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BOUNDING EXPECTATIONS OF FUNCTIONS OF RANDOM VECTORS WITH GIVEN MARGINALS AND SOME MOMENTS: APPLICATIONS OF THE MULTIVARIATE DISCRETE MOMENT PROBLEM

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Abstract. The paper shows how the bounding technique provided by the multivariate discrete moment problem can be used for bounding expectations of functions of random variables with known univariate marginals and some of the mixed moments. Four examples are presented. In the first one the function is a Monge or related type array, in the second one it is a pseudo-Boolean function. In the further examples bounds are given for values of multivariate generating functions and expectations of special utility functions of random vectors. Numerical results are presented.

1. Introduction

Recently, a number of papers have been published about the univariate and multivariate discrete moment problem (DMP, MDMP), where bounding formulas as well as algorithmic bounds for functions of random variables under moment information are presented (see Prékopa 1990, Prékopa 1998, Mádi-Nagy and Prékopa 2004, Mádi-Nagy 2009). In the univariate case the moments of order up to m of a random variable are supposed to be known and lower and upper bounds for various functions of the random variable have been proposed in Prékopa (1990). These include bounds for probabilities of the type $X \geqslant a$ or $X = a$. The paper by Prékopa (1998) deals with bounds for functions of random vectors, where the mixed moments of the components of total order up to m are known. The results are generalized in Mádi-Nagy and Prékopa (2004), where, in addition to the knowledge of moments of total order up to m, further moments of the univariate marginals are also known.

Sometimes all univariate marginals are completely known and the stochastic dependencies are characterized by mixed moments, e.g., covariances. This is the case in the paper by Hou and Prékopa (2007), where a bounding technique, different from the one in MDMP is used.

The purpose of the present paper is to give four examples for the application of the MDMP technique, for bounding expectations of functions of random vectors, where the univariate marginals and some of the mixed moments are known.

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Let $X=(X_1,\dots,X_s)$ be a random vector where the support of X_j is a known finite set $Z_j=\{z_{j0},\dots,z_{jn_j}\}$ with distinct elements, $j=1,\dots,s$ and introduce the notation:

$$p_{i_1\dots i_s}=P(X_1=z_{1i_1},\dots,X_s=z_{si_s}),\ 0\leqslant i_j\leqslant n_j,\ j=1,\dots,s. \tag{1.1}$$

We assume that the probability distribution of X is unknown, but known are the univariate marginals, i.e., the distributions of the components X_j's, $j=1,\dots,s$. For them we use the following notations:

$$P(X_j=z_{ji})=q_i^{(j)},\ i=0,\dots,n_j,\ j=1,\dots,s.$$

The $(\alpha_1,\dots,\alpha_s)$-order moment of the random vector $(X_1,\dots,X_s)$ is defined as

$$\mu_{\alpha_1\dots\alpha_s}=E[X_1^{\alpha_1}\cdots X_s^{\alpha_s}]=\sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s}z_{1i_1}^{\alpha_1}\cdots z_{si_s}^{\alpha_s}p_{i_1\dots i_s},$$

where $\alpha_1,\dots,\alpha_s$ are nonnegative integers. The sum $\alpha_1+\cdots+\alpha_s$ is called the total order of the moment. We assume that the mixed moments of order up to m are known.

Our aim is to give lower and upper bounds for

$$E[f(X_1,\dots,X_s)],$$

where $f(z),\ z\in Z_1\times\cdots\times Z_s$ is a discrete function about which we will introduce some assumptions. For simplicity let $f_{i_1\dots i_s}=f(z_{1i_1},\dots,z_{si_s})$.

The MDMP that we use in this paper is the following:

$$\min(\max)\sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s}f_{i_1\dots i_s}p_{i_1\dots i_s}$$

subject to

$$\begin{aligned}
&\sum_{i_1=0}^{n_1}\cdots\sum_{i_{j-1}=0}^{n_{j-1}}\sum_{i_{j+1}=0}^{n_{j+1}}\sum_{i_s=0}^{n_s}p_{i_1\dots i_{j-1}ii_{j+1}\dots i_s}=q_i^{(j)}\\
&\text{for } i=0,\cdots,n_j,\ j=1,\dots,s;\ \text{and}\\
&\sum_{i_1=0}^{n_1}\cdots\sum_{i_s=0}^{n_s}z_{1i_1}^{\alpha_1}\cdots z_{si_s}^{\alpha_s}p_{i_1\dots i_s}=\mu_{\alpha_1\dots\alpha_s}\\
&\text{for } 0\leqslant\alpha_j,\ j=1,\dots,s,\alpha_u\alpha_v\neq0 \text{ for some } u\neq v,\ \alpha_1+\cdots+\alpha_s\leqslant m;\\
&p_{i_1\dots i_s}\geqslant0,\ \text{all } i_1,\dots,i_s,
\end{aligned} \tag{1.2}$$

