}{p}}.$$

Obviously, $K(x,y)f_1(x)f_2(y)$ satisfies (1)–(2) in the definition of $\mathscr{H}_p$. Moreover,

$$\int_0^\infty K(1,y)f_1(1)f_2(y)y^{-\frac{1}{p^*}}\,dy = \int_0^\infty \frac{y^{\frac{\lambda}{2}-1}}{\alpha\min\{1,y^\lambda\} + \beta\max\{1,y^\lambda\}}dy$$
$$= \frac{1}{\lambda}\int_0^\infty \frac{t^{-\frac{1}{2}}}{\alpha\min\{1,t\} + \beta\max\{1,t\}}dt$$

and we can find that the integral above is equal to [1, Eq. (2.4)]. This confirms that $k_p = C_\lambda(\alpha,\beta) < \infty$, $K(x,y)f_1(x)f_2(y) \in \mathscr{H}_p$, and hence, (3.3) holds. $\square$

We remark that for these examples in this section, the corresponding inequalities for (2.2) and (2.3) can be obtained in a similar way (*cf.* [1, Theorem 3.5] and [8, Theorem 3.4]).

4. Integral analogues

Our results introduced in section 2 also have integral analogues. Suppose $p > 1$ and for any measurable function g defined on $(0,\infty)$, let the norm of g be $\|g\|_p = (\int_0^\infty |g(x)|^p dx)^{\frac{1}{p}}$. Set $L_p(0,\infty)$ the Banach space consisting of such g with $\|g\|_p < \infty$. The following theorem is the non-discrete form of Theorem 2.1.

THEOREM 4.1. *Let* $p > 1$, $1/p + 1/p^* = 1$, $\phi_i \in \mathscr{F}(\alpha_i)$ *for* $i = 1,2$. *If* $K(x,y)f_1(x)f_2(y)$ *is homogeneous of degree* -1 *and* $k_p = k_p(K(x,y)f_1(x)f_2(y)) < \infty$, *then*

$$\int_0^\infty \int_0^\infty K(\phi_1(x),\phi_2(y))|g(x)||h(y)|dxdy < \alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p \|\frac{g}{f_1(\phi_1)}\|_p \|\frac{h}{f_2(\phi_2)}\|_{p^*}, \quad (4.1)$$

$$\left(\int_0^\infty \left(\int_0^\infty K(\phi_1(x),\phi_2(y))f_2(\phi_2(y))|g(x)|dx\right)^p dy\right)^{\frac{1}{p}} < \alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p \|\frac{g}{f_1(\phi_1)}\|_p, \quad (4.2)$$

$$\left(\int_0^\infty \left(\int_0^\infty K(\phi_1(x),\phi_2(y))f_1(\phi_1(x))|h(y)|dy\right)^{p^*} dx\right)^{\frac{1}{p^*}} < \alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p \|\frac{h}{f_2(\phi_2)}\|_{p^*}, \quad (4.3)$$

where $\frac{g}{f_1(\phi_1)}(x) = \frac{g(x)}{f_1(\phi_1(x))}$ *and* $\frac{h}{f_2(\phi_2)}(y) = \frac{h(y)}{f_2(\phi_2(y))}$ *are two measurable functions defined on* $(0,\infty)$ *with* $0 < \|\frac{g}{f_1(\phi_1)}\|_p, \|\frac{h}{f_2(\phi_2)}\|_{p^*} < \infty$. *Moreover, if (2.4) is true, then the constant* $\alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p$ *is the best possible.*

Proof. The proof is similar to that of theorem 2.1, so we just sketch it. We may assume $f_1 = f_2 = 1$, $K(x,y)$ is homogeneous of degree -1 and $k_p(K) < \infty$. One may check that [3, Theorems 286 & 287] also have integral analogues and hence, it suffices to show (4.1) is true. The inequality in (4.1) can be established by the fact

$$\int_0^\infty \int_0^\infty K(\phi_1(x),\phi_2(y))|g(x)||h(y)|dxdy \leqslant P^{\frac{1}{p}} Q^{\frac{1}{p^*}}, \quad (4.4)$$

