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A CHARACTERIZATION OF THE SINE FUNCTION BY FUNCTIONAL INEQUALITIES

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Abstract. In the present paper we deal with the first generalization of Wilson's difference

$$f(x)g(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2,$$

assuming that its absolute value is majorized by some function in a single variable.

1. Introduction

We shall start with solutions of the classical Wilson functional equation (the sine equation):

THEOREM 1. (W. H. Wilson [13], see also [2]) *Let $(G,+)$ be a uniquely 2-divisible Abelian group. A function $f : G \to \mathbb{C}$ satisfies the Wilson functional equation (the sine equation):*

$$f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2=f(x)f(y) \quad \text{for all} \quad x,y\in G \tag{1}$$

if and only if f is additive or there exists a homomorphism $m : G \to \mathbb{C}$, i.e. a solution of the exponential functional equation

$$m(x+y)=m(x)m(y) \quad \text{for all} \quad x,y\in G, \tag{2}$$

such that

$$f(x)=\gamma(m(x)-m(-x)) \quad \text{for all} \quad x\in G$$

with some $\gamma \in \mathbb{C}$.

We shall also be using the following

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THEOREM 2. (Pl. Kannappan [11], also [2] and [6]) *Let* $(G,+)$ *be an Abelian group. A function* $f: G \to \mathbb{C}$ *satisfies d'Alembert functional equation (the cosine equation):*

$$f(x+y)+f(x-y)=2f(x)f(y) \quad \text{for all} \quad x,y\in G, \tag{3}$$

if and only if there exists an exponential function $m: G\to\mathbb{C}$ *such that*

$$f(x)=\frac{1}{2}(m(x)+m(-x)) \quad \text{for all} \quad x\in G.$$

From now on, f_o and f_e stand for the odd and the even part of a function $f: G\to \mathbb{C}$, that is,

$$f_o(x)=\frac{f(x)-f(-x)}{2}, \quad f_e(x)=\frac{f(x)+f(-x)}{2} \quad \text{for all} \quad x\in G.$$

In what follows we need the following lemma concerning *Wilson's first generalization of d'Alembert's functional equation* (see [1]):

$$f(x+y)+f(x-y)=2f(x)g(y) \quad \text{for all} \quad x,y\in G, \tag{4}$$

for functions $f,g: G\to\mathbb{C}$, where $(G,+)$ is a uniquely 2-divisible Abelian group.

LEMMA 1. *Let* $(G,+)$ *be a uniquely 2-divisible Abelian group and let functions* $f,g: G\to\mathbb{C}$, $f\neq 0$ *satisfy equation* (4)*. Then* g *satisfies the cosine functional equation* (3)*. Moreover* f_o *satisfies the sine functional equation* (1) *and* $f_e = C\cdot g$, *with some* $C\in\mathbb{C}$.

The proof of this lemma can be obtained by the method applied in [1], using the results of [11].

Proof. Since $f\neq 0$, there exists an $a\in G$ such that $f(a)\neq 0$. Putting $x=a$ in (4), we obtain

$$f(a+y)+f(a-y)=2f(a)g(y), \quad y\in G,$$

i.e.

$$g(y)=\frac{f(a+y)+f(a-y)}{2f(a)}, \quad y\in G. \tag{5}$$

Then using (4) and (5) we get

$$\begin{aligned} g(x+y)+g(x-y) &= \frac{f(a+x+y)+f(a-x-y)}{2f(a)} + \frac{f(a+x-y)+f(a-x+y)}{2f(a)} \\ &= \frac{[f(a+x+y)+f(a+x-y)]+[f(a-x+y)+f(a-x-y)]}{2f(a)} \\ &= \frac{2f(a+x)g(y)+2f(a-x)g(y)}{2f(a)} \\ &= 2\frac{f(a+x)+f(a-x)}{2f(a)}g(y) \\ &= 2g(x)g(y), \quad x,y\in G. \end{aligned}$$

Thus, function g satisfies the cosine functional equation

$$g(x+y)+g(x-y)=2g(x)g(y), \quad x,y\in G. \tag{6}$$

Now we shall distinguish two cases regarding the value of function f at zero.

1° Let $f(0)=0$. Put $x=0$ in (4); after simple calculation we get

$$f(-y)=-f(y), \quad y\in G. \tag{7}$$

Moreover we have

$$f_o=f, \quad f_e=0.$$

Replacing y by x in (4) we arrive at

$$f(2x)=2f(x)g(x), \quad x\in G. \tag{8}$$

Then, conditions (4), (7) and (8) imply that

$$\begin{aligned} f(x+y)^2-f(x-y)^2 &= [f(x+y)+f(x-y)][f(x+y)-f(x-y)] \\ &= 2f(x)g(y)[f(x+y)-f(x-y)] \\ &= f(x)[2f(x+y)g(y)-2f(x-y)g(y)] \\ &= f(x)[f(x+y+y)+f(x+y-y)-f(x-y+y)-f(x-y-y)] \\ &= f(x)[f(2y+x)+f(x)-f(x)-f(x-2y)] \\ &= f(x)[f(2y+x)+f(2y-x)]=f(x)2f(2y)g(x) \\ &= 2f(x)g(x)f(2y)=f(2x)f(2y), \quad x,y\in G, \end{aligned}$$

whence,

$$f(x+y)^2-f(x-y)^2=f(2x)f(2y), \quad x,y\in G.$$

In view of above equation and in the light of the unique 2-divisibility of G we conclude that $f=f_o$ satisfies the sine functional equation (1). Furthermore $f_e=f(0)\cdot g=0$.

2° Let $f(0)=C\neq 0$ and let $F:G\to\mathbb{C}$ be a function defined by the formula $F(x):=\frac{1}{C}f(x)$ for all $x\in G$. Then $F(0)=1$ and using (4) we get

$$F(x+y)+F(x-y)=2F(x)g(y), \quad x,y\in G. \tag{9}$$

Putting $x=0$ in (9) we can determine function g as even part of function F, i.e.

$$g(y)=\frac{F(y)+F(-y)}{2}=\frac{1}{C}\cdot\frac{f(y)+f(-y)}{2}, \quad y\in G,$$

or, equivalently,

$$f_e(x)=C\cdot g(x), \quad x\in G.$$

Let $F(x)=F_e(x)+F_o(x)$, $x\in G$, where F_e and F_o denotes the even and the odd part of F, respectively. Then

$$F(x)=g(x)+F_o(x), \quad x\in G, \tag{10}$$

and applying (10) in (9), we have

$$g(x+y)+F_o(x+y)+g(x-y)+F_o(x-y)=2[g(x)+F_o(x)]g(y), \quad x,y\in G,$$

or, equivalently,

$$g(x+y)+g(x-y)-2g(x)g(y)+F_o(x+y)+F_o(x-y)=2F_o(x)g(y), \quad x,y\in G,$$

which jointly with (6) gives

$$F_o(x+y)+F_o(x-y)=2F_o(x)g(y), \quad x,y\in G.$$

Now F_o and g satisfy the equation (4). Directly from the definition of F_o, we get the equality $F_o(0)=0$ which in case 1° implies that F_o satisfies the sine functional equation (1). Moreover, with the aid of (10), we arrive at

$$f(x)=Cg(x)+f_o(x), \quad x\in G,$$

where the odd part of f has a form $f_o=CF_o$ and satisfies the Wilson's functional equation (1) as well.