where $(q_0^{(j)},\dots,q_{n_j}^{(j)})$, $j=1,\dots,s$ are known vectors, and the moments $\mu_{\alpha_1\dots\alpha_s}$, that appear in (1.2), are known. The decision variables are $p_{i_1\dots i_s}$, $0\leqslant i_j\leqslant n_j$, $j=1,\dots,s$.

The objective function, the first set of constraints and the nonnegativity restrictions define an s-dimensional transportation problem (see Hou and Prékopa 2007). Problem (1.2) will be called extended s-dimensional transportation problem.

Since the cardinality of the support of X_j is n_j+1, it follows that the moments

$$E[X_j^k] = \sum_{i=0}^{n_j} z_{ji}^k q_i^{(j)},\ k=0,\dots n_j$$

uniquely determine the probability distribution of X_j. In view of this, problem (1.2) is equivalent to the following:

$$\min(\max) \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} f_{i_1\dots i_s} p_{i_1\dots i_s}$$

subject to

$$\sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1\dots i_s} = \mu_{\alpha_1\dots\alpha_s}$$

for $\alpha_j = 0,\ j=1,\dots,k-1,k+1,\dots,s,\ 0 \leqslant \alpha_k \leqslant n_k,\ k=1,\dots,s$ and

for $0 \leqslant \alpha_j,\ j=1,\dots,s, \alpha_u\alpha_v \neq 0$ for some $u \neq v,\ \alpha_1+\cdots+\alpha_s \leqslant m$;

$p_{i_1\dots i_s} \geqslant 0,$ all $i_1,\dots,i_s$.

(1.3)

The compact matrix form of problem (1.3) will be written as:

$$\begin{aligned} &\min(\max) \quad f^T p \\ &\text{subject to} \\ &\widehat{A}p = \widehat{b} \\ &p \geqslant 0. \end{aligned} \qquad (1.4)$$

The paper is organized as follows. In Section 2 we specialize our general theorems proved in Mádi-Nagy and Prékopa (2004) for the case of problem (1.3), suitable for our current applications. Sections 3-6 contain the four examples. In Section 3 we derive further results, by the use of problem (1.3), for the problem studied in Hou and Prékopa (2007), where the objective function enjoys the Monge or some related property. In Section 4 we apply the specialized MDMP technique to bounding the expectations of pseudo-Boolean functions under monotonicity conditions. The next example, bounding the values of moment generating functions, is presented in Section 5. In Section 6 we show how the MDMP technique applies to bounding expected utilities. Finally, we summarize the conclusions of our results.

2. Bounds when the univariate marginal distributions and moments of total order up to m are known

In this section we give lower and upper bounds for the objective function of problem (1.3). Let V_{min} (V_{max}) designate the minimum (maximum) value in problem (1.4). Let further B_1 (B_2) designate a dual feasible basis (i.e., a basis for which the optimality condition is satisfied) for the minimization (maximization) problem. Then, by linear programming theory, we know that

$$f_{B_1}^T B_1^{-1} b \leqslant V_{min} \leqslant E\left[f(X_1,\dots,X_s)\right] \leqslant V_{max} \leqslant f_{B_2}^T B_2^{-1} b. \qquad (2.1)$$

Hence, if a dual feasible basis is given then (2.1) yields bounds for the objective function. If B_1 (B_2) is an optimal basis in the minimization (maximization) problem, then the first (last) inequality holds with equality sign. We say that V_{min} and V_{max} are the sharp lower and upper bounds, respectively, for the expectation of $f(X_1,\dots,X_s)$. The theorems in the second part of this section give bounds by constructing dual feasible bases, using multivariate Lagrange interpolation methodology.

