Mathematical
Inequalities
& Applications
Volume 14, Number 1 (2011), 93–100

SOME INEQUALITIES OF DIFFERENTIAL POLYNOMIALS II

JUNFENG XU, HONGXUN YI AND ZHANLIANG ZHANG

(*Communicated by H. M. Srivastava*)

Abstract. In this paper, we consider the value distribution of the differential polynomials $f^2f^{(k)}-1$ where k is a positive integer, and obtain some estimates only by the reduced counting function. Our result answers a question in (Some inequalities of differential polynomials, Mathematical Inequalities and Applications, **12**, 1(2009), 99–113) completely.

1. Introduction and results

Let $\mathbb{C}$ be the open complex plane and $\mathscr{D} \in \mathbb{C}$ be a domain. Let f be a meromorphic function in the complex plane, we assumed that the reader is familiar with the notations of Nevanlinna theory (see, e.g., [4, 9, 10]).

DEFINITION 1.1. Let k be a positive integer, for any constant a in the complex plane. We denote by $N_{k)}(r,1/(f-a))$ the counting function of a-points of f with multiplicity $\leqslant k$, by $N_{(k}(r,1/(f-a))$ the counting function of a-points of f with multiplicity $\geqslant k$, by $N_k(r,1/(f-a))$ the counting function of a-points of f with multiplicity of k. and denote the reduced counting function by $\overline{N}_{k)}(r,1/(f-a))$, $\overline{N}_{(k}(r,1/(f-a))$ and $\overline{N}_k(r,1/(f-a))$, respectively.

Recently, Huang and Gu ([5]) have obtained a quantitative result about a differential polynomials $f^2f^{(k)}-1$. They proved the following theorem.

THEOREM A. *Let f be transcendental meromorphic in the complex plane and k be a positive integer, then*

$$T(r,f) \leqslant 6N\left(r,\frac{1}{f^2f^{(k)}-1}\right)+S(r,f). \tag{1.1}$$

REMARK 1.2. In fact, Q. Zhang [11] proved the case of $k=1$. X. Huang and Y. Gu proved the case of $k \geqslant 2$.

Mathematics subject classification (2010): 30D35, 26D10.
Keywords and phrases: Meromorphic function; differential polynomials; Nevanlinna theory; value distribution.

As we all known, the second fundamental theorem in Nevanlinna's theory of value distribution uses the reduced counting function to estimate the Nevanlinna characteristic function(cf. [8]). Naturally, we can pose the following important question.

Whether one can give some quantitative estimates on the generally differential polynomials by the reduced counting function?

In [7], the authors give some affirmative answers.

THEOREM B. *Let f be a transcendental meromorphic function, $L[f] = a_k f^{(k)} + a_{k-2} f^{(k-2)} + \cdots + a_0 f$, where $a_0, a_1, \cdots, a_k (\not\equiv 0)$ are small functions, for $c(\neq 0, \infty)$, let $F = f^2 L[f] - c$, then there exists a constant $M > 0$, which does not depend on f, such that*

$$T(r,f) \leqslant M\overline{N}\left(r, \frac{1}{F}\right) + S(r,f).$$

REMARK 1.3. We know F has infinitely many zeros, and the constant M is at least 6 from Theorem A. But the method of Theorem B can't give the certain coefficient. Hence, we want to get the more precise estimate of the coefficient. In fact, we proved the following result in [7] by giving some restriction on the zeros of f.

THEOREM C. *Let f be a transcendental meromorphic function, and let k be a positive integer. If $N_1\left(r, \frac{1}{F}\right) = S(r,f)$, then*

$$T(r,f) \leqslant 2\overline{N}\left(r, \frac{1}{f^2 f^{(k)} - 1}\right) + S(r,f). \tag{1.2}$$

In the paper, we continue to investigate the problem in this direction. Though Theorem C has the smaller coefficient 2, we know the condition of the simple zero is not necessary from Theorem B. Hence it is an important question how to remove the condition and get a precise estimation. We prove the following theorem.

