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AN ALTERNATIVE PROOF OF ELEZOVIĆ–GIORDANO–PEČARIĆ'S THEOREM

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Abstract. In this paper, an alternative proof is supplied for monotonicity and convexity of the function $z_{s,t}(x)=\left[\Gamma(x+t)/\Gamma(x+s)\right]^{1/(t-s)}-x$ with $z_{s,s}(x)=e^{\psi(x+s)}-x$, where Γ is the classical Euler's gamma function, s and t are real numbers with $t-s\neq\pm1$, $\alpha=\min\{s,t\}$ and $x\in(-\alpha,\infty)$.

1. Introduction

Let s and t be real numbers with $t-s\neq\pm1$ and define

$$z_{s,t}(x)=\begin{cases}\left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}-x, & s\neq t\\ e^{\psi(x+s)}-x, & s=t\end{cases} \tag{1}$$

for $x\in(-\alpha,\infty)$ with $\alpha=\min\{s,t\}$, where

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}t \tag{2}$$

for $x>0$ stands for the classical Euler's gamma function, and $\psi(x)$ denotes the psi or digamma function, the logarithmic derivative of $\Gamma(x)$.

The implicit or explicit origins and backgrounds of the function $z_{s,t}(x)$ defined by (1) may be traced back to [3, 6, 18, 20], especially [7, Theorem 2]. The monotonicity and convexity of the function $z_{s,t}(x)$ and its special cases have been proved several times by different approaches in, for example, [1, 7, 13, 15, 17, 18]. We observe that a complete solution to monotonicity and convexity of the function $z_{s,t}(x)$ was first given in [2, Theorem 1]. For detailed information on the history, please refer to the survey and expository papers [9, 10] and plenty of references therein.

We recite the monotonicity and convexity of the function $z_{s,t}(x)$ as follows.

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THEOREM 1. *The function* $z_{s,t}(x)$ *is*

1. *convex and decreasing for* $|t-s|<1$,
2. *concave and increasing for* $|t-s|>1$.

The purpose of this paper is to present an alternative proof of Theorem 1.

2. Lemmas

In order to prove Theorem 1 alternatively, the following lemmas are necessary.

LEMMA 1. ([8, p. 16]) *The polygamma functions* $\psi^{(n)}(x)$ *can be expressed for* $x>0$ *and* $n\in\mathbb{N}$ *as*

$$\psi^{(n)}(x)=(-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}\,\mathrm{d}t. \tag{3}$$

LEMMA 2. ([19]) *Let* $f_i(t)$ *for* $i=1,2$ *be piecewise continuous in arbitrary finite intervals included on* $(0,\infty)$, *suppose there exist some constants* $M_i>0$ *and* $c_i\geqslant 0$ *such that* $|f_i(t)|\leqslant M_ie^{c_it}$ *for* $i=1,2$. *Then*

$$\int_0^\infty\left[\int_0^t f_1(u)f_2(t-u)\,\mathrm{d}u\right]e^{-st}\,\mathrm{d}t=\int_0^\infty f_1(u)e^{-su}\,\mathrm{d}u\int_0^\infty f_2(v)e^{-sv}\,\mathrm{d}v. \tag{4}$$

LEMMA 3. *For* $x\in(0,\infty)$,

$$\ln x-\frac{1}{x}<\psi(x)<\ln x-\frac{1}{2x} \tag{5}$$

and

$$\frac{1}{x}+\frac{1}{2x^2}<\psi'(x)<\frac{1}{x}+\frac{1}{x^2}. \tag{6}$$

Proof. This may be derived easily from the fact [16, p. 82] that a completely monotonic function which is non-identically zero cannot vanish at any point on $(0,\infty)$ and the complete monotonicity obtained in [11, Theorem 2]: The function $\psi(x)-\ln x+\frac{\alpha}{x}$ is completely monotonic on $(0,\infty)$ if and only if $\alpha\geqslant 1$ and so is the function $\ln x-\frac{\alpha}{x}-\psi(x)$ if and only if $\alpha\leqslant\frac{1}{2}$.