Stability problems concerning classical functional equations have been treated by several authors (see e.g. [3, 4, 6, 7, 11, 12]). It is known that equation (1) and equation (3) for complex functions defined on an Abelian group are stable in the sense of Hyers-Ulam (resp. P. Cholewa [8] and J.A. Baker [6]). Generalizations of this result appeared in various directions. It turned out that equation (1) for complex functions defined on an Abelian group is superstable in the sense of Ger, too. Namely, the following theorems hold true:

THEOREM 3. (R. Badora, R. Ger [5]) *Let $(G,+)$ be a uniquely 2-divisible Abelian group and let $f:G\to\mathbb{C}$, $\varphi:G\to\mathbb{R}$ satisfy the inequality*

$$\left| f(x)f(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2\right| \leqslant \varphi(x) \quad \textit{for all} \quad x,y\in G.$$

Then either f is bounded or f satisfies Wilson's equation (1).

THEOREM 4. (R. Badora, R. Ger [5])*Let $(G,+)$ be a uniquely 2-divisible Abelian group and let $f:G\to\mathbb{C}$, $\varphi:G\to\mathbb{R}$ satisfy the inequality*

$$\left| f(x)f(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2\right| \leqslant \varphi(y) \quad \textit{for all} \quad x,y\in G.$$

Then either f is bounded or f satisfies Wilson's equation (1).

2. Main results

DEFINITION 1. Let $(G,+)$ be a uniquely 2-divisible Abelian group. We say that $f,g: G \to \mathbb{C}$ satisfy the *first generalization of Wilson's functional equation* if and only if for all $x,y \in G$ we have

$$f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = f(x)g(y). \tag{11}$$

Solutions of the first generalization of Wilson's functional equation yields the following

THEOREM 5. *Let $(G,+)$ be a uniquely 2-divisible Abelian group and let $f,g: G \to \mathbb{C}$ satisfy* (11). *Then:*

(i) *if $f=0$, then g is arbitrary,*
(ii) *if $g=0$, then $f(G) \subset \{-C,C\}$, where $C \in \mathbb{C}$,*
(iii) *if $f \neq 0$, then g satisfies equation* (1) *and moreover, in the case $f(0)=0$, $f=g$,*
(iv) *if $f \neq 0$ and $f(0)=C \neq 0$, then g is additive and*

$$f(x) = \begin{cases} C + \frac{1}{2}g(x) & \text{for all} \quad x \in G \setminus Z \\ C\varepsilon(x) & \text{for all} \quad x \in Z \end{cases},$$

or

$$g(x) = \gamma(m(x) - m(-x)) \quad \text{for all} \quad x,y \in G$$

and

$$f(x) = \begin{cases} \alpha m(x) + \beta m(-x) & \text{for all} \quad x \in G \setminus Z \\ C\varepsilon(x) & \text{for all} \quad x \in Z \end{cases},$$

with some $\alpha,\beta,\gamma \in \mathbb{C}$, where $\varepsilon: G \to \{-1,1\}$ and $Z := \{x \in G : g(x) = 0\}$.

Proof. Ad (ii). For $g=0$ putting $y=x$ in (11) we get

$$f(x)^2 = C^2 \quad \text{for all} \quad x \in G,$$

where $C := f(0)$. Hence

$$f(x) \in \{-C,C\} \quad \text{for all} \quad x \in G,$$

which ends the proof of (ii)

Replace x by $2x$ and y by $2y$ in (11). Then, for all $x,y \in G$, we obtain

$$f(x+y)^2 - f(x-y)^2 = f(2x)g(2y). \tag{12}$$

We will prove the following implication

$$f \neq 0 \Rightarrow g\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = g(x)g(y) \quad \text{for all} \quad x,y \in G.$$

Assume that $f \neq 0$. Putting $y=0$ in (12) we obtain

$$0 = f(x)^2 - f(x)^2 = f(2x)g(0) \quad \text{for all} \quad x \in G,$$

whence

$$g(0) = 0. \tag{13}$$

Now, replacing y by $-y$ in (12) we get

$$f(x-y)^2 - f(x+y)^2 = f(2x)g(-2y), \quad x, y \in G.$$

From above and (12) we have

$$f(2x)g(-2y) = -f(2x)g(2y) \quad \text{for all} \quad x, y \in G.$$

Hence, in the light of the unique 2-divisibility of G we conclude that

$$g(-y) = -g(y) \quad \text{for all} \quad y \in G. \tag{14}$$

Moreover, since by assumption, $f \neq 0$, there exists an element $a \in G$ such that $f(2a) \neq 0$. Put $x = a$ in (12), then

$$g(2y) = \frac{f(a+y)^2 - f(a-y)^2}{f(2a)} \quad \text{for all} \quad y \in G. \tag{15}$$

By (15) and (11) for all $x, y \in G$ we get

$$\begin{aligned}
g(x+y) + g(x-y) &= g\left(2\frac{x+y}{2}\right) + g\left(2\frac{x-y}{2}\right) \\
&= \frac{f(a+\frac{x+y}{2})^2 - f(a-\frac{x+y}{2})^2}{f(2a)} + \frac{f(a+\frac{x-y}{2})^2 - f(a-\frac{x-y}{2})^2}{f(2a)} \\
&= \frac{f(\frac{2a+x+y}{2})^2 - f(\frac{2a-x-y}{2})^2 + f(\frac{2a+x-y}{2})^2 - f(\frac{2a-x+y}{2})^2}{f(2a)} \\
&= \frac{f(\frac{2a+y+x}{2})^2 - f(\frac{2a+y-x}{2})^2 + f(\frac{2a-y+x}{2})^2 - f(\frac{2a-y-x}{2})^2}{f(2a)} \\
&= \frac{f(2a+y)g(x) + f(2a-y)g(x)}{f(2a)} \\
&= \frac{f(2a+y) + f(2a-y)}{f(2a)} g(x).
\end{aligned}$$