where

$$\begin{aligned} P &= \int_0^\infty |g(x)|^p \int_0^\infty K(1,\frac{\phi_2(y)}{\phi_1(x)}) \left(\frac{\phi_2(y)}{\phi_1(x)}\right)^{-\frac{1}{p^*}} (\phi_2'(y))^{-1} \left(\frac{\phi_2'(y)}{\phi_1(x)}\right) dydx \\ &\leqslant \int_0^\infty |g(x)|^p \alpha_2^{-1} \int_0^\infty K(1,u)u^{-\frac{1}{p^*}} dudx \\ &= \alpha_2^{-1} k_p \|g\|_p^p \end{aligned} \quad (4.5)$$

and

$$Q \leqslant \int_0^\infty |h(y)|^{p^*} \alpha_1^{-1} \int_0^\infty K(u,1) u^{-\frac{1}{p}} du dy \leqslant \alpha_1^{-1} k_p \|h\|_{p^*}^{p^*}.$$

It is not difficulty to verify that all equalities in (4.5) hold only if $\phi_2'(y) = \alpha_2$ for almost every $y > 0$. It tells us that $\phi_2(y) = \alpha_2 y + \phi_2(0^+)$ since ϕ_2 is absolutely continuous. Moreover, if the equality in (4.4) holds, then

$$|g(x)|^p \left(\frac{\phi_1(x)}{\phi_2(y)} \right)^{\frac{1}{p^*}} = A |h(y)|^{p^*} \left(\frac{\phi_2(y)}{\phi_1(x)} \right)^{\frac{1}{p}}$$

for almost every $(x,y) \in (0,\infty) \times (0,\infty)$ and for some $A > 0$. This implies $|h(y)|^{p^*} = C\phi_2(y)^{-1}$ for almost every $y > 0$, where $C > 0$, and this contradicts to $\|h\|_{p^*} < \infty$. In addition, assume (2.4) holds. To show the constant is the best possible, one may take the functions $g_k(x) = (\phi_1(x))^{-\frac{(1+\varepsilon_k)}{p}} \chi_{[1,\infty)}(x)$ and $h_k(y) = (\phi_2(y))^{-\frac{(1+\varepsilon_k)}{p^*}} \chi_{[1,\infty)}(y)$, where $\varepsilon_k \to 0^+$ as $k \to \infty$ and χ_E denote the characteristic function of E. The details are left to the readers. □

As in section 3, one can test some special kernels in Theorem 4.1 and obtain corresponding well-known results (*cf.* [1, Theorems 2.3 & 2.5] and [8, Theorems 2.1 & 2.2]). The case $f_1 = f_2 = 1$ is also under consideration, and the statement is as follows.

COROLLARY 4.2. *Let $p > 1$, $1/p + 1/p^* = 1$, and $\phi_i \in \mathscr{F}(\alpha_i)$ for $i = 1,2$. If $K(x,y)$ is homogeneous of degree -1 and $k_p = k_p(K) < \infty$, then*

$$\int_0^\infty \int_0^\infty K(\phi_1(x), \phi_2(y)) |g(x)||h(y)| dx dy < \alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p \|g\|_p \|h\|_{p^*},$$

$$\left(\int_0^\infty \left(\int_0^\infty K(\phi_1(x), \phi_2(y)) |g(x)| dx \right)^p dy \right)^{\frac{1}{p}} < \alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p \|g\|_p,$$

$$\left(\int_0^\infty \left(\int_0^\infty K(\phi_1(x), \phi_2(y)) |h(y)| dy \right)^{p^*} dx \right)^{\frac{1}{p^*}} < \alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p \|h\|_{p^*},$$

where g and h are two measurable functions defined on $(0,\infty)$ with $0 < \|g\|_p, \|h\|_{p^*} < \infty$. Moreover, if (2.4) is true, then the constant $\alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p$ is the best possible.

Finally, we give a remark here. Suppose T is a linear operator on $L_p(0,\infty)$ defined by $Tg(y) = \int_0^\infty K(\phi_1(x), \phi_2(y)) g(x) dx$, where $g \in L_p(0,\infty)$ and K, ϕ_1, ϕ_2 satisfy the hypotheses in Corollary 4.2. As a consequence, we conclude that T is bounded on $L_p(0,\infty)$ and the corresponding operator norm is exactly the constant $\alpha_1^{-\frac{1}{p^*}} \alpha_2^{-\frac{1}{p}} k_p(K)$.

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