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INCREASING CO–RADIANT FUNCTIONS AND HERMITE–HADAMARD TYPE INEQUALITIES

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Abstract. In this paper, we study Hermite-Hadamard type inequalities for increasing co-radiant functions. Some examples of such inequalities for functions defined on a special domains are given.

1. Introduction

It is well-known that if $f:[a,b]\to R$ is convex function on $[a,b]$, then the inequality

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{1}{2}(f(a)+f(b)) \tag{1}$$

holds, and both inequalities in (1) are sharp. These inequalities are well-known as the Hermite-Hadamard inequalities. There are many different generalizations and companion results of these inequalities for convex functions, see [3] and [9].

There are also many generalizations of these inequalities for such nonconvex functions as quasiconvex functions [4, 8, 11], P-functions [5, 8], multiplicative convex functions [7], r-convex functions [6], Godunova-Levin type functions [5], increasing convex-along-rays functions [2, 10], increasing radiant functions [12], increasing positively homogeneous functions [1] etc.

For instance, in [6], if $f\in Q(I)$ (I is an interval in R), i.e.,

$$f(\lambda x+(1-\lambda)y)\leqslant\frac{f(x)}{\lambda}+\frac{f(y)}{1-\lambda},\ x,y\in I,\ \lambda\in(0,1)$$

and $f\in L_1[a,b]$, then

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{4}{b-a}\int_a^b f(x)dx$$

and the inequality is sharp.

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In the present paper, we consider one generalization of Hermite-Hadamard inequalities for the class of increasing co-radiant functions defined on the $R^n_{++} = \{x \in R^n | x_i > 0,\ i = 1,2,...,n\}$.

In Section 2 we give certain concepts of abstract convexity and definition of increasing co-radiant functions and we recall some results related to these functions. In Section 3 we consider Hermite-Hadamard type inequalities for the class increasing co-radiant functions. Some examples of such inequalities for functions defined on R_{++} and R^2_{++} are given in Section 4. In Section 5 the results are summarized.

2. Preliminaries

2.1. Abstract convexity and Hermite-Hadamard type inequalities

Let R be a real line and $R_{+\infty} = R \cup \{+\infty\}$. Consider a set X and a set H of function $h : X \to R$ defined on X. We assume that H is equipped with the pointwise order relation: if $f,g \in Y$, then $f \geqslant g$ if and only if $f(x) \geqslant g(x)$ for all $x \in X$.

A function $f : X \to R_{+\infty}$ is called abstract convex with respect to H (or H*-convex*) if there exists a set $U \subset H$ such that

$$f(x) = \sup\{h(x) : h \in U\} \text{ for all } x \in X.$$

Clearly f is H-convex if and only if

$$f(x) = \sup\{h(x) | h \leqslant f\} \text{ for all } x \in X.$$

Let Y be a set of functions $f : X \to R_{+\infty}$. A set $H \subset Y$ is called a supremal generator of the set Y if each function $f \in Y$ is abstract convex with respect to H.

A lot of authors study of Hermite-Hadamard type inequalities is based on the following Principle of Preservation of inequalities.

PROPOSITION 1. *Let H be a supremal generator of Y and let ψ be increasing functional defined on Y, that is, $(f,g \in Y,\ f \geqslant g\ ) \Longrightarrow \psi(f) \geqslant \psi(g)$. Then*

$$(h(u) \leqslant \psi(h) \textit{ for all } h \in H) \Leftrightarrow (f(u) \leqslant \psi(f) \textit{ for all } f \in Y).$$

The proof of Proposition 1 is easy and can be found in [10].

A function f is called increasing (with respect to the coordinate-wise order relation) if $x \geqslant y$ implies $f(x) \geqslant f(y)$.

The function f is positively homogeneous of degree one if $f(\lambda x) = \lambda f(x)$ for all $x \in R^n_+$ and $\lambda > 0$.

Let L be the set of all min-type functions defined on R^n_{++}, i.e. the set L consists of all functions of the form:

$$l(x) = \langle l,x \rangle := \min_i \frac{x_i}{l_i},\ x \in R^n_{++}$$

with $l \in R^n_{++}$. We know that (see [10]) a function $f : R^n_{++} \to R$ is L-convex if and only if f is increasing and positively homogeneous degree one (shortly IPH).

Let f be a function on R^n_{++}. For each $y \in R^n_{++}$ consider the ray $\{\lambda y | \lambda > 0\}$ and the restriction f_y of f on this ray. By definition

$$f_y(\lambda) = f(\lambda y),\ \lambda \geqslant 0$$

The function f is called convex-along-rays if the function f_y is convex for all $y \in R^n_{++}$.

For the increasing convex-along-rays (shortly ICAR) functions on R^2_{++}, the following proposition is considered in the study of Hermite-Hadamard inequalities (see [2]).

PROPOSITION 2. *Let f be an ICAR function defined on R^2_{++}. Then for each $(\bar{x}, \bar{y}) \in R^2_{++}$ there exists a number $b > 0$ such that*

$$b\left[\min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right) - 1\right] \leqslant f(x,y) - f(\bar{x}, \bar{y})$$

for all $(x,y) \in R^2_{++}$.

Proof of this theorem can be found in [10].

Recall that a function $f : R^n_{++} \to \bar{R}_+ = [0,\infty]$ is called radiant if $f(\lambda x) \leqslant \lambda f(x)$ for all $\lambda \in (0,1)$ and $x \in R^n_{++}$.

Denote by φ_l the function defined on R^n_{++} by the formula

$$\varphi_l(x) = \begin{cases} 0, & \text{if } \langle l,x\rangle < 1 \\ \langle l,x\rangle, & \text{if } \langle l,x\rangle \geqslant 1 \end{cases}$$

where $\langle l,x\rangle$ is min-type function.

It is known (see [12]) that the set

$$\mathscr{H} = \{c\varphi_l | l \in R^n_{++}, c \in [0,\infty)$$

is supremal generator of the class increasing radiant (shortly InR) functions defined on R^n_{++}.