Hence,

$$g(x+y) + g(x-y) = \frac{f(2a+y) + f(2a-y)}{f(2a)} g(x) \quad \text{for all} \quad x, y \in G. \tag{16}$$

Putting $y = x$ in (16) and using (13) we have

$$g(2x) = \frac{f(2a+x) + f(2a-x)}{f(2a)} g(x) \quad \text{for all} \quad x \in G. \tag{17}$$

Interchanging the roles of x and y in (16) and making use of the oddness of g we obtain

$$g(x+y)-g(x-y)=\frac{f(2a+x)+f(2a-x)}{f(2a)}g(y) \quad \text{for all} \quad x,y\in G. \tag{18}$$

Hence, by (16), (17) and (18), for $x,y\in G$, we get

$$\begin{aligned} g(x+y)^2-g(x-y)^2 &= [g(x+y)+g(x-y)][g(x+y)-g(x-y)] \\ &= \frac{f(2a+y)+f(2a-y)}{f(2a)}g(x)\frac{f(2a+x)+f(2a-x)}{f(2a)}g(y) \\ &= g(2x)g(2y). \end{aligned}$$

Thus, we have proved that the function g satisfies Wilson's equation.

Ad (iii). Now, assume that, additionally, $f(0)=0$. To prove (iii), we need only to show that

$$f(x)=g(x) \quad \text{for all} \quad x\in G.$$

Putting $x=0$ in (12) we obtain

$$f(y)^2-f(-y)^2=0 \quad \text{for all} \quad y\in G. \tag{19}$$

whereas replacing x by y in (12) gives

$$f(y+x)^2-f(y-x)^2=f(2y)g(2x) \quad \text{for all} \quad x,y\in G. \tag{20}$$

Subtracting (12) and (20) we get

$$-f(x-y)^2+f(-(x-y))^2=f(2x)g(2y)-f(2y)g(2x)$$

for all $x,y\in G$. Then by (19) we conclude

$$f(2x)g(2y)-f(2y)g(2x)=0 \quad \text{for all} \quad x,y\in G,$$

or, equivalently (in the light of the unique 2-divisibility of G),

$$f(x)g(y)=f(y)g(x) \quad \text{for all} \quad x,y\in G. \tag{21}$$

Since $f\neq 0$, there exists an element $b\in G$ such that $f(b)\neq 0$. Set $x=b$ in (21). Then, for all $x,y\in G$, we get

$$g(y)=\frac{g(b)}{f(b)}f(y).$$

Let $\alpha := \frac{g(b)}{f(b)}$, then $g(y)=\alpha f(y)$ for all $y\in G$. Since g satisfies Wilson's equation we have

$$\alpha^2 f(x+y)^2-\alpha^2 f(x-y)^2=\alpha f(2x)\alpha f(2y) \quad \text{for all} \quad x,y\in G.$$

Now, consider two cases

1° Let $\alpha \neq 0$. Then

$$f(x+y)^2 - f(x-y)^2 = f(2x)f(2y) \quad \text{for all} \quad x,y \in G.$$

Thus from (12) we obtain

$$f(x)[f(y)-g(y)] = 0 \quad \text{for all} \quad x,y \in G.$$

Putting $x = b$ we have

$$f(y) = g(y),$$

for all $y \in G$.

2° If $\alpha = 0$, then $g = 0$. Moreover, from (11) we get

$$f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = 0 \quad \text{for all} \quad x,y \in G.$$

Now, setting $y = x$ we arrive at

$$f(x)^2 = 0 \quad \text{for all} \quad x \in G,$$

whence

$$f(x) = 0 \quad \text{for all} \quad x \in G,$$

which is impossible

Ad (iv). Assume that $f(0) =: C \neq 0$ and $g \neq 0$. Define functions $F, h : G \to \mathbb{C}$ by

$$F(x) = \frac{f(x)}{C}, \quad h(x) = \frac{g(x)}{C},$$

for all $x \in G$. Then

$$F(x+y)^2 - F(x-y)^2 = F(2x)h(2y) \quad \text{for all} \quad x,y \in G \tag{22}$$

and from (14) we conclude that

$$h(-x) = -h(x) \quad \text{for all} \quad x \in G. \tag{23}$$

Directly from the definition, we see that $F(0) = 1$ and h satisfies Wilson's equation. Putting $y = x$ in (22) we have

$$F(x)^2 - 1 = F(x)h(x) \quad \text{for all} \quad x \in G. \tag{24}$$

From (24) for all $x,y \in G$ we obtain

$$\begin{aligned} F(x+y)^2 - F(x-y)^2 &= F(x+y)h(x+y) + 1 - [F(x-y)h(x-y)+1] \\ &= F(x+y)h(x+y) - F(x-y)h(x-y). \end{aligned}$$

Jointly with (22) this implies that

$$F(x+y)h(x+y) - F(x-y)h(x-y) = F(2x)h(2y), \quad x,y \in G. \tag{25}$$

Applying (23) for $x = 0$ in (25) we have

$$[F(y) + F(-y)]h(y) = h(2y)$$

for all $x,y \in G$, whence, in particular, we get

$$2F_e(x)h(x) = h(2x) \quad \text{for all} \quad x \in G. \tag{26}$$

Let

$$Z := \{x \in G : h(x) = 0\}.$$

From definition of the function h and of the set Z, we see that $Z = \{x \in G : g(x) \ = 0\}$. By (26) we conclude that

$$F_e(x) = \frac{h(2x)}{2h(x)} \quad \text{for all} \quad x \in G \setminus Z. \tag{27}$$

Since $F = F_o + F_e$, from (24) we conclude

$$F_o(x)^2 + F_e(x)^2 + 2F_o(x)F_e(x) - 1 = F_o(x)h(x) + F_e(x)h(x) \quad \text{for all} \quad x \in G.$$

Replacing x by $-x$ we arrive at

$$F_o(x)^2 + F_e(x)^2 - 2F_o(x)F_e(x) - 1 = F_o(x)h(x) - F_e(x)h(x) \quad \text{for all} \quad x \in G.$$