LEMMA 4. *For* $u\in\mathbb{R}$ *and* $\beta>\alpha\geqslant 0$ *with* $(\alpha,\beta)\neq(0,1)$, *let*

$$q_{\alpha,\beta}(u)=\begin{cases}\dfrac{e^{-\alpha u}-e^{-\beta u}}{1-e^{-u}}, & u\neq 0;\\ \beta-\alpha, & u=0.\end{cases} \tag{7}$$

1. *The function* $q_{\alpha,\beta}(u)$ *is logarithmically convex for* $\beta-\alpha>1$ *and logarithmically concave for* $0<\beta-\alpha<1$ *on* $(-\infty,\infty)$.

2. *For $\beta-\alpha>1$, the function*

$$Q_{s,t;\lambda}(u)=q_{\alpha,\beta}(u)q_{\alpha,\beta}(\lambda-u) \tag{8}$$

is increasing on $\left(\frac{\lambda}{2},\infty\right)$ and decreasing on $\left(-\infty,\frac{\lambda}{2}\right)$, where λ is any real constant; For $0<\beta-\alpha<1$, it is decreasing on $\left(\frac{\lambda}{2},\infty\right)$ and increasing on $\left(-\infty,\frac{\lambda}{2}\right)$.

Proof. It is clear that the function $q_{\alpha,\beta}(u)$ can be rewritten as

$$q_{\alpha,\beta}(u)=\frac{\sinh((\beta-\alpha)u/2)}{\sinh(u/2)}\exp\frac{(1-\alpha-\beta)u}{2}\triangleq p_{\alpha,\beta}\left(\frac{u}{2}\right).$$

Since the functions $q_{\alpha,\beta}(u)$ and $p_{\alpha,\beta}(u)$ are positive for $\beta>\alpha$, taking the logarithm of $p_{\alpha,\beta}(u)$ and differentiating yield

$$\begin{aligned}
\ln p_{\alpha,\beta}(u)&=\ln\sinh((\beta-\alpha)u)-\ln\sinh u+(1-\alpha-\beta)u,\\
[\ln p_{\alpha,\beta}(u)]'&=(\beta-\alpha)\coth((\beta-\alpha)u)-\coth u-\alpha-\beta+1,\\
[\ln p_{\alpha,\beta}(u)]''&=\frac{1}{u^2}\left\{\left(\frac{u}{\sinh u}\right)^2-\left[\frac{(\beta-\alpha)u}{\sinh((\beta-\alpha)u)}\right]^2\right\}\\
&\triangleq\frac{[h(u)]^2-[h((\beta-\alpha)u)]^2}{u^2}.
\end{aligned}$$

It is clear that the functions $h(u)$ and $[\ln p_{\alpha,\beta}(u)]''$ are even and the former is positive on $(-\infty,\infty)$, increasing on $(-\infty,0)$, and decreasing on $(0,\infty)$. As a result,

1. for $\beta-\alpha>1$, if $u>0$, then $(\beta-\alpha)u>u>0$ and $h((\beta-\alpha)u)<h(u)$, and so $[\ln p_{\alpha,\beta}(u)]''>0$ on $(0,\infty)$;
2. for $\beta-\alpha>1$, if $u<0$, then $(\beta-\alpha)u<u<0$ and $h((\beta-\alpha)u)<h(u)$, and so $[\ln p_{\alpha,\beta}(u)]''>0$ on $(-\infty,0)$;
3. for $0<\beta-\alpha<1$, if $u>0$, then $0<(\beta-\alpha)u<u$ and $h((\beta-\alpha)u)>h(u)$, and so $[\ln p_{\alpha,\beta}(u)]''<0$ on $(0,\infty)$;
4. for $0<\beta-\alpha<1$, if $u<0$, then $0>(\beta-\alpha)u>u$ and $h((\beta-\alpha)u)>h(u)$, and so $[\ln p_{\alpha,\beta}(u)]''<0$ on $(-\infty,0)$.

From the obvious relationship $p_{\alpha,\beta}(u)=q_{\alpha,\beta}(2u)$ on $(-\infty,\infty)$, the logarithmically convex properties in Lemma 4 follows readily.