In order to see the relationship between multivariate Lagrange interpolation and dual feasible bases of problem (1.3), let $U=\{u_1,\dots,u_M\}$ be a set of distinct points in $\mathbb{R}^s$ and $H=\{(\alpha_1,\dots,\alpha_s)\}$ a finite set of s-tuples of nonnegative integers $(\alpha_1,\dots,\alpha_s)$. We say that the set U admits an H-type Lagrange interpolation if for any real function $f(z)$, $z\in U$, there exists a polynomial $p(z)$ of the form

$$p(z)=\sum_{(\alpha_1,\dots,\alpha_s)\in H} c(\alpha_1,\dots,\alpha_s)z_1^{\alpha_1}\cdots z_s^{\alpha_s}, \tag{2.2}$$

where all $c(\alpha_1,\dots,\alpha_s)$ are real, such that

$$p(u_i)=f(u_i),\ i=1,\dots,M. \tag{2.3}$$

Let us define $\widehat{b}(z_1,\dots,z_s)$ in a similar way as we have defined $\widehat{b}$ but we remove the expectation and replace z_j for X_j, $j=1,\dots,s$. In connection with problem (1.3) we define H, I and U as follows:

$$\begin{aligned} H = \{(\alpha_1,\dots,\alpha_s)|\ &0\leqslant\alpha_j,\ \alpha_j \text{ integer},\ \alpha_1+\cdots+\alpha_s\leqslant m,\ j=1,\dots,s;\\ &\text{or } \alpha_j=0,\ j=1,\dots,k-1,k+1,\dots,s,\ m\leqslant\alpha_k\leqslant n_k,\ k=1,\dots,s\}, \end{aligned}$$

$$I=\{(i_1,\dots,i_s)|\ \widehat{a}_{i_1\cdots i_s}\in\widehat{B}\},$$

$$U=\{(z_{1i_1},\dots,z_{si_s})|\ (i_1,\dots,i_s)\in I\}.$$

Then

$$L_I(z_1,\dots,z_s)=f_{\widehat{B}}^T\widehat{B}^{-1}\widehat{b}(z_1,\dots,z_s))$$

is the unique H-type Lagrange polynomial corresponding to the set U.

The dual feasibility of the basis $\widehat{B}$ in the minimization (maximization) problem means that

$$\begin{aligned} &f(z_1,\dots,z_s)\geqslant L_I(z_1,\dots,z_s),\ \text{all } (z_1,\dots,z_s)\in Z\\ &(f(z_1,\dots,z_s)\leqslant L_I(z_1,\dots,z_s),\ \text{all } (z_1,\dots,z_s)\in Z), \end{aligned} \tag{2.4}$$

where equality holds in case of $(z_1,\dots,z_s)\in U$. Relation (2.4) is called the condition of optimality of the minimization (maximization) problem (1.3).

Replacing $(X_1,\dots,X_s)$ for $(z_1,\dots,z_s)$ and taking expectations in (2.4) we obtain bounds for $E[f(X_1,\dots,X_s)]$. If the basis is also primal feasible, then it is optimal and thus, the obtained bound is sharp.

To state the results of this section we need the following definitions.

DEFINITION 2.1. Let $f(z),\ z\in\{z_0,\dots,z_n\}$ be a univariate discrete function, where $z_0,\dots,z_n$ are distinct real numbers. We define

$$[z_i;f]=f(z_i),\ \text{where } z_i\in\{z_0,\dots,z_n\}.$$

The *kth order (univariate) divided differences* ($k\geqslant 1$) are defined recursively as:

$$[z_i,\dots,z_{i+k};f]=\frac{[z_{i+1},\dots,z_{i+k};f]-[z_i,\dots,z_{i+k-1};f]}{z_{i+k}-z_i},0\leqslant i\leqslant n-k.$$

DEFINITION 2.2. Let $f(z),\ z\in Z=Z_1\times\cdots\times Z_s$ be a multivariate discrete function and

$$\begin{aligned} Z_{I_1\dots I_s} &= \{z_{1i},\ i\in I_1\}\times\cdots\times\{z_{si},\ i\in I_s\} \\ &= Z_{1I_1}\times\cdots\times Z_{sI_s}, \end{aligned} \tag{2.5}$$

where $|I_j|=k_j+1,\ j=1,\dots,s$. We define the $(k_1,\dots,k_s)$*-order (multivariate) divided difference of* f on the set (2.5) in an iterative way. First we take the k_1th divided difference with respect to the first variable, then the k_2th divided difference with respect to the second variable etc. These operations can be executed in any order even in a mixed manner, the result is always the same. Let