Subtracting the last two equations yields

$$2F_o(x)F_e(x) = F_e(x)h(x) \quad \text{for all} \quad x \in G. \tag{28}$$

By equation (28) we get

$$F_o(x) = \frac{1}{2}h(x) \quad \text{for all} \quad x \in G \setminus Z. \tag{29}$$

In the light of (27) and (29) function F has a form

$$F(x) = \frac{h(2x)}{2h(x)} + \frac{h(x)}{2} \quad \text{for all} \quad x \in G \setminus Z. \tag{30}$$

Moreover, from (24) we infer that

$$F(x)^2 = 1 \quad \text{for all} \quad x \in Z.$$

In other words, we have

$$F(x) = \begin{cases} \frac{h(2x)}{2h(x)} + \frac{h(x)}{2}, & x \in G \setminus Z \\ \varepsilon(x), & x \in Z \end{cases},$$

where ε is a function on a uniquely 2-divisible Abelian group G into a set $\{-1,1\}$. Now, for the function h by Theorem 1 we obtain two cases

1° The function h is additive, then we have

$$F(x)=\frac{2h(x)}{2h(x)}+\frac{h(x)}{2}=1+\frac{h(x)}{2},$$

for all $x\in G\setminus Z$.

2° There exists an exponential function $m:G\to\mathbb{C}$ such that

$$h(x)=\gamma(m(x)-m(-x))$$

with some $\gamma\in\mathbb{C}$. Hence, by (30) and (2), for all $x\in G\setminus Z$, we get

$$\begin{aligned}F(x)&=\frac{\gamma(m(2x)-m(-2x)}{2\gamma(m(x)-m(-x))}+\frac{\gamma(m(x)-m(-x)}{2}\\&=\frac{1}{2}\frac{m(x)^2-\frac{1}{m(x)^2}}{m(x)-\frac{1}{m(x)}}+\frac{\gamma}{2}\left(m(x)-\frac{1}{m(x)}\right)\\&=\frac{1}{2}\frac{m(x)^4-1}{(m(x)^2-1)m(x)}+\frac{\gamma}{2}\frac{m(x)^2-1}{m(x)}\\&=\frac{m(x)^2+1+\gamma m(x)^2-\gamma}{2m(x)}\\&=\frac{1+\gamma}{2}m(x)+\frac{1-\gamma}{2}m(-x).\end{aligned}$$

Because of the equalities $f=CF$ and $g=Ch$, the proof has been completed.

The main idea of the proof of the next theorem has already been used by some authors, for example P. Găvrută (see [7]), R. Badora (see [3], [4]), R. Badora, R. Ger (see [5]) and the monograph [10].

Let $(G,+)$ be a uniquely 2-divisible Abelian group. Fix a function $\varphi:G\to\mathbb{R}$ (not necessarily constant nor bounded). Now, for function $f,g:G\to\mathbb{C}$ we shall consider the following inequality

$$\left|f(x)g(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2\right|\leqslant\varphi(x)\quad\text{for all}\quad x,y\in G.\tag{31}$$

THEOREM 6. *Let $(G,+)$ be a uniquely 2-divisible Abelian group and let functions $f,g:G\to\mathbb{C}$ and $\varphi:G\to\mathbb{R}$ satisfy inequality* (31). *Then:*

(i) *if $f=0$, then g is arbitrary,*
(ii) *if $g=0$, then $|f(x)|\leqslant\sqrt{\varphi(x)+|f(0)|^2}$ for all $x\in G$,*
(iii) *if $f\neq 0\neq g$ and f is bounded, then g is bounded, too,*
(iv) *if $f\neq 0$ and g is unbounded, then there exists a solution $h:G\to\mathbb{C}$ of the d'Alembert equation such that*

$$f(x+y)+f(x-y)=2f(x)h(y)\quad\textit{for all}\quad x,y\in G,$$

$f_e = Ch$, f_o satisfies equation (1)*, and*

$$|C||g(x) - 2f_o(x)| \leqslant \varphi(0) \quad \textit{for all} \quad x \in G.$$

Moreover, if $C = 0$, then

$$|g(x) - f_o(x)| \leqslant N$$

for all $x \in G$, where $N := \inf\left\{ \frac{\varphi(x)}{|f_o(x)|} : x \in G,\ f_o(x) \neq 0 \right\}$.

REMARK 1. The omitted case: f-unbounded and g-bounded, remains unresolved.

We shall prove Theorem 6.

Proof. Ad (ii). For $g = 0$ inequality (31) assumes the form

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 \right| \leqslant \varphi(x)$$

for all $x, y \in G$. Put here $y = x$ to get

$$|f(x)|^2 \leqslant \varphi(x) + |f(0)|^2 \quad \text{for all} \quad x \in G.$$

Ad (iii). If $0 \neq f \neq g$ is bounded, then $M := \sup\{|f(x)| : x \in G\} \in (0, +\infty)$, whence by (31) we obtain

$$|f(x)g(y)| \leqslant \varphi(x) + 2M^2 \quad \text{for all} \quad x, y \in G. \tag{32}$$

Since $f \neq 0$, there exists an element $a \in G$ such that $f(a) \neq 0$. Setting $x = a$ in (32) we conclude that g is bounded.

Ad (iv). Assume that $f \neq 0$ and g is unbounded. Putting $x = 2x$ and $y = 2y$ in (31) we have

$$|f(2x)g(2y) - f(x+y)^2 + f(x-y)^2| \leqslant \varphi(2x) \quad \text{for all} \quad x, y \in G. \tag{33}$$

Because of the unboundedness of g, there exists a sequence $(t_n)_{n \in \mathbb{N}}$ of elements of G such that

$$0 \neq |g(2t_n)| \longrightarrow \infty \quad \text{as} \quad n \to \infty. \tag{34}$$