THEOREM 1.4. *Let f be a transcendental meromorphic function, and let k be a positive integer. Then*

$$T(r,f) \leqslant M\overline{N}\left(r, \frac{1}{f^2 f^{(k)} - 1}\right) + S(r,f). \tag{1.3}$$

where M is 6 if $k = 1$ or $k \geqslant 3$, M is 10 if $k = 2$.

2. Proof of the theorem

In order to prove our result, we need to the following lemmas.

LEMMA 2.1. *Let f be transcendental meromorphic function, and let k be a positive integer. Then*

$$\begin{aligned} 3T(r,f) \leqslant\ & \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{F}\right) + N_{k)}\left(r, \frac{1}{F}\right) + k\overline{N}_{(k+1}\left(r, \frac{1}{F}\right) \\ & + \overline{N}\left(r, \frac{1}{f^2 f^{(k)} - 1}\right) - N_0\left(r, \frac{1}{(f^2 f^{(k)})'}\right) + S(r,f). \end{aligned} \tag{2.1}$$

where $N_0(r,\frac{1}{(f^2f^{(k)})'})$ *denotes the counting function of the zeros of* $(f^2f^{(k)})'$*, not of* $f(f^2f^{(k)}-1)$*. Especially, if* $k=1$*, we get*

$$3T(r,f)\leqslant \overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{f^2f'-1}\right)-N_0\left(r,\frac{1}{(f^2f')'}\right)+S(r,f). \quad (2.2)$$

Proof. We claim first that $f^2f^{(k)}\not\equiv$ constant. If $f^2f^{(k)}\equiv C$, where C is a constant. Obviously, $C\neq 0$. Hence f has no zero and $\frac{1}{f^3}=\frac{1}{C}\frac{f^{(k)}}{f}$. Therefore,

$$3T(r,f)=m\left(r,\frac{1}{f^3}\right)+N\left(r,\frac{1}{f^3}\right)+O(1)=m\left(r,\frac{f^{(k)}}{f}\right)+O(1)=S(r,f).$$

a contradiction. Hence $f^2f^{(k)}$ is not equivalent to a constant.

Let

$$\frac{1}{f^3}\equiv\frac{f^2f^{(k)}}{f^3}-\frac{(f^2f^{(k)})'}{f^3}\frac{f^2f^{(k)}-1}{(f^2f^{(k)})'},$$

we have

$$\begin{aligned}3m\left(r,\frac{1}{F}\right) &= m\left(r,\frac{1}{f^3}\right)\\ &\leqslant m\left(r,\frac{f^2f^{(k)}-1}{(f^2f^{(k)})'}\right)+m\left(r,\frac{f^{(k)}}{f}\right)+m\left(r,\frac{(f^2f^{(k)})'}{f^3}\right)+O(1)\\ &\leqslant N\left(r,\frac{(f^2f^{(k)})'}{f^2f^{(k)}-1}\right)-N\left(r,\frac{f^2f^{(k)}-1}{(f^2f^{(k)})'}\right)+S(r,f)\\ &= N(r,(f^2f^{(k)})')+N\left(r,\frac{1}{f^2f^{(k)}-1}\right)-N\left(r,\frac{1}{(f^2f^{(k)})'}\right)\\ &\quad -N(r,f^2f^{(k)}-1)+S(r,f)\\ &= \overline{N}(r,f)+N\left(r,\frac{1}{f^2f^{(k)}-1}\right)-N\left(r,\frac{1}{(f^2f^{(k)})'}\right)+S(r,f).\end{aligned}$$

Hence

$$\begin{aligned}3T(r,f) &= 3m\left(r,\frac{1}{F}\right)+3N\left(r,\frac{1}{F}\right)+O(1)\\ &= \overline{N}(r,f)+3N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{f^2f^{(k)}-1}\right)-N\left(r,\frac{1}{(f^2f^{(k)})'}\right)+S(r,f).\end{aligned} \quad (2.3)$$