Using the properties of *IPH* functions related to min-type functions [1] and the properties of *InR* functions related to φ_l functions [12], Hermite-Hadamard inequalities are studied.

2.2. Increasing co-radiant functions and its some properties

A function $f : K \to R_{+\infty}$ defined on a cone $K \subset R^n$ is called co-radiant if

$$f(\lambda x) \geqslant \lambda f(x) \text{ for all } x \in K,\ \lambda \in [0,1]$$

It is easy to check that f is co-radiant if and only if

$$f(\nu x) \leqslant \nu f(x) \text{ for all } x \in K,\ \nu \geqslant 1$$

We shall consider increasing co-radiant (shortly ICR) functions defined on the cone R^n_{++}.

Since an ICR function f is increasing and $f(0) \geqslant 0$ it follows that $f(x) \geqslant 0$ for all $x \in R^n_{++}$.

It is easy to check that a finite ICR function f is continuous on the cone R^n_{++}.

Let us give some examples of ICR functions:

- An increasing positively homogeneous function f of degree μ, where $\mu \in (0,1]$ is ICR. In particular, a Cobb-Douglas function

$$f(x) = kx_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \text{ with } \sum_{i=1}^{n} \alpha_n = 1,\ \alpha_i \geqslant 0$$

is ICR.

- Let P be a polynomial degree m with nonnegative coefficients. Then the function

$$f(x) = [P(x)]^{\frac{1}{m}}$$

is ICR.

- Let $f_i,\ i = 1,2,\ldots,m$ be an increasing positively homogeneous function of degree $0 \leqslant \mu_i \leqslant 1$. Then the sum, minimum and maximum of a family $f_1, f_2, \ldots, f_m$ are ICR functions.

- For each $f \in ICR$ its conjugate function

$$f^*(x) = \frac{1}{f(\frac{1}{x})}$$

where $\frac{1}{x} = (\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n})$, is also ICR.

Consider the function ψ_l defined on R^n_{++}

$$\psi_l(x) = \begin{cases} \langle l,x \rangle & \text{if } \langle l,x \rangle \leqslant 1 \\ 1 & \text{if } \langle l,x \rangle > 1 \end{cases}$$

where $l \in R^n_{++}$.

Recall that the set

$$H = \left\{ c\psi_l : l \in R^n_{++}, c \in [0,\infty] \right\}$$

is supremal generator of the class ICR functions defined on R^n_{++} (see [10]).

PROPOSITION 3. *Let f be an ICR function defined on R^n_{++}. Then the following inequality holds for all $x, l \in R^n_{++}$:*

$$f(l)\psi_l(x) \leqslant f(x) \tag{2}$$

Proof. If $\langle l,x\rangle > 1$, then $\psi_l(x) = 1$ and $\frac{x_i}{l_i} > 1$ for all $i = 1,2,...,n$. Consequently, we can write $x > l$. Since f is increasing functions, then (2) holds.

If $\langle l,x\rangle \leqslant 1$, then $\psi_l(x) = \langle l,x\rangle$. Since $\langle l,x\rangle = \min\limits_{1\leqslant i\leqslant n} \frac{x_i}{l_i} \implies x_i \geqslant l_i\psi_l(x)$ for $i = 1,2,...,n \implies x \geqslant l\psi_l(x)$. Because the function f is ICR, we have

$$f(x) \geqslant f(l\psi_l(x)) \geqslant \psi_l(x)f(l)$$

Hence inequality (2) holds for all $x,l \in R^n_{++}$. $\square$

Let f be an ICR function defined on R^n_{++} and $D \subset R^n_{++}$. It can be easily shown by Proposition 3 such that,

$$f_D(x) = \sup_{l\in D}(f(l)\psi_l(x)) \tag{3}$$

is ICR function and possesses the properties:

$$f_D(x) \leqslant f(x) \text{ for all } x \in R^n_{++},\ f_D(x) = f(x) \text{ for all } x \in D.$$

A function $f : D \to [0,\infty]$ is called ICR on D, if there exists an ICR function F defined on R^n_{++} such that $F|_D = f$, that is, $F(x) = f(x)$ for all $x \in D$.

PROPOSITION 4. *Let $f : D \to [0,\infty]$ be a function on $D \in R^n_{++}$, than the following assertions are equivalent:*

1) f is abstract convex with respect to the set of functions $c\psi_l : D \to [0,\infty]$ with $l \in D$, $c \in [0,\infty]$;

2) f is increasing co-radiant on D;

3) $f(l)\psi_l(x) \leqslant f(x)$ for all $l,x \in D$.

Proof. $1) \Rightarrow 2)$. It is obvious since any function $c\psi_n$ defined on D can be considered as elementary function $c\psi_n \in H$ defined R^n_{++}.

$2) \Rightarrow 3)$. By definition, there exists an ICR function $F : R^n_{++} \to [0,\infty]$ such that $F(x) = f(x)$ for all $x \in D$. Then by (3), we have

$$f(x) = F_D(x) = \sup_{l\in D} F(l)\psi_l = \sup_{l\in D} f(x)\psi_l(x)$$

for all $x \in D$, what implies the assertion 3).

$3) \Rightarrow 1)$. Consider the function f_D defined on D $f_D(x) = \sup\limits_{l\in D} f(x)\psi_l(x)$. It is clear that f_D is abstract convex with respect to the set of functions defined on D, i.e., $\left\{c\psi_l : l \in R^n_{++}, c \in [0,\infty)\right\}$. Further, using $3)$ we get for all $x \in D$

$$f_D(x) \leqslant f(x) = f(x)\psi_x(x) \leqslant \sup_{l\in D} f(l)\psi_l(x) = f_D(x)$$

Hence $f_D(x) = f(x)$ for all $x \in D$ and we have the defined statement $1)$. $\square$

The result of this proposition given below implies that the class ICR is broad enough.