Set $y = t_n$ in (33). Thus

$$|f(2x) - \frac{f(x+t_n)^2 - f(x-t_n)^2}{g(2t_n)}| \leqslant \frac{\varphi(2x)}{|g(2t_n|}$$

for all $x \in G$ and $n \in \mathbb{N}$. Now, passing to the limit as $n \to \infty$ and taking (34) into account, we infer that

$$\lim_{n \to \infty} \frac{f(x+t_n)^2 - f(x-t_n)^2}{g(2t_n)} = f(2x), \quad x \in G. \tag{35}$$

for all $x \in G$. On setting $y = y + 2t_n$ in (31) we get

$$\left| f(x)g(y+2t_n) - f\left(\frac{x+y+2t_n}{2}\right)^2 + f\left(\frac{x-y-2t_n}{2}\right)^2 \right| \leqslant \varphi(x) \tag{36}$$

for all $x, y \in G$ and $n \in \mathbb{N}$. Similarly, putting $y = -y + 2t_n$ in (31) we obtain

$$\left| f(x)g(-y+2t_n) - f\left(\frac{x-y+2t_n}{2}\right)^2 + f\left(\frac{x+y-2t_n}{2}\right)^2 \right| \leqslant \varphi(x) \tag{37}$$

for all $x, y \in G$ and $n \in \mathbb{N}$. From (36) and (37) for all $x, y \in G$ and $n \in \mathbb{N}$ we have

$$\left| 2f(x)\frac{g(y+2t_n)+g(-y+2t_n)}{2g(2t_n)} - \frac{f\left(\frac{x+y}{2}+t_n\right)^2 - f\left(\frac{x+y}{2}-t_n\right)^2}{g(2t_n)} - \frac{f\left(\frac{x-y}{2}+t_n)\right)^2 - f\left(\frac{x-y}{2}-t_n)\right)^2}{g(2t_n)} \right| \leqslant \frac{2\varphi(x)}{|g(2t_n)|}.$$

In the light of (35), we conclude that there exists the limit

$$h(y) := \lim_{n\to\infty} \frac{g(y+2t_n)+g(-y+2t_n)}{2g(2t_n)} \quad \text{for all} \quad y \in G. \tag{38}$$

Relation (38) defines a function $h : G \to \mathbb{C}$ which satisfies the first generalization of Wilson's functional equation, i.e.

$$f(x+y) + f(x-y) = 2f(x)h(y) \quad \text{for all} \quad x, y \in G. \tag{39}$$

From Lemma 1, for a non-zero function f, we obtain that h satisfies d'Alembert's equation:

$$h(x+y) + h(x-y) = 2h(x)h(y) \quad \text{for all} \quad x, y \in G. \tag{40}$$

Moreover $f_e = Ch$ and f_o satisfies Wilson's equation (1). Fix $x \in G$ and set

$$f(x) = Ch(x) + f_o(x) \tag{41}$$

in (31). Therefore for all $x, y \in G$, we get

$$\left| [Ch(x)+f_o(x)]g(y) - \left[Ch\left(\frac{x+y}{2}\right) + f_o\left(\frac{x+y}{2}\right)\right]^2 + \left[Ch\left(\frac{x-y}{2}\right) + f_o\left(\frac{x-y}{2}\right)\right]^2 \right| \leqslant \varphi(x). \tag{42}$$

Put $x = 0$ in (42); then for all $y \in G$ we have

$$\left| [Ch(0) + f_o(0)]g(y) - \left[Ch\left(\frac{y}{2}\right) + f_o\left(\frac{y}{2}\right)\right]^2 + \left[Ch\left(-\frac{y}{2}\right) + f_o\left(-\frac{y}{2}\right)\right]^2 \right| \leqslant \varphi(0)$$

or, equivalently,

$$\left| [C+0]g(y) - \left[Ch\left(\frac{y}{2}\right) + f_o\left(\frac{y}{2}\right)\right]^2 + \left[Ch\left(\frac{y}{2}\right) - f_o\left(\frac{y}{2}\right)\right]^2 \right| \leqslant \varphi(0)$$

for all $y \in G$. Hence

$$\left|Cg(y) - 4Ch\left(\frac{y}{2}\right) f_o\left(\frac{y}{2}\right)\right| \leqslant \varphi(0) \quad \text{for all} \quad y \in G. \tag{43}$$

By (39), (40) and (41) we get relation

$$f_o(x+y) + f_o(x-y) = 2f_o(x)h(y) \quad \text{for all} \quad x, y \in G. \tag{44}$$

Put $y = x$ in (44) to get

$$f_o(2x) = 2f_o(x)h(x), \quad x \in G,$$

which together with (43) gives

$$|C||g(x) - 2f_o(x)| \leqslant \varphi(0) \quad \text{for all} \quad x \in G.$$

Now, if $C \neq 0$, then

$$|g(x) - 2f_o(x)| \leqslant \frac{\varphi(0)}{|C|}, \quad x \in G;$$

in the case where $C = 0$, we have $f_e = 0$ and $f = f_o$ satisfies equation (1). Thus from (31) we conclude that

$$|f_o(x)g(y) - f_o(x)f_o(y)| \leqslant \varphi(x) \quad \text{for all} \quad x, y \in G,$$

whence

$$|f_o(x)||g(y) - f_o(y)| \leqslant \varphi(x) \quad \text{for all} \quad x, y \in G.$$

Since $f = f_o$ is a non-zero function, we may put

$$N := \inf\left\{ \frac{\varphi(x)}{|f_o(x)|} : x \in G,\ f_o(x) \neq 0 \right\},$$

which ends the proof.

Below, we shall present two examples of functions which satisfy inequality (31).

EXAMPLE 1. *Let* $f, g : G \to \mathbb{C}$ *have the forms*

$$f(x) = \frac{m(x) - m(-x)}{2}, \quad g(x) = \frac{m(x) - m(-x)}{2} + 1 \quad \textit{for all} \quad x \in G,$$

where $m : G \to \mathbb{C}$ *is exponential function, i.e.* m *satisfies equation* (2)*. Then, for all* $x, y \in G$ *we have*

$$\left| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| = \frac{|m(x) - m(-x)|}{2} =: \varphi(x).$$

EXAMPLE 2. *Let* $f,g:G\to\mathbb{C}$ *be functions defined by*

$$f(x)=m(x),\quad g(x)=m(x)-m(-x)+1\quad \textit{for all}\quad x\in G,$$

where $m:G\to\mathbb{C}$ *is exponential function, i.e.* m *satisfies equation* (2). *Thus*

$$\left|f(x)g(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2\right|=|m(x)|=:\varphi(x)\quad \textit{for all}\quad x,y\in G.$$

From Theorem 6 we have the following

COROLLARY 1. (Theorem 3) *Let* $(G,+)$ *be a uniquely 2-divisible Abelian group and let* $f:G\to\mathbb{C}$, $\varphi:G\to\mathbb{R}$ *satisfy the inequality*

$$\left|f(x)f(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2\right|\leqslant\varphi(x)\quad \textit{for all}\quad x,y\in G.$$

Then either f *is bounded or* f *satisfies equation*

$$f(x)f(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2\quad \textit{for all}\quad x,y\in G.$$