Let

$$N\left(r,\frac{1}{(f^2f^{(k)})'}\right)=N_{000}\left(r,\frac{1}{(f^2f^{(k)})'}\right)+N_{00}\left(r,\frac{1}{(f^2f^{(k)})'}\right)+N_0\left(r,\frac{1}{(f^2f^{(k)})'}\right) \quad (2.4)$$

where $N_{000}(r,\frac{1}{(f^2f^{(k)})'})$ denotes the counting function of the zeros of $(f^2f^{(k)}-1)'$, which come from the zeros of $f^2f^{(k)}-1$, $N_{00}(r,\frac{1}{(f^2f^{(k)})'})$ denotes the counting function of the zeros of $(f^2f^{(k)}-1)'$, which come from the zeros of f. Hence we have

$$N\left(r,\frac{1}{f^2f^{(k)}-1}\right)-N_{000}\left(r,\frac{1}{(f^2f^{(k)})'}\right)=\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right). \tag{2.5}$$

Supposed that z_0 is a zero of f with multiplicity q, if $q\leqslant k$, then z_0 is a zero of $(f^2f^{(k)})'$ with multiplicity at least $2q-1$; if $q\geqslant k+1$, then z_0 is a zero of $(f^2f^{(k)})'$ with multiplicity at least $3q-(k+1)$. Hence we have

$$\begin{aligned}3N\left(r,\frac{1}{F}\right)-N_{00}\left(r,\frac{1}{(f^2f^{(k)})'}\right)&\leqslant N_{k)}\left(r,\frac{1}{F}\right)+\overline{N}_{k)}\left(r,\frac{1}{F}\right)+(k+1)\overline{N}_{(k+1}\left(r,\frac{1}{F}\right)\\&=N_{k)}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F}\right)+k\overline{N}_{(k+1}\left(r,\frac{1}{F}\right).\end{aligned} \tag{2.6}$$

Combining (2.3)–(2.6), we have

$$\begin{aligned}3T(r,f)\leqslant\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{F}\right)+N_{k)}\left(r,\frac{1}{F}\right)+k\overline{N}_{(k+1}\left(r,\frac{1}{F}\right)\\+\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right)-N_0\left(r,\frac{1}{(f^2f^{(k)})'}\right)+S(r,f).\end{aligned}$$

This completes the proof of the lemma.

LEMMA 2.2. [11] *Let f be a transcendental meromorphic function, and let k be a positive integer. Let*

$$G(z)=13\left(\frac{F'}{F}\right)^2+20\left(\frac{F'}{F}\right)'-24\frac{F'}{F}\frac{l'}{l}+8\left(\frac{l'}{l}\right)^2-88\left(\frac{l'}{l}\right)', \tag{2.7}$$

then we have (1) $G(z)\not\equiv 0$; (2) The simple poles of $f(z)$ are the zeros of $G(z)$.

Now we begin to prove Theorem 1.4.

(I) If $k=1$, Q. D. Zhang proved the inequality (1.1) by using the auxiliary function. Here we use his method to construct the function $G(z)$, and can obtain a better result if $k=1$.

Let $F(z)=f^2f'-1$ and $l(z)=\frac{F'}{f}=2(f')^2+ff''$. Obviously $l(z)\not\equiv 0$. Also let

$$G(z)=13\left(\frac{F'}{F}\right)^2+20\left(\frac{F'}{F}\right)'-24\frac{F'}{F}\frac{l'}{l}+8\left(\frac{l'}{l}\right)^2-88\left(\frac{l'}{l}\right)'. \tag{2.8}$$