$$[z_{1i},\ i\in I_1;\cdots;z_{si},\ i\in I_s;f] \tag{2.6}$$

designate the $(k_1,\dots,k_s)$-order divided difference and call the sum $k_1+\cdots+k_s$ its total order. For example,

$$\begin{aligned}[z_{10},z_{11};z_{20},z_{21};f] &= [z_{20},z_{21};\frac{f(z_{11},z_2)-f(z_{10},z_2)}{z_{11}-z_{10}}] \\ &= \frac{\frac{f(z_{11},z_{21})-f(z_{10},z_{21})}{z_{11}-z_{10}}-\frac{f(z_{11},z_{20})-f(z_{10},z_{20})}{z_{11}-z_{10}}}{z_{21}-z_{20}}.\end{aligned}$$

If $f(z),\ z\in Z$ is derived from a function $\overline{f}(z)$ defined in $\overline{Z}=[z_{10},z_{1n_1}]\times\cdots\times[z_{s0},z_{sn_s}]$ by taking $f(z)=\overline{f}(z),\ z\in Z$ and $\overline{f}(z)$ has continuous, nonnegative derivatives of order $(k_1,\dots,k_s)$ in the interior of $\overline{Z}$, then all divided differences of $f(z),\ z\in Z$ of order $(k_1,\dots,k_s)$ are nonnegative. For further results in this respect see Popoviciu (1944).

In what follows we will use the notations

$$\begin{aligned} Z_{ji} &= \{z_{j0},\dots,z_{ji}\} \\ Z'_{ji} &= \{z_{j0},\dots,z_{ji},z_j\}, \\ & i=0,\dots,n_j,\ j=1,\dots,s. \end{aligned}$$

Consider the set of subscripts

$$I=I_0\cup\left(\cup_{j=1}^{s}I_j\right), \tag{2.7}$$

where

$$I_0=\left\{(i_1,\dots,i_s)|\ 0\leqslant i_j\leqslant m-1,\ \text{integers},\ j=1,\dots,s,\ i_1+\ldots+i_s\leqslant m\right\} \tag{2.8}$$

and

$$I_j = \{(i_1,\ldots,i_s)|\ m \leqslant i_j \leqslant n_j,\ i_l = 0 \text{ for every } l \neq j\},\ j = 1,\ldots,s. \tag{2.9}$$

Corresponding to the points

$$Z_I = \{(z_{1i_1},\ldots,z_{si_s})|\ (i_1,\ldots,i_s) \in I\} \tag{2.10}$$

we assign the Lagrange polynomial, given by its Newton's form:

$$\begin{aligned} &L_I(z_1,\ldots,z_s) \\ &= \sum_{\substack{i_1+\ldots+i_s \leqslant m \\ 0 \leqslant i_j \leqslant m-1,\ j=1,\ldots,s}} [Z_{1i_1};\cdots;Z_{si_s};f] \prod_{j=1}^{s} \prod_{k=0}^{i_j-1} (z_j - z_{jk}) \\ &+ \sum_{j=1}^{s} \sum_{i=m}^{n_j} \left[Z_{10};\cdots;Z_{(j-1)0};Z_{ji};Z_{(j+1)0};\cdots;Z_{s0};f\right] \prod_{k=0}^{i-1} (z_j - z_{jk}), \\ &\text{where, by definition, } \prod_{k=0}^{i_j-1} (z_j - z_{jk}) = 1, \text{ for } i_j = 0. \end{aligned} \tag{2.11}$$

In (2.11) the function f is not necessarily restricted to the set Z as its domain of definition; it may be defined on any subset of $\mathbb{R}^s$ that contains Z.