Proof. From the previous theorem applied for unbounded function $g=f$, there exists a function $h:G\to\mathbb{C}$ such that

$$f(x+y)+f(x-y)=2f(x)h(y)\quad \text{for all}\quad x,y\in G. \tag{45}$$

By Lemma 1 and equation (45) we infer that

$$h(x+y)+h(x-y)=2h(x)h(y)\quad \text{for all}\quad x,y\in G.$$

Moreover, $f_e(x)=Ch(x)$ for all $x\in G$ and f_o satisfies Wilson's equation. Assume that $C\neq 0$. Then, for all $x\in G$, we have

$$|f(x)-2f_o(x)|\leqslant\frac{\varphi(0)}{|C|}$$

whence

$$|f_e(x)-f_o(x)|\leqslant\frac{\varphi(0)}{|C|}\quad \text{for all}\quad x\in G. \tag{46}$$

Putting $y=-y$ in (1) we get

$$\left|f(x)f(-y)-f\left(\frac{x-y}{2}\right)^2+f\left(\frac{x+y}{2}\right)^2\right|\leqslant\varphi(x)\quad \text{for all}\quad x,y\in G. \tag{47}$$

In view of (1) and (47) we deduce that

$$|f(x)||f_e(y)|\leqslant\varphi(x)\quad \text{for all}\quad x,y\in G.$$

Since f is a non-zero function, we infer that f_e is bounded. Now, from (46) we get that f_o is bounded, as well, which is a contradiction. Thus $C=0$ and $f=f_o$ satisfies Wilson's equation.
Observe that inequality (31) fails to be symmetric with respect to x and y, we should consider also the following inequality

$$\left|f(x)g(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2\right|\leqslant\varphi(y)\quad\text{for all}\quad x,y\in G. \tag{48}$$

for all $x,y\in G$, where $(G,+)$ is a uniquely 2-divisible Abelian group. Here, the function $\varphi: G\to\mathbb{R}$ is given (not necessarily constant nor bounded) whereas functions $f,g: G\to\mathbb{C}$ are considered to be "unknown".

THEOREM 7. *Let $(G,+)$ be a uniquely 2-divisible Abelian group and let functions $f,g: G\to\mathbb{C}$ and $\varphi: G\to\mathbb{R}$ satisfy inequality* (48)*. Then:*

(i) *if $f=0$, then g is arbitrary,*
(ii) *if $g=0$, then $|f(x)|\leqslant\sqrt{\varphi(x)+|f(0)|^2}$ for all $x\in G$,*
(iii) *if $f\neq0\neq g$ and f is bounded, then*

$$|g(x)|\leqslant\frac{\varphi(x)+2M^2}{M}\quad\text{for all}\quad x\in G,$$

where $M:=\sup\{|f(x)|:x\in G\}$,
(iv) *if $g\neq0$ and f is unbounded, then g satisfies Wilson's equation* (1) *and*

$$|g(x)f(x)-f(x)^2|\leqslant\varphi(x)+|f(0)|^2\quad\text{for all}\quad x\in G.$$

Proof. Ad (ii). If $g=0$, then (48) has a form

$$\left|f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2\right|\leqslant\varphi(y)\quad\text{for all}\quad x,y\in G.$$

Put $y=x$, to get

$$|f(x)|\leqslant\sqrt{\varphi(x)+|f(0)|^2}\quad\text{for all}\quad x\in G.$$

Ad (iii). Assume that $f\neq0$, $g\neq0$ and f is bounded; then

$$M:=\sup\{|f(x)|:x\in G\}\in(0,+\infty).$$

By (48) we conclude that

$$|g(y)|\leqslant\frac{\varphi(y)+2M^2}{M}\quad\text{for all}\quad y\in G.$$

Ad (iv). Setting $x=2x$ and $y=2y$ in (48), we obtain

$$|f(2x)g(2y)-f(x+y)^2+f(x-y)^2|\leqslant\varphi(2y)\quad\text{for all}\quad x,y\in G. \tag{49}$$

Since f is unbounded, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ of elements from G such that

$$0 \neq |f(2t_n)| \longrightarrow \infty \quad \text{as} \quad n \to \infty. \tag{50}$$

Set $x = t_n$ in (49) to obtain

$$\left| g(2y) - \frac{f(t_n+y)^2 - f(t_n-y)^2}{f(2t_n)} \right| \leqslant \frac{\varphi(2y)}{|f(2t_n|} \quad \text{for all} \quad y \in G,\ n \in \mathbb{N}.$$

Now, passing to the limit as $n \to \infty$, and taking (50) into account, we infer that

$$g(2y) = \lim_{n\to\infty} \frac{f(t_n+y)^2 - f(t_n-y)^2}{f(2t_n)} \quad \text{for all} \quad y \in G. \tag{51}$$

Put $x = 2t_n + x$ in (48). Thus

$$\left| f(2t_n+x)g(y) - f\left(\frac{2t_n+x+y}{2}\right)^2 + f\left(\frac{2t_n+x-y}{2}\right)^2 \right| \leqslant \varphi(y),\ x,y \in G,\ n \in \mathbb{N}. \tag{52}$$

Setting $x = 2t_n - x$ in (48), we get

$$\left| f(2t_n-x)g(y) - f\left(\frac{2t_n-x+y}{2}\right)^2 + f\left(\frac{2t_n-x-y}{2}\right)^2 \right| \leqslant \varphi(y),\ x,y \in G,\ n \in \mathbb{N}. \tag{53}$$

Adding inequalities (52) and (53) side by side we infer that for all $x,y \in G$ and $n \in \mathbb{N}$ one has

$$\left| \frac{f(2t_n+x)+f(2t_n-x)}{f(2t_n)} g(y) - \frac{f\left(t_n+\frac{x+y}{2}\right)^2 - f\left(t_n-\frac{x+y}{2}\right)^2}{f(2t_n)} + \frac{f\left(t_n+\frac{x-y}{2}\right)^2 - f\left(t_n-\frac{x-y}{2}\right)^2}{f(2t_n)} \right| \leqslant \frac{2\varphi(y)}{|f(2t_n)|}.$$

In the light of (51), there exists the limit

$$h(x) := \lim_{n\to\infty} \frac{f(2t_n+x) + f(2t_n-x)}{f(2t_n)} \quad \text{for all} \quad x \in G. \tag{54}$$