By Lemma 2.2, we know $G(z)\not\equiv 0$, and the simple poles of f are the zeros of $G(z)$. Note that the poles of $G(z)$ whose multiplicity is at most two come from the multiple poles of f, F or the zeros of l. But it is still difficult to deal with the zeros of

l. We consider the poles of $\beta^2 G(z)$. By differentiating the equation $F(z) = f^2 f' - 1$, we get

$$f\beta = -\frac{F'}{F}, \tag{2.9}$$

where

$$\beta = 2(f')^2 + ff'' - ff'\frac{F'}{F}, \quad l = -\beta F. \tag{2.10}$$

We can see the zeros of l either is the zeros of F, or the zeros of β. From the above we know that the multiple poles of f with the multiplicity $q(\geqslant 2)$ is the zeros of β with the multiplicity of $q-1$. Hence the poles of $\beta^2 G(z)$ only come from the zeros of F, and the multiplicity is at most 4. Hence,

$$N(r, \beta^2 G) \leqslant 4\overline{N}(r, 1/F).$$

Note that $m(r,G) = S(r,f)$, therefore $m(r, \beta^2 G) = S(r,f)$. Hence

$$T(r, \beta^2 G) \leqslant 4\overline{N}(r, 1/F).$$

Since the multiple zeros of f with the multiplicity $p(\geqslant 2)$ are the multiple zeros of β with multiplicity at least $2p-2$, therefore, are at least the zeros of $\beta^2 G$ with the multiplicity $2(2p-2)-2 = 4p-6$. Also note that the simple poles of f are the zeros of $\beta^2 G$. Hence we have

$$\overline{N}_{1)}(r,f) + 2N\left(r, \frac{1}{F}\right) - 2\overline{N}\left(r, \frac{1}{F}\right) \leqslant N\left(r, \frac{1}{\beta^2 G}\right) \leqslant T(r, \beta^2 G) \leqslant 4\overline{N}\left(r, \frac{1}{F}\right). \tag{2.11}$$

Combining (2.2) and (2.15), we have

$$T(r,f) + N_{(2}(r,f) - 2\overline{N}_{(2}(r,f) + m(r,f) + 4m\left(r, \frac{1}{F}\right) + 6N\left(r, \frac{1}{F}\right) - 6\overline{N}\left(r, \frac{1}{F}\right)$$
$$\leqslant 6\overline{N}\left(r, \frac{1}{f^2 f' - 1}\right) + S(r,f),$$

Hence we have

$$T(r,f) < 6\overline{N}\left(r, \frac{1}{f^2 f' - 1}\right) + S(r,f). \tag{2.12}$$

Obviously, our result improves the conclusion of Q.D. Zhang greatly.

(II) If $k \geqslant 2$, X. J. Huang and Y. X. Gu constructed the similar function $G_1(z)$. Let $F_1(z) = f^2 f^{(k)} - 1$ and $l_1(z) = \frac{F_1'}{f} = 2(f')^2 + ff''$. Obviously $l_1(z) \not\equiv 0$. Let

$$G_1(z) = a_1\left(\frac{F_1'}{F_1}\right)^2 + a_2\left(\frac{F_1'}{F_1}\right)' + a_3\frac{F_1'}{F_1}\frac{l_1'}{l_1} + a_4\left(\frac{l_1'}{l_1}\right)^2 + a_5\left(\frac{l_1'}{l_1}\right)'. \tag{2.13}$$

Where

$$
\begin{aligned}
a_1 &= 2(k+1)^2 - \frac{(3k+7)(k^2-4k-29)}{(k+3)}\\
a_2 &= -(k+5)(k^2-4k-29);\\
a_3 &= 4(k^2-4k-29);\\
a_4 &= -4(k+3)(k+1)\\
a_5 &= 2(k+2)(k+3)(k+5).
\end{aligned}
$$

By Lemma 3 in [5], we know $G_1(z)\not\equiv 0$, and Lemma 4 of [5], we know the simple poles of f are the zeros of $G_1(z)$. Note the poles of $G_1(z)$ come from the multiple poles of f, F_1 or the zeros of l_1, whose multiple is at most two. But it is also difficult to deal with the zeros of l_1.