Next, we define the residual function:

$$R_I(z_1,\ldots,z_s) = R_{1I}(z_1,\ldots,z_s) + R_{2I}(z_1,\ldots,z_s), \tag{2.12}$$

where

$$\begin{aligned} &R_{1I}(z_1,\ldots,z_s) \\ &= \sum_{j=1}^{s} \left[z_{10};\cdots;z_{(j-1)0};Z'_{jn_j};z_{(j+1)0};\cdots;z_{s0};f\right] \prod_{k=0}^{n_j} (z_j - z_{jk}) \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} &R_{2I}(z_1,\ldots,z_s) \\ &= \sum_{h=1}^{s-1} \sum_{\substack{i_h+\cdots+i_s=m \\ 0 \leqslant i_j \leqslant m-1,\ j=h,\ldots,s}} \left[z_1;\cdots;z_{h-1};Z'_{hi_h};Z_{(h+1)i_{h+1}};\cdots;Z_{si_s};f\right] \prod_{l=0}^{i_h} (z_h - z_{hl}) \\ &\times \prod_{h+1}^{s} \prod_{k=0}^{i_j-1} (z_j - z_{jk}) \\ &+ \sum_{j=h+1}^{s} \left[z_1;\cdots;z_{h-1};Z'_{h0};Z_{(h+1)0};\cdots;Z_{(j-1)0};Z'_{j(m-1)};Z_{(j+1)0};\cdots;Z_{s0}\right] (z_h - z_{h0}) \\ &\times \prod_{k=0}^{m-1} (z_j - z_{jk}). \end{aligned} \tag{2.14}$$

THEOREM 2.1. *Consider the Lagrange polynomial* (2.11)*, corresponding to the points in* Z_I*. For any* $z = (z_1,\ldots,z_s)$ *for which the function* f *is defined, we have the equality*

$$L_I(z_1,\ldots,z_s) + R_I(z_1,\ldots,z_s) = f(z_1,\ldots,z_s). \tag{2.15}$$

Based on this we prove two theorems that provide us with bounds for the expectations of $f(X_1,\dots,X_s)$.

THEOREM 2.2. *Let* $z_{j0} < z_{j1} < \cdots < z_{jn_j}$, $j = 1,\dots,s$. *Suppose that the function* $f(z)$, $z \in Z$ *has nonnegative mixed divided differences of total order* $m+1$.

Under this condition $L_I(z_1,\dots,z_s)$, *defined by* (2.11), *is a unique* H*-type Lagrange polynomial on* Z_I *and satisfies the relation*

$$f(z_1,\dots,z_s) \geqslant L_I(z_1,\dots,z_s),\ (z_1,\dots,z_s) \in Z, \tag{2.16}$$

i.e., the set of columns $\widehat{B}$ *of* $\widehat{A}$ *in problem* (1.4)*, with the subscript set* I*, is a dual feasible basis in the minimization problem* (1.4)*, and*

$$E[f(X_1,\dots,X_s)] \geqslant E[L_I(X_1,\dots,X_s)]. \tag{2.17}$$

If $\widehat{B}$ *is also a primal feasible basis in problem* (1.4)*, then the inequality* (2.17) *is sharp.*

If all the above mentioned divided differences are nonpositive, then (2.16) *and* (2.17) *hold with reversed inequality signs.*

Proof. The proof is similar to that of Theorem 4.1 in Mádi-Nagy and Prékopa (2004).

A sketch of a direct proof is as follows. Both terms in (2.14) are nonnegative because of the nonnegativity of the divided differences involved and the special structure of the basis implies the nonnegativity of products of the differences of the values. Further, the relations

$$\prod_{k=0}^{n_j}(z_j - z_{jk}) = 0 \text{ for } z_j \in Z_j, \tag{2.18}$$

imply that $R_{1I}(z_1,\dots,z_s)$ in (2.13) is zero. □

In the next theorem we give both lower and upper bounds for the function $f(z_1,\dots,z_s)$, $(z_1,\dots,z_s) \in Z$ and the expectation $E[f(X_1,\dots,X_s)]$.

THEOREM 2.3. *Let* $z_{j0} > z_{j1} > \cdots > z_{jn_j}$, $j = 1,\dots,s$. *Suppose that the function* $f(z)$, $z \in Z$ *has nonnegative mixed divided differences of total order* $m+1$. *Under this condition we have the following assertions:*

(a) If $m+1$ *is even, then the Lagrange polynomial* $L_I(z_1,\dots,z_s)$*, defined by* (2.11)*, satisfies*

$$f(z_1,\dots,z_s) \geqslant L_I(z_1,\dots,z_s),\ (z_1,\dots,z_s) \in Z, \tag{2.19}$$

i.e., the set of columns $\widehat{B}$ *in* $\widehat{A}$*, corresponding to the subscripts* I*, is a dual feasible basis in the minimization problem* (1.4)*. We also have the inequality*

$$E[f(X_1,\dots,X_s)] \geqslant E[L_I(X_1,\dots,X_s)]. \tag{2.20}$$

If $\widehat{B}$ *is also a primal feasible basis in the LP* (1.4)*, then the lower bound* (2.20) *for* $E[f(X_1,\dots,X_s)]$ *is sharp.*