COROLLARY 5. *Let $D \subset R^n_{++}$ be a set such that every point $x \in D$ is maximal in D, i.e., for all $x, y \in D : (y \geqslant x) \Rightarrow (y = x)$. Then for any function $f : D \to [0, \infty]$ there exists an ICR function $F : R^n_{++} \to [0, \infty]$, for which $F|_D = f$.*

Proof. By proposition 4, it is sufficiently to check only that $f(l)\psi_l(x) \leqslant f(x)$ for all $l, x \in D$. If $l = x$, it is clear. If $l \neq x$, then $\langle l, x\rangle < 1$ since l is maximal element in D, hence $\psi_l(x) = 0$ and $f(l)\psi_l(x) = 0 \leqslant f(x)$. □

3. Hermite-Hadamard type inequalities for ICR functions

Using the properties of ICR functions given in the second section, we examine Hermite-Hadamard type inequalities for ICR functions.

Let's state an important proposition which can be proved easily:

PROPOSITION 6. *Let $D \subset R^n_{++}$, $f : D \to [0, \infty]$ is ICR function and integrable on D. then the following inequality holds for all $u \in D$*

$$f(u)\int_D \psi_u(x)dx \leqslant \int_D f(x)dx \tag{4}$$

Proof. It can be easily shown by Proposition 4. □

Now let's survey Hermite-Hadamard type inequalities for ICR functions via sets $Q_k(D)$. Let $D \subset R^n_{++}$ be closed domain, i.e., D is bounded set such that $cl(intD) = D$ and k is positive number. Let $Q_k(D)$ denote the set of all points $\bar{x} \in D$ such that

$$\frac{k}{A(D)}\int_D \psi_{\bar{x}}(x)dx = 1 \tag{5}$$

where $A(D) = \int_D dx$.

In the case of $k = 1$, $Q_1(D)$ will be the set $Q(D)$ in [2] and [12].

PROPOSITION 7. *Let f be an ICR function on D. If the set $Q_k(D)$ is nonempty and f is integrable on D, then*

$$\sup_{\bar{x}\in Q_k(D)} f(\bar{x}) \leqslant \frac{k}{A(D)}\int_D f(x)dx \tag{6}$$

Proof. If $f(\bar{x}) = +\infty$, it is clear from Proposition 4 that f is not integrable. Hence, $f(\bar{x}) < +\infty$. According to Proposition 4, $f(\bar{x})\psi_{\bar{x}}(x) \leqslant f(x)$ for all $x \in D$. Since $\bar{x} \in Q_k(D)$, then by (4), we get

$$f(\bar{x}) = f(\bar{x})\frac{k}{A(D)}\int_D \psi_{\bar{x}}(x)dx = \frac{k}{A(D)}\int_D \psi_{\bar{x}}(x)f(\bar{x})dx \leqslant \frac{k}{A(D)}\int_D f(x)dx$$

whence (6) is derived. □

Now that for each $\bar{x} \in Q_k(D)$, we have also the following inequality

$$f(\bar{x}) \leqslant \frac{k}{A(D)} \int_D f(x)dx \tag{7}$$

and it is sharp. For example, if $f(\bar{x}) = \psi_{\bar{x}}(x)$, then (7) holds as the equality.

Now let us examine the right hand side of (1) for ICR functions. First, let's prove the auxiliary proposition. Let's denote maximum type functions by $\langle l,x\rangle^+$, i.e., $\langle l,x\rangle^+ = \max\limits_{1\leqslant i\leqslant n} \frac{x_i}{l_i}$.

PROPOSITION 8. *Let f be an ICR function on D. Then the following inequalities hold for all $l,x \in D$*

$$f(l) \leqslant \psi_x^+(l)f(x) \tag{8}$$

where

$$\psi_x^+(l) = \begin{cases} \langle x,l\rangle^+ & \text{if } \langle x,l\rangle^+ \geqslant 1 \\ 1 & \text{if } \langle x,l\rangle^+ \leqslant 1. \end{cases}$$

Proof. Since f is ICR function on D, then $f(l)\psi_l(x) \leqslant f(x)$ for all $x,l \in D$, that is,

$$\begin{array}{ll} f(l)\langle l,x\rangle \leqslant f(x), & \text{if } \langle l,x\rangle \leqslant 1 \\ f(l) \leqslant f(x), & \text{if } \langle l,x\rangle \geqslant 1 \end{array}$$

whence

$$\begin{array}{ll} f(l) \leqslant \frac{f(x)}{\langle l,x\rangle} = \langle x,l\rangle^+ f(x), & \text{if } \langle x,l\rangle^+ \geqslant 1 \\ f(l) \leqslant f(x), & \text{if } \langle x,l\rangle^+ \leqslant 1 \end{array}$$

thus

$$f(l) \leqslant \psi_x^+(l)f(x) \text{ for all } x,l \in D. \qquad \square$$

PROPOSITION 9. *Let function f be an ICR function and integrable on D. Then for all $u \in D$*

$$\int_D f(x)dx \leqslant f(u)\int_D \psi_u^+(x)dx. \tag{9}$$

Proof. It follows from Proposition 8 □

Inequality (9) is sharp since we get equality for $f(x) = \psi_u^+(x)$. The inequality (9) can be shown in a more general way as follows:

$$\int_D f(x)dx \leqslant \inf_{u\in D}\left\{ f(u)\int_D \psi_u^+(x)dx \right\} \tag{10}$$

4. Examples

Hermite-Hadamard type inequalities have been studied for ICAR functions [2], InR functions [12] and IPH functions [1] on the special domains of R_{++} and R^2_{++}. Let's derive (4), (6), (9) inequalities and the sets $Q_k(D)$ for ICR functions.

EXAMPLE 10. Let $D=[a,b]\subset R_{++}$. Since

$$\int_a^b \psi_u(x)dx=\int_a^u \frac{x}{u}dx+\int_u^b dx=\frac{2bu-u^2-a^2}{2u}$$

the formula (4) is as follows:

$$f(u)\leqslant \frac{2u}{2bu-u^2-a^2}\int_u^b f(x)dx \tag{11}$$

which are sharp in the class of all ICR functions $f\in L_1[a,b]$, and holds for any $u\in[a,b]$.