We consider the poles of the function $\beta^2 G_1(z)$. Similar with the proof of the (2.9),

$$\beta = 2(f')^2 + ff'' - ff'\frac{F_1'}{F_1}, \quad l_1 = -\beta F_1.$$

Then we can see the zero of l_1 either is the zero of F_1, or the zero of β. From the above, we know the multiple zeros of f with the multiplicity $q(\geqslant 2)$ are the zeros of β with the multiplicity $q-1$. Hence the poles of $\beta^2 G_1(z)$ only come from the zeros of F, and the multiplicity are at most four. Therefore,

$$N(r,\beta^2 G) \leqslant 4\overline{N}(r,1/F).$$

Note that $m(r,G) = S(r,f)$, therefore $m(r,\beta^2 G) = S(r,f)$. Hence

$$T(r,\beta^2 G) \leqslant 4\overline{N}(r,1/F).$$

Then by the zeros of f with multiplicity $p(\geqslant k)$ are at least the zeros of β with the multiplicity $2p-2$, therefore are at least the zeros of $\beta^2 G$ with the multiplicity $2(2p-2)-2 = 4p-6$. Note that the simple poles of f are also the zeros of $\beta^2 G$. Hence we have

$$\overline{N}_{1)}(r,f) + 4N_{(k}\left(r,\frac{1}{F}\right) - 6\overline{N}_{(k}\left(r,\frac{1}{F}\right) \leqslant N\left(r,\frac{1}{\beta^2 G}\right) \leqslant T(r,\beta^2 G) \leqslant 4\overline{N}\left(r,\frac{1}{F}\right). \tag{2.14}$$

Next we divide two cases:

Case (1). If $k \geqslant 3$, the we have

$$\overline{N}_{1)}(r,f) + 2N_{(k}\left(r,\frac{1}{F}\right) \leqslant 4\overline{N}(r,\frac{1}{F}). \tag{2.15}$$

Combining the doubled (2.1) and (2.15), we have

$$
\begin{aligned}
6T(r,f) + \overline{N}_{1)}(r,f) + 2N_{(k}\left(r,\frac{1}{F}\right) - 2\overline{N}(r,f) - 2\overline{N}\left(r,\frac{1}{F}\right) - 2N_{k)}\left(r,\frac{1}{F}\right) - 2k\overline{N}_{(k+1}\left(r,\frac{1}{F}\right)\\
\leqslant 6\overline{N}\left(r,\frac{1}{f^2 f^{(k)}-1}\right) - N_0\left(r,\frac{1}{(f^2 f^{(k)})'}\right) + S(r,f).
\end{aligned}
\tag{2.16}
$$

Then

$$\begin{aligned}
6T(r,f)&+\overline{N}_{1)}(r,f)+2N_{(k}\left(r,\frac{1}{F}\right)-2\overline{N}(r,f)-2\overline{N}\left(r,\frac{1}{F}\right)\\
&-2N_{k)}\left(r,\frac{1}{F}\right)-2k\overline{N}_{(k+1}\left(r,\frac{1}{F}\right)\\
&\geqslant T(r,f)+m(r,f)+N(r,f)+\overline{N}_{1)}(r,f)-2\overline{N}(r,f)+4m\left(r,\frac{1}{F}\right)+4N\left(r,\frac{1}{F}\right)\\
&+2N_{(k}\left(r,\frac{1}{F}\right)-2\overline{N}\left(r,\frac{1}{F}\right)-2N_{k)}\left(r,\frac{1}{F}\right)-2k\overline{N}_{(k+1}\left(r,\frac{1}{F}\right).
\end{aligned}\tag{2.17}$$

In (2.17), we first consider the case of the pole

$$\begin{aligned}
N(r,f)+\overline{N}_{1)}(r,f)-2\overline{N}(r,f)&\geqslant N(r,f)+\overline{N}_{1)}(r,f)-2\overline{N}_{1)}(r,f)-2\overline{N}_{(2}(r,f)\\
&\geqslant N(r,f)-\overline{N}_{1)}(r,f)-2\overline{N}_{(2}(r,f)\\
&=N_{1)}(r,f)+N_{(2}(r,f)-\overline{N}_{1)}(r,f)-2\overline{N}_{(2}(r,f)\\
&>0.
\end{aligned}\tag{2.18}$$