Taking the logarithm of $Q_{s,t;\lambda}(u)$ and differentiating give

$$[\ln Q_{s,t;\lambda}(u)]'=\frac{q'_{\alpha,\beta}(u)}{q_{\alpha,\beta}(u)}-\frac{q'_{\alpha,\beta}(\lambda-u)}{q_{\alpha,\beta}(\lambda-u)}.$$

For $\beta-\alpha>1$, by the logarithmic convexities of $q_{\alpha,\beta}(u)$, it follows that the function $\frac{q'_{\alpha,\beta}(u)}{q_{\alpha,\beta}(u)}$ is increasing and $\frac{q'_{\alpha,\beta}(\lambda-u)}{q_{\alpha,\beta}(\lambda-u)}$ is decreasing on $(-\infty,\infty)$; From the obvious fact that

$[\ln Q_{s,t;\lambda}(u)]'|_{u=\lambda/2}=0$, it follows that $[\ln Q_{s,t;\lambda}(u)]'>0$ for $u>\frac{\lambda}{2}$ and $[\ln Q_{s,t;\lambda}(u)]'<0$ for $u<\frac{\lambda}{2}$; Hence, the function $Q_{s,t;\lambda}(u)$ is increasing for $u>\frac{\lambda}{2}$ and decreasing for $u<\frac{\lambda}{2}$. Similarly, for $0<\beta-\alpha<1$, the function $Q_{s,t;\lambda}(u)$ is decreasing for $u>\frac{\lambda}{2}$ and increasing for $u<\frac{\lambda}{2}$. The proof of Lemma 4 is proved.

3. An alternative proof of Theorem 1

Since $z_{s,t}(x)=z_{t,s}(x)$, without loss of generality, we can assume $t>s\geqslant 0$ and $t-s\neq 1$ in what follows.

Differentiation of $z_{s,t}(x)$, utilization of (3) and application of Lemma 2 yield

$$z'_{s,t}(x)=\frac{[z_{s,t}(x)+x][\psi(x+t)-\psi(x+s)]}{t-s}-1 \tag{9}$$

and

$$\begin{aligned}
\frac{z''_{s,t}(x)}{z_{s,t}(x)+x}&=\left[\frac{\psi(x+t)-\psi(x+s)}{t-s}\right]^2+\frac{\psi'(x+t)-\psi'(x+s)}{t-s}\\
&=\left[\frac{1}{t-s}\int_s^t\psi'(x+u)\mathrm{d}u\right]^2+\frac{1}{t-s}\int_s^t\psi''(x+u)\mathrm{d}u\\
&=\left[\frac{1}{t-s}\int_s^t\int_0^\infty\frac{ve^{-(x+u)v}}{1-e^{-v}}\mathrm{d}v\mathrm{d}u\right]^2-\frac{1}{t-s}\int_s^t\int_0^\infty\frac{v^2e^{-(x+u)v}}{1-e^{-v}}\mathrm{d}v\mathrm{d}u\\
&=\left(\int_0^\infty\frac{ve^{-xv}}{1-e^{-v}}\cdot\frac{1}{t-s}\int_s^t e^{-uv}\mathrm{d}u\mathrm{d}v\right)^2\\
&\quad-\int_0^\infty\frac{v^2e^{-xv}}{1-e^{-v}}\cdot\frac{1}{t-s}\int_s^t e^{-uv}\mathrm{d}u\mathrm{d}v\\
&=\left(\int_0^\infty\frac{e^{-xv}}{1-e^{-v}}\cdot\frac{e^{-sv}-e^{-tv}}{t-s}\mathrm{d}v\right)^2-\int_0^\infty\frac{ve^{-xv}}{1-e^{-v}}\cdot\frac{e^{-sv}-e^{-tv}}{t-s}\mathrm{d}v\\
&=\int_0^\infty\left[\frac{1}{(t-s)u}\int_0^u q_{s,t}(r)q_{s,t}(u-r)\mathrm{d}r-q_{s,t}(u)\right]ue^{-xu}\mathrm{d}u\\
&=\int_0^\infty\left[\frac{1}{(t-s)u}\int_0^u Q_{s,t;u}(r)\mathrm{d}r-q_{s,t}(u)\right]ue^{-xu}\mathrm{d}u.
\end{aligned}\tag{10}$$