By definition, the set $Q_k(D)$ consists of all points $\bar{x}\in[a,b]$, for which,

$$\frac{k}{A(D)}\int_D \psi_{\bar{x}}(x)dx=\frac{k}{b-a}\int_a^b \psi_{\bar{x}}(x)dx=\frac{k(b\bar{x}-\bar{x}^2-a^2)}{2\bar{x}(b-a)}=1$$

So, a point $\bar{x}\in[a,b]$ belongs to $Q_k(D)$ if and only if

$$k\bar{x}^2+2[b-a-kb]\bar{x}+a^2k=0 \tag{12}$$

We get

$$\bar{x}=b-\frac{b-a}{k}\pm\sqrt{\left(\frac{b-a}{k}-b\right)^2-a^2}$$

If $k=1$, then $\bar{x}=a$. If $k<1$, it is easily shown that discriminant is negative, i.e., $\left(\frac{b-a}{k}-b\right)^2-a^2\leqslant 0$, so there is no real root of (12). If $k>1$, then one of the roots is $\bar{x}=b-\frac{b-a}{k}-\sqrt{\left(\frac{b-a}{k}-b\right)^2-a^2}<a$ and it is not in $[a,b]$. It is easily seen that if $k\leqslant\frac{2b}{b-a}$, then second root is in $[a,b]$. If $k\in[1,\frac{2b}{b-a}]$, $Q_k(D)$ is not empty and is as follows:

$$Q_k(D)=\left\{b-\frac{b-a}{k}+\sqrt{\left(\frac{b-a}{k}-b\right)^2-a^2}\right\}$$

Now let's derive the inequality (9).

$$\int_a^b f(x)dx\leqslant f(u)\int_a^b \psi_u^+(x)dx=f(u)\left(\int_a^u dx+\int_u^b \frac{x}{u}dx\right)=f(u)\frac{b^2+u^2-2au}{2u} \tag{13}$$

By combining the inequalities (11) and (13), for all $u \in [a,b]$, the following inequality is derived:

$$f(u)\frac{2bu-u^2-a^2}{2u(b-a)} \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant f(u)\frac{b^2+u^2-2au}{2u(b-a)} \tag{14}$$

In order to examine Hermite-Hadamard type inequalities on the domains $D \subset R^2_{++}$, let's derive the formula to compute the integrals $\int_D \psi_u(x)dx$ and $\int_D \psi_u^+(x)dx$ on these domains. Let $D \subset R^2_{++}$ and $u \in D$. In order to calculate integral, we represent the set D as $D_1(u) \cup D_2(u) \cup D_3(u)$, where

$$\begin{aligned} D_1(u) &= \left\{x \in D : \tfrac{x_2}{u_2} \leqslant \tfrac{x_1}{u_1},\ x_2 \leqslant u_2\right\}, \\ D_2(u) &= \left\{x \in D : \tfrac{x_2}{u_2} \geqslant \tfrac{x_1}{u_1},\ x_1 \leqslant u_1\right\}, \\ D_3(u) &= \{x \in D : x_1 \geqslant u_1,\ x_2 \geqslant u_2\}. \end{aligned}$$

Then

$$\begin{aligned} \int_D \psi_u(x)dx &= \int_{D_1(u)} \langle u,x\rangle\, dx + \int_{D_2(u)} \langle u,x\rangle\, dx + \int_{D_3(u)} dx \\ &= \frac{1}{u_2}\int_{D_1(u)} x_2 dx_1 dx_2 + \frac{1}{u_1}\int_{D_2(u)} x_1 dx_1 dx_2 + \int_{D_3(u)} dx_1 dx_2 \end{aligned} \tag{15}$$

In a similar way, $\int_D \psi_u^+(x)dx$ can be computed. For this case, $D_1^+(u) \cup D_2^+(u) \cup D_3^+(u)$, where

$$\begin{aligned} D_1^+(u) &= \left\{(x_1,x_2) \in D : \tfrac{x_2}{u_2} \leqslant \tfrac{x_1}{u_1},\ u_1 \leqslant x_1\right\}, \\ D_2^+(u) &= \left\{(x_1,x_2) \in D : \tfrac{x_2}{u_2} \geqslant \tfrac{x_1}{u_1},\ u_2 \leqslant x_2\right\}, \\ D_3^+(u) &= \{(x_1,x_2) \in D : x_1 \leqslant u_1,\ x_2 \leqslant u_2\} \end{aligned}$$

and we have

$$\begin{aligned} \int_D \psi_u^+(x)dx &= \int_{D_1^+(u)} \langle u,x\rangle^+ dx + \int_{D_2^+(u)} \langle u,x\rangle^+ dx + \int_{D_3^+(u)} dx \\ &= \frac{1}{u_1}\int_{D_1^+(u)} x_1 dx_1 dx_2 + \frac{1}{u_2}\int_{D_2^+(u)} x_2 dx_1 dx_2 + \int_{D_3^+(u)} dx_1 dx_2 \end{aligned} \tag{16}$$

EXAMPLE 11. Let $D \subset R^2_{++}$ be the triangle, that is

$$D = \left\{(x_1,x_2) \in R^2_{++} : 0 < x_1 \leqslant a,\ 0 < x_2 \leqslant vx_1\right\}$$

If $u \in D$, then we have

$$\begin{aligned} D_1(u) &= \left\{(x_1,x_2) \in R^2_{++} : 0 < x_2 \leqslant u_2,\ \tfrac{u_1}{u_2}x_2 < x_1 \leqslant a\right\}, \\ D_2(u) &= \left\{(x_1,x_2) \in R^2_{++} : 0 < x_1 < u_1,\ \tfrac{u_2}{u_1}x_1 \leqslant x_2 \leqslant vx_1\right\}, \\ D_3(u) &= \left\{(x_1,x_2) \in R^2_{++} : u_1 \leqslant x_1 \leqslant a,\ u_2 \leqslant x_2 \leqslant vx_1\right\} \end{aligned}$$