In (2.17), we consider the case of the zero

$$\begin{aligned}
4N&\left(r,\frac{1}{F}\right)+2N_{(k}\left(r,\frac{1}{F}\right)-2\overline{N}\left(r,\frac{1}{F}\right)-2N_{k)}\left(r,\frac{1}{F}\right)-2k\overline{N}_{(k+1}\left(r,\frac{1}{F}\right)\\
&\geqslant 4N\left(r,\frac{1}{F}\right)+2N_{k}\left(r,\frac{1}{F}\right)+N_{(k+1}\left(r,\frac{1}{F}\right)-2\overline{N}\left(r,\frac{1}{F}\right)\\
&\quad-2N_{k)}\left(r,\frac{1}{F}\right)-\frac{2k}{k+1}N_{(k+1}\left(r,\frac{1}{F}\right)\\
&\geqslant 4N\left(r,\frac{1}{F}\right)+N_{k}\left(r,\frac{1}{F}\right)+\frac{2}{k+1}N_{(k+1}\left(r,\frac{1}{F}\right)-2\overline{N}\left(r,\frac{1}{F}\right)-2N_{k)}\left(r,\frac{1}{F}\right)\\
&>0.
\end{aligned}\tag{2.19}$$

From (2.16)–(2.19), we have

$$T(r,f)<6\overline{N}\left(r,\frac{1}{f^2f^{(k)}-1}\right)+S(r,f).\tag{2.20}$$

Case (2). If $k=2$, by (2.14), we have

$$\overline{N}_{1)}(r,f)+N_{(k}\left(r,\frac{1}{F}\right)\leqslant 4\overline{N}\left(r,\frac{1}{F}\right).$$

Similar with the above discussion, we have

$$T(r,f)<10\overline{N}\left(r,\frac{1}{f^2f''-1}\right)+S(r,f).\tag{2.21}$$

This completes the proof of Theorem 1.4.

Acknowledgements. Supported by NNSF of China(10771121, 10871076), the NSF of Guangdong Province (9452902001003278) and Excellent Young Fund of Department of Education of Guangdong (LYM08097).

REFERENCES

[1] W. BERGEILER AND A. EREMENKO, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoamericana., **11** (1995), 355–373.

[2] J. CLUNIE, *On integral and meromorphic functions*, J. London Math. Soc., **37** (1962), 17–27.

[3] F. GACKSTATTER, I. LAINE, *Zur Theorie der gewöhnlichen Differentialgleichungen im Komplexen*, Ann. Polon. Math., **38** (1980), 259–287.

[4] W. K. HAYMAN, *Meromorphic functions*, Clarendon Press, Oxford, 1964.

[5] X. J. HUANG AND Y. X. GU, *On the value distribution of* $f^2f^{(k)}$, J. Aust. Math. Soc., **78**(2005), 17–26.

[6] I. LAINE, *Nevanlinna theory and complex differential equations*, Walter de Gruyter, Berlin-New York, 1993.

[7] J. F. XU, H. X. YI AND Z. L. ZHANG, *Some inequalities of differential polynomials*, Mathematical Inequalities and Applications, **12**, 1 (2009), 99–113.

[8] K. YAMANOI, *The second main theorem for small functions and related problems*, Acta Math., **192** (2004), 225–294.

[9] C. C. YANG AND H.X. YI, *Uniqueness Theory of Meromorphic Functions*, New York, Dordrecht, Boston, London, 2003.

[10] L. YANG, *Value distribution theory*, Springer, Berlin, Heidelberg, New York, 1993.

[11] Q. D. ZHANG, *A growth theorem for meromorphic functions*, J. Chengdu Inst. Meteor., **20** (1992), 12–20.

(Received April 1, 2009)

Junfeng Xu
Department of Mathematics, Wuyi University
Jiangmen 529020, P.R.China
e-mail: xujunf@gmail.com

Hongxun Yi
Department of Mathematics, Shandong University
Jinan 250100, Shandong
P.R.China
e-mail: hxyi@sdu.edu.cn

Zhanliang Zhang
Department of Mathematics, Zhaoqing University
Zhaoqing 526061, P.R.China
e-mail: zlzhang@zqu.edu.cn