Moreover, the function $h: G \to \mathbb{C}$ obtained in that way has to satisfies the equation

$$h(x)g(y) = g(x+y) - g(x-y) \quad \text{for all} \quad x,y \in G. \tag{55}$$

By definition (54) we have $h(0) = 2$. Now, from (55) putting $x = y = 0$ we get $g(0) = 0$. Setting $x = 0$ and $y = x$ in (55) we conclude that g is odd. Thus

$$h(x)g(x) = g(2x), \quad x \in G. \tag{56}$$

Applying (55), (56) and oddness of g we have

$$\begin{aligned} g(x+y)^2 - g(x-y)^2 &= [g(x+y)+g(x-y)][g(x+y)-g(x-y)] \\ &= [g(x+y)+g(x-y)]h(x)g(y) \\ &= [h(x)g(x+y)+h(x)g(x-y)]g(y) \\ &= [g(x+x+y)-g(x-x-y) \\ &\quad + g(x+x-y)-g(x-x+y)]g(y) \\ &= [g(y+2x)-g(-y)+g(-y+2x)-g(y)]g(y) \\ &= [g(y+2x)-g(y-2x)]g(y) = h(y)g(2x)g(y) \\ &= g(2x)g(2y) \quad \text{for all} \quad x,y \in G. \end{aligned}$$

On account of the unique 2-divisibility of G, the above equation states that g yields a solution to the Wilson's equation. The last inequality of our theorem results from (48) putting $y = x$.

Observe that the estimation spoken of in (iv) cannot essentially improved. This is visualized by the following

EXAMPLE 3. *Let $f, g : G \to \mathbb{C}$ have a form*

$$f(x) = a(x) + 1, \quad g(x) = a(x) \quad \textit{for all} \quad x \in G,$$

where $a : G \to \mathbb{C}$ is additive. Thus

$$\left| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| = |a(y)| =: \varphi(y) \quad \textit{for all} \quad x,y \in G.$$

From Theorem 7 for $g = f$ we obtain

COROLLARY 2. *Let $(G,+)$ be a uniquely 2-divisible Abelian group and let functions $f : G \to \mathbb{C}$, $\varphi : G \to \mathbb{R}$ satisfy the inequality*

$$\left| f(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \leqslant \varphi(y) \quad \textit{for all} \quad x,y \in G.$$

Then either f is bounded or f satisfies equation

$$f(x)f(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 \quad \textit{for all} \quad x,y \in G.$$

Now, we consider a very special case $\varphi(x) \equiv \varepsilon = const$ in Theorem 6 and Theorem 7.

COROLLARY 3. *Let $(G,+)$ be a uniquely 2-divisible Abelian group and let $\varepsilon \geqslant 0$ be a given number. Let $f,g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \leqslant \varepsilon \quad \text{for all} \quad x,y \in G.$$

Then

(i) *if $f=0$, then g is arbitrary,*
(ii) *if $g=0$, then f is bounded,*
(iii) *if $f \neq 0 \neq g$ and f is bounded, then g is bounded, too;*
(iv) *if $g \neq 0$ is bounded and f is unbounded, then g satisfies Wilson's equation* (1). *Moreover,*

$$|g(x)f(x) - f(x)^2| \leqslant \alpha$$

for all $x \in G$ and for some $\alpha \in \mathbb{R}$;
(v) *if $f \neq 0$ and g is unbounded, then f is unbounded, too. Moreover, there exists a function $h : G \to \mathbb{C}$ such that*

$$f(x+y) + f(x-y) = 2f(x)h(y) \quad \text{for all} \quad x,y \in G,$$

For $f = f_e + f_o$ we have $f_e(x) = Ch(x)$ for all $x \in G$ and for some $C \in \mathbb{R}$ and f_o satisfies Wilson's equation. Moreover, if $C \neq 0$, then

$$|g(x) - 2f_o(x)| \leqslant \frac{\varepsilon}{|C|} \quad \text{for all} \quad x \in G.$$

If $C=0$, then $g = f = f_o$.

Proof. Ad (ii). From (ii) in Theorem 6 (or (ii) in Theorem 7) for $g=0$ and $\varphi(x) \equiv \varepsilon$ we infer, that

$$|f(x)| \leqslant \sqrt{\varepsilon + |f(0)|^2} \quad \text{for all} \quad x \in G.$$

Thus, f is bounded.

Ad (iii). For $f \neq 0$, $g \neq 0$ and for bounded function f, by (iii) in Theorem 7, we obtain the estimation

$$|g(x)| \leqslant \frac{\varepsilon + 2M^2}{M} \quad \text{for all} \quad x \in G,$$

where $M := \sup\{|f(x)| : x \in G\}$; thus g is bounded.

Ad (iv). If $g \neq 0$ is bounded and f unbounded, then from (iv) in Theorem 7 we get

$$g\left(\frac{x+y}{2}\right)^2 - g\left(\frac{x-y}{2}\right)^2 = g(x)g(y) \quad \text{for all} \quad x,y \in G$$

and

$$|g(x)f(x) - f(x)^2| \leqslant \alpha$$

for all $x \in G$ and $\alpha := \varepsilon + |f(0)|^2$.

Ad (v). If we assume that $f \neq 0$ and g is unbounded, then from (iv) in Theorem 6 we conclude, that there exists a function $h : G \to \mathbb{C}$ such that

$$f(x+y) + f(x-y) = 2f(x)h(y) \quad \text{for all} \quad x, y \in G.$$

Moreover, $f_e(x) = Ch(x)$ for all $x \in G$ and some $C \in \mathbb{R}$ and

$$f_o(x)f_o(y) = f_o\left(\frac{x+y}{2}\right)^2 - f_o\left(\frac{x-y}{2}\right)^2 \quad \text{for all} \quad x, y \in G.$$

If $C \neq 0$, then

$$|g(x) - 2f_o(x)| \leqslant \frac{\varepsilon}{|C|} \quad \text{for all} \quad x \in G,$$

whereas in the case where $C = 0$, we get $f = f_o$ and

$$|g(x) - f_o(x)| \leqslant N, \quad x \in G,$$

with $N = \inf\left\{\frac{\varepsilon}{|f_o(x)|} : x \in G,\ f_o(x) \neq 0\right\}$. By (iii) we infer that whenever g is unbounded, then so is f. Now, for $f = f_o$ we have $N = 0$ which means that $f = g$ which finishes the proof.

3. Characterizations of the sine function

From the results of the previous section we derive some new characterizations of the sine function mapping the real line $\mathbb{R}$ into the complex plane $\mathbb{C}$. The basic idea in the proof was motivated by R. Badora and R. Ger [5]. Let us first recall a result that was proved in more general version by R. Ger in [9].