By (15) we get

$$\begin{aligned} \int_D \psi_u(x)dx &= \int_{D_1(u)} \langle u,x\rangle\, dx + \int_{D_2(u)} \langle u,x\rangle\, dx + \int_{D_3(u)} dx \\ &= \frac{1}{u_2}\int_0^{u_2}\int_{\frac{u_1x_2}{u_2}}^{a} x_2dx_1dx_2 + \frac{1}{u_1}\int_0^{u_1}\int_{\frac{u_2x_1}{u_1}}^{vx_1} x_1dx_2dx_1 + \int_{u_1}^{a}\int_{u_2}^{vx_1} dx_2dx_1 \\ &= \frac{1}{6}\left(2u_1u_2 + 3a^2v - vu_1^2 - 3au_2\right) \end{aligned}$$

Thus the inequality (4) will be as follows:

$$f(u_1,u_2) \leqslant \frac{6}{2u_1u_2 + 3a^2v - vu_1^2 - 3au_2}\int_D f(x_1,x_2)dx_1dx_2 \tag{17}$$

A point $\bar{x} \in D$ belongs to $Q_k(D)$ if and only if

$$\frac{2k}{va^2}\,\frac{3va^2 + 2\bar{x}_1\bar{x}_2 - 3a\bar{x}_2 - v\bar{x}_1^2}{6} = 1 \Leftrightarrow 2\bar{x}_1\bar{x}_2 - 3a\bar{x}_2 - v\bar{x}_1^2 = 3va^2k^{-1}$$

Consider now inequality (9) for domain D. Let's calculate the integral of the function $\psi_u^+(x)$ on D. In this case, $D_1^+(u), D_2^+(u)$ and $D_3^+(u)$ are as follows:

$$\begin{aligned} D_1^+(u) &= \left\{(x_1,x_2) \in D : u_1 \leqslant x_1 \leqslant a,\ 0 < x_2 \leqslant \tfrac{u_2}{u_1}x_1\right\}, \\ D_2^+(u) &= \left\{(x_1,x_2) \in D : \tfrac{u_2}{v} \leqslant x_1 \leqslant u_1,\ u_2 \leqslant x_2 \leqslant vx_1\right\} \cup \\ &\quad \left\{(x_1,x_2) \in D : u_1 \leqslant x_1 \leqslant a,\ \tfrac{u_2}{u_1}x_1 \leqslant x_2 \leqslant vx_1\right\}, \\ D_3^+(u) &= \left\{(x_1,x_2) \in D : 0 < x_2 \leqslant u_2,\ \tfrac{x_2}{v} \leqslant x_1 \leqslant u_1\right\}. \end{aligned}$$

By (16) we get

$$\begin{aligned} \int_D \psi_u^+(x)dx &= \frac{1}{u_1}\int_{u_1}^{a}\int_0^{\frac{u_2x_1}{u_1}} x_1dx_2dx_1 + \frac{1}{u_2}\int_{\frac{u_2}{v}}^{u_1}\int_{u_2}^{vx_1} x_2dx_2dx_1 + \frac{1}{u_2}\int_{u_1}^{a}\int_{\frac{u_2x_1}{u_1}}^{vx_1} x_2dx_2dx_1 + \int_0^{u_2}\int_{\frac{x_2}{v}}^{u_1} dx_1dx_2 \\ &= \frac{a^3vu_2^2 - u_1^2u_2^3 + 2vu_1^3u_2^2 + a^3v^3u_1^2}{6u_2u_1^2v} \end{aligned}$$

thus

$$\int_D f(x_1,x_2)dx_1dx_2 \leqslant \frac{6u_2u_1^2v}{a^3vu_2^2 - u_1^2u_2^3 + 2vu_1^3u_2^2 + a^3v^3u_1^2} f(u_1,u_2) \tag{18}$$

EXAMPLE 12. Consider the triangle $D \subset R^n_{++}$ defined as

$$D = \left\{ (x_1, x_2) \in R^2_{++} : \frac{x_1}{a} + \frac{x_2}{b} \leqslant 1 \right\}$$

Let $u \in D$. Then we get

$$\begin{aligned} D_1(u) &= \left\{ (x_1,x_2) \in R^2_{++} : 0 < x_2 \leqslant u_2,\ \tfrac{u_1}{u_2} x_2 \leqslant x_1 \leqslant a - \tfrac{a}{b} x_2 \right\}, \\ D_2(u) &= \left\{ (x_1,x_2) \in R^2_{++} : 0 < x_1 < u_1,\ \tfrac{u_2}{u_1} x_1 \leqslant x_2 \leqslant b - \tfrac{b}{a} x_1 \right\}, \\ D_3(u) &= \left\{ (x_1,x_2) \in R^2_{++} : u_1 \leqslant x_1 \leqslant a - \tfrac{a}{b} u_2,\ u_2 \leqslant x_2 \leqslant b - \tfrac{b}{a} x_1 \right\} \end{aligned}$$

By (15) we have

$$\begin{aligned} \int_D \psi_u(x)dx &= \frac{1}{u_2} \int_0^{u_2} \int_{\frac{u_1 x_2}{u_2}}^{a - \frac{a x_2}{b}} x_2 dx_1 dx_2 + \frac{1}{u_1} \int_0^{u_1} \int_{\frac{x_1 u_2}{u_1}}^{b - \frac{b x_1}{a}} x_1 dx_2 dx_1 + \int_{u_1}^{a - \frac{a u_2}{b}} \int_{u_2}^{b - \frac{b x_1}{a}} dx_2 dx_1 \\ &= \frac{ab}{6} \left[\left(\frac{u_1}{a} + \frac{u_2}{b} \right)^2 - 3 \left(\frac{u_1}{a} + \frac{u_2}{b} \right) + 3 \right] \end{aligned}$$

In this domain, the inequality (4) is as follows:

$$f(u_1,u_2) \leqslant \frac{6}{ab \left[\left(\frac{u_1}{a} + \frac{u_2}{b} \right)^2 - 3 \left(\frac{u_1}{a} + \frac{u_2}{b} \right) + 3 \right]} \int_D f(x_1,x_2) dx_1 dx_2 \tag{19}$$

or

$$f(u_1,u_2) \leqslant \frac{6ab}{(u_1 b + u_2 a)^2 - 3ab(u_1 b + u_2 a) + 3a^2 b^2} \int_D f(x_1,x_2) dx_1 dx_2$$