If $t-s>1$, by the monotonicity of $Q_{s,t;\lambda}(u)$ in Lemma 4, it follows easily that

$$Q_{s,t;u}(r)\leqslant Q_{s,t;u}(0)=Q_{s,t;u}(u)=q_{s,t}(0)q_{s,t}(u)=(t-s)q_{s,t}(u),$$

consequently, the bracketed term in the line (10) is negative on $(0,\infty)$, and so $z''_{s,t}(x)<0$. If $0<t-s<1$, the similar argument leads to $z''_{s,t}(x)>0$. The convex and concave properties of $z_{s,t}(x)$ are proved.

By the mean value theorem, it is immediate that

$$\begin{aligned} z_{s,t}'(x)+1 &= \left[\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{1/(t-s)}\frac{\psi(x+t)-\psi(x+s)}{t-s}\right] \\ &= \frac{\psi(x+t)-\psi(x+s)}{t-s}\exp\frac{\ln\Gamma(x+t)-\ln\Gamma(x+s)}{t-s} \\ &= \psi'(x+\xi_1)e^{\psi(x+\xi_2)} \end{aligned}$$

for $\xi_i \in (s,t)$ and $i=1,2$. By inequalities in (5) and (6), it is ready to obtain

$$\left[\frac{x+\xi_2}{x+\xi_1}+\frac{x+\xi_2}{2(x+\xi_1)^2}\right]\frac{1}{e^{1/(x+\xi_2)}} < z_{s,t}'(x)+1 < \left[\frac{x+\xi_2}{x+\xi_1}+\frac{x+\xi_2}{(x+\xi_1)^2}\right]\frac{1}{e^{1/2(x+\xi_2)}}$$

which means $\lim_{x\to\infty} z_{s,t}'(x)=0$. For $t-s>1$, the conclusion that $z_{s,t}''(x)\leqslant 0$ obtained above implies $z_{s,t}'(x)$ is decreasing, and so $z_{s,t}'(x)>0$ and $z_{s,t}(x)$ is increasing. For $0<t-s<1$, the result that $z_{s,t}''(x)\geqslant 0$ obtained above implies $z_{s,t}'(x)$ is increasing, and so $z_{s,t}'(x)<0$ and $z_{s,t}(x)$ is decreasing. The proof of Theorem 1 is complete.

4. Some remarks

REMARK 1. The logarithmically convex properties of $q_{\alpha,\beta}(u)$ on $(-\infty,0)$ in Lemma 4 of this paper corrects some mistakes appeared in [17, Lemma 1]. However, these mistakes did not affect the correctness of the proof provided in [17] for Theorem 1, since properties of $q_{\alpha,\beta}(u)$ on $(-\infty,0)$ are idle there.

REMARK 2. The logarithmically convex properties in Lemma 4 of this paper were also proved in [4] and related references therein by using different techniques.

REMARK 3. It is well-known that a positive and k-times differentiable function $f(x)$ is said to be k-log-convex (or k-log-concave, respectively) on an interval I with $k\geqslant 2$ if and only if $[\ln f(x)]^{(k)}$ exists and $[\ln f(x)]^{(k)}\geqslant 0$ (or $[\ln f(x)]^{(k)}\leqslant 0$, respectively) on I. The 3-log-convex properties of $q_{\alpha,\beta}(u)$ were already obtained in [12, Theorem 1.1]: For $1>\beta-\alpha>0$, the function $q_{\alpha,\beta}(u)$ is 3-log-convex on $(0,\infty)$ and 3-log-concave on $(-\infty,0)$; For $\beta-\alpha>1$, it is 3-log-concave on $(0,\infty)$ and 3-log-convex on $(-\infty,0)$.

REMARK 4. Some double inequalities of polygamma functions $\psi^{(i)}(x)$ for $i\in\mathbb{N}$ were proved in [5, p. 107, Lemma 3] by the similar approach to that in Lemma 3 above.

REMARK 5. This is a modified version of the preprint [14].

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