THEOREM 8. (A. Ostrowski) *Every not identically vanishing a Lebesgue measurable solution* $m : \mathbb{R} \to \mathbb{C}$ *of the exponential equation* (2) *has to have the form* $m(x) = e^{ax}, \quad x \in \mathbb{R}$, *where* a *is a complex constant.*

Combining that fact with Theorem 1, we conclude that every not identically vanishing a Lebesgue measurable solution $f : \mathbb{R} \to \mathbb{C}$ of the Wilson's equation (1) is either linear or has to have a form

$$f(x) = \beta \sin \alpha x \quad \text{for all} \quad x \in \mathbb{R},$$

with some $\alpha, \beta \in \mathbb{C}$.

In the light of these remarks we get the following corollaries.

COROLLARY 4. *A function $f:\mathbb{R}\to\mathbb{C}$ is of the form*

$$f(x)=\beta\sin\alpha x,\quad x\in\mathbb{R},$$

with some constants $\alpha\in\mathbb{C}\backslash\mathbb{R}$ and $\beta\in\mathbb{C}\backslash\{0\}$, if and only if

(i) *f is Lebesgue measurable,*
(ii) *$f(0)=0$ and f is unbounded*
(iii) *the function*

$$\mathbb{R}\backslash\{0\}\ni x\longmapsto\frac{f(x)}{x}\in\mathbb{C}$$

is nonconstant,
(iv) *there exists a function $\varphi:\mathbb{R}\to\mathbb{R}$ and a function $g:\mathbb{R}\to\mathbb{C}$ such that*

$$\left|f(x)g(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2\right|\leqslant\varphi(x)$$

for all $x,y\in G$.

Proof. $(\Rightarrow)$ If $f(x)=\beta\sin\alpha x$ for all $x\in\mathbb{R}$ and some $\alpha\in\mathbb{C}\backslash\mathbb{R}$, $\beta\in\mathbb{C}\backslash\{0\}$, then we define $g:\mathbb{R}\to\mathbb{C}$ and $\varphi:\mathbb{R}\to\mathbb{R}$ by

$$g(x)=f(x),\quad\varphi(x)=0\quad\text{for all}\quad x\in\mathbb{R}.$$

$(\Leftarrow)$ Condition (iv) jointly with Theorem 7 yields that there exists a function $h:G\to\mathbb{C}$ such that

$$f(x+y)+f(x-y)=2f(x)h(y)\quad\text{for all}\quad x,y\in G.$$

With the aid of the equality $f(0)=0$ and Lemma 1 we deduce that f satisfies Wilson's equation. Because of (iii) we rule out that f is linear. Therefore

$$f(x)=\beta\sin\alpha x$$

for every $x\in\mathbb{R}$, with some $\alpha,\beta\in\mathbb{C}$. Clearly, a function of that form is unbounded if and only if the constant α falls into the set $\mathbb{C}\backslash\mathbb{R}$ and $\beta\in\mathbb{C}\backslash\{0\}$.

COROLLARY 5. *A function $g:\mathbb{R}\to\mathbb{C}$ is of the form*

$$g(x)=\beta\sin\alpha x,\quad x\in\mathbb{R},$$

with some constant $\alpha\in\mathbb{C}\backslash\mathbb{R}$ and $\beta\in\mathbb{C}\backslash\{0\}$ if and only if

(i) *g is Lebesgue measurable,*
(ii) *the function*

$$\mathbb{R}\backslash\{0\}\ni x\longmapsto\frac{g(x)}{x}\in\mathbb{C}$$

is nonconstant,
(iii) *there exists a function $\varphi:\mathbb{R}\to\mathbb{R}$ and a unbounded function $f:\mathbb{R}\to\mathbb{C}$ such that*

$$\left|f(x)g(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2\right|\leqslant\varphi(y)\quad\textit{for all}\quad x,y\in G.$$

Proof. ($\Rightarrow$) Let $g(x) = \beta \sin \alpha x$ for all $x \in \mathbb{R}$ and for some $\alpha \in \mathbb{C} \setminus \mathbb{R}$, $\beta \in \mathbb{C} \setminus \{0\}$. Define $f : \mathbb{R} \to \mathbb{C}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = g(x), \quad \varphi(x) = 0 \quad \text{for all} \quad x \in \mathbb{R}$$

Because α falls into the set $\mathbb{C} \setminus \mathbb{R}$, the function g is unbounded. We get (iii), which completes the proof.

($\Leftarrow$) By assumption (iii) and Theorem 7 we deduce that g satisfies Wilson's equation whence

$$g(x) = \beta \sin \alpha x$$

for all $x \in \mathbb{R}$ and some $\alpha \in \mathbb{C} \setminus \mathbb{R}$, $\beta \in \mathbb{C} \setminus \{0\}$.

REFERENCES

[1] J. ACZÉL, *Functional equations and their applications*, Academic Press, 1966.
[2] J. ACZÉL, J. DHOMBRES, *Functional equations in several variables*, Cambridge University Press, 1989.
[3] R. BADORA, *On the stability of the cosine functional equation*, Rocznik Naukowo-Dydaktyczny WSP w Krakowie, **196** (1998), 1–14.
[4] R. BADORA, *On Hyers-Ulam stability of Wilson's functional equation*, Aequationes Math., **60** (2000), 211–218.
[5] R. BADORA, R. GER, *On some trigonometric functional inequalities*, Functional Equations-Results and Advances, Kluwer Academic Publishers (2002), 3–15.
[6] J.A. BAKER, *The stability of the cosine equation*, Proc. Amer. Math. Soc., **80** (1980), 411–416.
[7] P. GĂVRUTĂ, *A generalization of Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431–436.
[8] P. W. CHOLEWA, *The stability of the sine equation*, Proc. Amer. Math. Soc., **88** (1983), 631–634.
[9] R. GER, *Christensen measurability and functional equations*, Berichte der Mathematisch-Statistischen Sektion in der Forschungsgessellschaft Joanneum-Graz, Bericht **289** (1988), 1–17.
[10] D.H. HYERS, G. ISAC, TH.M. RASSIAS, *Stability of functional equations in several variables*, Birkhauser-Boston-Basel-Berlin, 1998.
[11] PL. KANNAPPAN, *The functional equations $f(xy) + f(xy)^{-1} = 2f(x)f(y)$ for groups*, Proc. Amer. Math. Soc., **19** (1968), 69–74.
[12] Z. MOSZNER, *On the stability of functional equations*, Aequationes Math., **77** (2009), 33–88.
[13] W.H. WILSON, *On certain related functional equations*, Bull. Amer. Math. Soc., **26** (1919), 300–312.

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