Let's derive the set $Q_k(D)$ for the given triangular domain D. Since $A(D) = \frac{ab}{2}$, then for $\bar{x} \in D$

$$\begin{aligned} \bar{x} \in Q_k(D) &\Leftrightarrow \frac{k}{3} \left[\left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right)^2 - 3 \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right) + 3 \right] = 1 \\ &\Leftrightarrow \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right)^2 - 3 \left(\frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} \right) + 3 - \frac{3}{k} = 0 \end{aligned}$$

whence

$$\bar{x} \in Q_k(D) \Leftrightarrow \frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} = \frac{3}{2} - \sqrt{\frac{3}{k} - \frac{3}{4}}.$$

If $0 \leqslant \frac{3}{2} - \sqrt{\frac{3}{k} - \frac{3}{4}} \leqslant 1$, $Q_k(D)$ is not empty. Hence, $k \in [1,3]$. Thus for $k \in [1,3]$,

$$Q_k(D) = \left\{ (\bar{x}_1, \bar{x}_2) \in D : \frac{\bar{x}_1}{a} + \frac{\bar{x}_2}{b} = \frac{3}{2} - \sqrt{\frac{3}{k} - \frac{3}{4}} \right\}$$

and for $k \in R_+ \setminus [1,3]$, $Q_k(D)$ is empty set.

For the same domain D, let's compute $\int\limits_D \psi_u^+(x)dx$ in order to derive the inequality (9). To do this, let's express the sets $D_1^+(u), D_2^+(u)$ and $D_3^+(u)$ first:

$$\begin{aligned}
D_1^+(u) &= \left\{(x_1,x_2) \in R_{++}^2 : u_1 \leqslant x_1 \leqslant \tfrac{abu_1}{au_2+bu_1},\ 0 < x_2 \leqslant \tfrac{u_2}{u_1}x_1\right\} \cup \\
&\quad \left\{(x_1,x_2) \in R_{++}^2 : \tfrac{abu_1}{au_2+bu_1} \leqslant x_1 \leqslant a,\ 0 < x_2 \leqslant b - \tfrac{b}{a}x_1\right\}, \\
D_2^+(u) &= \left\{(x_1,x_2) \in R_{++}^2 : u_2 \leqslant x_2 \leqslant \tfrac{abu_2}{au_2+bu_1},\ 0 < x_1 \leqslant \tfrac{u_1}{u_2}x_2\right\} \cup \\
&\quad \left\{(x_1,x_2) \in R_{++}^2 : \tfrac{abu_2}{au_2+bu_1} \leqslant x_2 \leqslant b,\ 0 < x_1 \leqslant a - \tfrac{a}{b}x_2\right\}, \\
D_3^+(u) &= \{(x_1,x_2) \in D : 0 < x_1 \leqslant u_1,\ 0 \leqslant x_2 \leqslant u_2\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int\limits_D \psi_u^+(x)dx &= \frac{1}{u_1}\int\limits_{u_1}^{\frac{abu_1}{au_2+bu_1}}\int\limits_0^{\frac{u_2x_1}{u_1}} x_1dx_2dx_1 + \frac{1}{u_1}\int\limits_{\frac{abu_1}{au_2+bu_1}}^{a}\int\limits_0^{b-\frac{bx_1}{a}} x_1dx_2dx_1 \\
&\quad + \frac{1}{u_2}\int\limits_{u_2}^{\frac{abu_2}{au_2+bu_1}}\int\limits_0^{\frac{u_1x_2}{u_2}} x_2dx_1dx_2 + \frac{1}{u_2}\int\limits_{\frac{abu_2}{au_2+bu_1}}^{b}\int\limits_0^{a-\frac{ax_2}{b}} x_2dx_1dx_2 + \int\limits_0^{u_1}\int\limits_0^{u_2} dx_2dx_1 \\
&= \frac{u_1u_2}{3} + \frac{ab(au_2+bu_1)}{6u_1u_2} - \frac{a^2b^2}{6(au_2+bu_1)}.
\end{aligned}$$

Therefore

$$\int\limits_D f(x_1,x_2)dx_1dx_2 \leqslant \frac{ab}{6}\left(\frac{2u_1u_2}{ab} + \frac{au_2+bu_1}{u_1u_2} - \frac{ab}{au_2+bu_1}\right) f(u_1,u_2) \tag{20}$$

By combining (19) and (20), for all $u \in D$, the following inequality holds:

$$\begin{aligned}
\tfrac{f(u_1,u_2)}{3}\left[\left(\tfrac{u_1}{a}+\tfrac{u_2}{b}\right)^2 - 3\left(\tfrac{u_1}{a}+\tfrac{u_2}{b}\right) + 3\right] &\leqslant \tfrac{1}{A(D)}\int\limits_D f(x_1,x_2)dx_1dx_2 \\
&\leqslant \tfrac{f(u_1,u_2)}{3}\left(\tfrac{2u_1u_2}{ab} + \tfrac{au_2+bu_1}{u_1u_2} - \tfrac{ab}{au_2+bu_1}\right)
\end{aligned} \tag{21}$$

EXAMPLE 13. Let $D \subset R_{++}^2$ be the rectangle defined as

$$D = \left\{(x_1,x_2) \in R_{++}^2 : x_1 \leqslant a,\ x_2 \leqslant b\right\}$$

If $u \in D$, then

$$\begin{aligned}
D_1(u) &= \left\{(x_1,x_2) \in R_{++}^2 : 0 < x_2 \leqslant u_2,\ \tfrac{u_1}{u_2}x_2 \leqslant x_1 \leqslant a\right\}, \\
D_2(u) &= \left\{(x_1,x_2) \in R_{++}^2 : 0 < x_1 < u_1,\ \tfrac{u_2}{u_1}x_1 \leqslant x_2 \leqslant b\right\}, \\
D_3(u) &= \left\{(x_1,x_2) \in R_{++}^2 : u_1 \leqslant x_1 \leqslant a,\ u_2 \leqslant x_2 \leqslant b\right\}
\end{aligned}$$

By (15),

$$\int_D \psi_u(x)dx = \frac{1}{u_2}\int_0^{u_2}\int_{\frac{u_1x_2}{u_2}}^{a} x_2dx_1dx_2 + \frac{1}{u_1}\int_0^{u_1}\int_{\frac{x_1u_2}{u_1}}^{b} x_1dx_2dx_1 + \int_{u_1}^{a}\int_{u_2}^{b} dx_2dx_1$$
$$= ab - \frac{1}{2}(au_2+bu_1) + \frac{1}{3}u_1u_2$$

In this rectangular domain D the inequality (4) is as follows:

$$f(u_1,u_2) \leqslant \frac{1}{ab-\frac{1}{2}(au_2+bu_1)+\frac{1}{3}u_1u_2}\int_D f(x_1,x_2)dx_1dx_2 \tag{22}$$

Since $A(D)=ab$, then we get the equation for $\bar{x}\in Q_k(D)$

$$\frac{k}{ab}(ab-\frac{1}{2}(a\bar{x}_2+b\bar{x}_1)+\frac{1}{3}\bar{x}_1\bar{x}_2) = 1 \Leftrightarrow 2\bar{x}_1\bar{x}_2 - 3(a\bar{x}_2+b\bar{x}_1) = (1-k)ab$$

whence

$$Q_k(D) = \{(\bar{x}_1,\bar{x}_2)\in D : 2\bar{x}_1\bar{x}_2 - 3(a\bar{x}_2+b\bar{x}_1) = (1-k)ab\}$$

Consider now inequality (9). Let's study two cases:
a) If $\frac{u_2}{u_1}\leqslant\frac{b}{a}$ then $D_1^+(u)$, $D_2^+(u)$ and $D_3^+(u)$ will be as follows:

$$\begin{aligned}
D_1^+(u) &= \left\{(x_1,x_2)\in R^2_{++} : u_1\leqslant x_1\leqslant a,\ 0<x_2\leqslant \tfrac{u_2}{u_1}x_1\right\},\\
D_2^+(u) &= \left\{(x_1,x_2)\in R^2_{++} : 0<x_1\leqslant u_1,\ u_2\leqslant x_2\leqslant b\right\}\cup\\
&\quad \left\{(x_1,x_2)\in R^2_{++} : u_1\leqslant x_1\leqslant a,\ \tfrac{u_2}{u_1}x_1\leqslant x_2\leqslant b\right\},\\
D_3^+(u) &= \left\{(x_1,x_2)\in R^2_{++} : 0<x_1\leqslant u_1,\ 0<x_2\leqslant u_2\right\}.
\end{aligned}$$

By (16) we have

$$\int_D \psi_u^+(x)dx = \frac{1}{u_1}\int_0^{a}\int_0^{\frac{u_2x_1}{u_1}} x_1dx_2dx_1 + \frac{1}{u_2}\int_0^{u_1}\int_{u_2}^{b} x_2dx_2dx_1 + \frac{1}{u_2}\int_{u_1}^{a}\int_{\frac{u_2}{u_1}x_1}^{b} x_2dx_2dx_1 + \int_0^{u_1}\int_0^{u_2} dx_2dx_1$$
$$= \frac{2}{3}u_1u_2 + \frac{a^3u_2}{6u_1^2} + \frac{b^2a}{2u_2}$$

b) If $\frac{u_2}{u_1}\geqslant\frac{b}{a}$, then $D_1^+(u)$, $D_2^+(u)$ and $D_3^+(u)$ are follows:

$$\begin{aligned}
D_1^+(u) &= \left\{(x_1,x_2)\in R^2_{++} : 0<x_2\leqslant u_2,\ u_1\leqslant x_1\leqslant a\right\}\cup\\
&\quad \left\{(x_1,x_2)\in R^2_{++} : x_2\leqslant u_2\leqslant b,\ \tfrac{u_1}{u_2}x_2\leqslant x_1\leqslant a\right\},\\
D_2^+(u) &= \left\{(x_1,x_2)\in R^2_{++} : u_2\leqslant x_2\leqslant b,\ 0\leqslant x_1\leqslant \tfrac{u_1}{u_2}x_2\right\},\\
D_3^+(u) &= \left\{(x_1,x_2)\in R^2_{++} : 0<x_2\leqslant u_2,\ 0\leqslant x_1\leqslant u_1\right\}
\end{aligned}$$

and after necessary computations, the following inequality is derived:

$$\int_D \psi_u^+(x)dx = \frac{2}{3}u_1u_2 + \frac{b^3u_1}{6u_2^2} + \frac{ba^2}{2u_1}.$$

By combining two cases, for $u \in D$, the following equality is derived:

$$\int_D \psi_u^+(x)dx = \Phi(u_1,u_2) = \begin{cases} \frac{2}{3}u_1u_2 + \frac{a^3u_2}{6u_1^2} + \frac{b^2a}{2u_2}, & \text{if } \frac{u_2}{u_1} \leqslant \frac{b}{a} \\ \frac{2}{3}u_1u_2 + \frac{b^3u_1}{6u_2^2} + \frac{ba^2}{2u_1}, & \text{if } \frac{u_2}{u_1} \geqslant \frac{b}{a}. \end{cases}$$

Hence, for the rectangular region D, (9) inequality is as follows:

$$\int_D f(x_1,x_2)dx \leqslant \Phi(u_1,u_2)f(u_1,u_2) \tag{23}$$

EXAMPLE 14. We shall now consider the case where the set D is part of the disk defined as

$$D = \left\{(x_1,x_2) \in R_{++}^2 : x_1^2 + x_2^2 \leqslant r^2\right\}$$

If $u \in D$, we get

$$\begin{aligned} D_1(u) &= \left\{(x_1,x_2) \in R_{++}^2 : 0 < x_2 \leqslant u_2,\ \tfrac{u_1}{u_2}x_2 \leqslant x_1 \leqslant \sqrt{r^2 - x_2^2}\right\}, \\ D_2(u) &= \left\{(x_1,x_2) \in R_{++}^2 : 0 < x_1 \leqslant u_1,\ \tfrac{u_2}{u_1}x_1 \leqslant x_2 \leqslant \sqrt{r^2 - x_1^2}\right\}, \\ D_3(u) &= \left\{(x_1,x_2) \in R_{++}^2 : u_1 \leqslant x_2 \leqslant r,\ u_2 \leqslant x_2 \leqslant \sqrt{r^2 - x_1^2}\right\}. \end{aligned}$$

By (15) we have

$$\begin{aligned} \int_D \psi_u(x)dx &= \frac{1}{u_2}\int_0^{u_2}\int_{\frac{u_1x_2}{u_2}}^{\sqrt{r^2-x_2^2}} x_2dx_1dx_2 + \frac{1}{u_1}\int_0^{u_1}\int_{\frac{u_2x_1}{u_1}}^{\sqrt{r^2-x_1^2}} x_1dx_2dx_1 + \int_{u_1}^{r}\int_{u_2}^{\sqrt{r^2-x_1^2}} dx_2dx_1 \\ &= \frac{\pi r^2}{4} - \frac{r^2}{2}\arcsin\frac{u_1}{r} - \frac{u_1(r^2-u_1^2)^{\frac{1}{2}}}{2} - \frac{(r^2-u_2^2)^{\frac{3}{2}}}{3u_2} \\ &\quad - \frac{(r^2-u_1^2)^{\frac{3}{2}}}{3u_1} + \frac{u_1u_2}{3} + \frac{r^3}{3u_2} + \frac{r^3}{3u_1} - u_2r \equiv \varphi(u_1,u_2). \end{aligned}$$

By using the equality above, the inequality (4) will be as follows:

$$f(u_1,u_2) \leqslant \frac{1}{\varphi(u_1,u_2)}\int_D f(x_1,x_2)dx_1dx_2. \tag{24}$$

Since $A(D) = \frac{\pi r^2}{4}$, then we get

$$Q_k(D) = \left\{(\bar{x}_1,\bar{x}_2) \in D : \varphi(\bar{x}_1,\bar{x}_2) = \frac{\pi r^2}{4k}\right\}$$

For the same domain D, let's compute $\int_D \psi_u^+(x)dx$ in order to derive the inequality (9). Let's determine the sets $D_1^+(u)$, $D_2^+(u)$ and $D_3^+(u)$. Let $u \in D$. Then

$$D_1^+(u) = \left\{(x_1,x_2) \in R_{++}^2 : 0 < x_2 \leqslant u_2,\ u_1 \leqslant x_1 \leqslant \sqrt{r^2 - x_2^2}\right\} \cup \left\{(x_1,x_2) \in R_{++}^2 : u_2 \leqslant x_2 \leqslant \frac{r}{\sqrt{1+\left(\frac{u_1}{u_2}\right)^2}},\ \frac{u_1}{u_2}x_2 \leqslant x_1 \leqslant \sqrt{r^2 - x_2^2}\right\},$$
$$D_2^+(u) = \left\{(x_1,x_2) \in R_{++}^2 : 0 < x_1 \leqslant u_1,\ u_2 \leqslant x_2 \leqslant \sqrt{r^2 - x_1^2}\right\} \cup \left\{(x_1,x_2) \in R_{++}^2 : u_1 \leqslant x_1 \leqslant \frac{r}{\sqrt{1+\left(\frac{u_2}{u_1}\right)^2}},\ \frac{u_2}{u_1}x_1 \leqslant x_2 \leqslant \sqrt{r^2 - x_1^2}\right\},$$
$$D_3^+(u) = \left\{(x_1,x_2) \in R_{++}^2 : 0 < x_1 \leqslant u_1,\ 0 < x_2 \leqslant u_2\right\}.$$

By (16) we get

$$\int_D \psi_u^+(x)dx = \frac{1}{u_1}\int_0^{u_2}\int_{u_1}^{\sqrt{r^2-x_2^2}} x_1 dx_1 dx_2 + \frac{1}{u_1}\int_{u_2}^{\frac{r}{\sqrt{1+(\frac{u_1}{u_2})^2}}}\int_{\frac{u_1 x_2}{u_2}}^{\sqrt{r^2-x_2^2}} x_1 dx_1 dx_2$$
$$+\frac{1}{u_2}\int_0^{u_1}\int_{u_2}^{\sqrt{r^2-x_1^2}} x_2 dx_2 dx_1 + \frac{1}{u_2}\int_{u_1}^{\frac{r}{\sqrt{1+(\frac{u_2}{u_1})^2}}}\int_{\frac{u_2 x_1}{u_1}}^{\sqrt{r^2-x_1^2}} x_2 dx_2 dx_1 + \int_0^{u_1}\int_0^{u_2} dx_2 dx_1$$
$$= \frac{r^3(\sqrt{u_1^2+u_2^2}+u_1^2u_2^2)}{3u_1u_2}.$$

Hence

$$\int_D f(x_1,x_2)dx_1dx_2 \leqslant \frac{r^3(\sqrt{u_1^2+u_2^2}+u_1u_2)}{3u_1u_2} f(u_1,u_2) \tag{25}$$

for all $(u_1,u_2) \in D$.

5. Conclusion

Hermite-Hadamard inequality, which is derived for convex functions and denoted by (1) is generalized by some authors (see [1], [2] and [12]) for some certain classes of abstract convex functions (for IPH, ICAR and InR functions)

In this article, same inequalities are studied for ICR functions, a class of abstract convex functions and considerable results are derived. $Q(D)$ which presented in the previous works is defined for more general cases and the inequality (6) which is more general is derived. In order to derive the similar inequality of the right hand side of

(1) for ICR functions, some studies are done and the inequalities like (9) and (10) are derived. Special domains which are studied for IPH, ICAR and InR functions are taken into consideration for ICR functions (in some examples, more general domains are studied) and some inequalities like (14), (17), (18), (21), (22), (23), (24) and (25) changing with respect to domain are derived.

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