(b) If $m+1$ is odd, then the Lagrange polynomial, defined by (2.11), satisfies

$$f(z_1,\ldots,z_s) \leqslant L_I(z_1,\ldots,z_s),\ (z_1,\ldots,z_s) \in Z, \tag{2.21}$$

i.e., the basis $\widehat{B}$ is dual feasible in the maximization problem (1.4). We also have the inequality

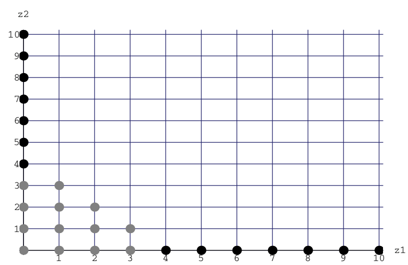
$$E[f(X_1,\ldots,X_s)] \leqslant E[L_I(X_1,\ldots,X_s)]. \tag{2.22}$$

If $\widehat{B}$ is also a primal feasible basis in the LP (1.4), then the upper bound (2.22) for $E[f(X_1,\ldots,X_s)]$ is sharp.

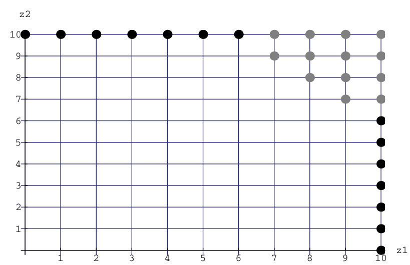
If all the above mentioned divided differences are nonpositive, then (2.19), (2.20), (2.21) and (2.22) hold with reversed inequality signs.

Proof. The proof is similar to that of Theorem 4.2 in Mádi-Nagy and Prékopa (2004). A sketch of a direct proof, similar to that of Theorem 2.2, can be given here, too. □

The dual feasible basis structures mentioned in the above theorems are illustrated in Figure 1.



(a)



(b)

Figure 1: Dual feasible bases corresponding to Theorems 2.2 (in (a)) and 2.3 (in (b)) in case of $n_1 = n_2 = 10$, $m = 3$, $Z_1 = Z_2 = \{0,1,\ldots,10\}$. The elements of I_0 are gray, those of I_j's are black.

For the case of $s = 2$ we can create a larger variety of dual feasible bases for problem (1.4), and produce better bounds than what we can obtain by the use of the dual feasible basis structures presented in the previous theorems.

All coefficients in the expression of $R_{2I}(z_1,z_2)$ are mixed divided differences of order $m+1$. Assume all of them are nonnegative. Our aim is to combine the elements of Z_1 and Z_2 such that the products in $R_{2I}(z_1,z_2)$ turn out to be nonnegative. This can be produced by slight modifications of the Min and Max Algorithms of Mádi-Nagy and Prékopa (2004). This way we can get a variety of dual feasible bases that give tight bounds on $E[f(X_1,X_2)]$. Below we summarize them for the special case of problem (1.3), where $s = 2$.

We may assume, without loss of generality, that the ordered sets Z_1 and Z_2 are the following: $Z_1 = \{0,1,\ldots,n_1\}$, $Z_2 = \{0,1,\ldots,n_2\}$. The procedure is presented below.

Min Algorithm

Algorithm to find $z_{10},\dots,z_{1(m-1)};z_{20},\dots,z_{2(m-1)}$.

Step 0. Initialize $t=0$, $-1\leqslant q_1\leqslant m-1$, $L=(0,1,\dots,q_1)$, $U=(n_1,n_1-1,\dots,n_1-(m-q_1-2))$, $V^0=\{\text{arbitrary merger of the sequences } L,U\}=(v^0,v^1,\dots,v^{m-1})$. If $|U|$ is even, then $h^0=0$, $l^0=1$, $u^0=n_2$, and if $|U|$ is odd, then $h^0=n_2$, $l^0=0$, $u^0=n_2-1$. Go to Step 1.

Step 1. If $t=m$, then go to Step 3. Otherwise go to Step 2.

Step 2. Let $V^t=(v^0,v^1,\dots,v^{m-1-t})$, $H^t=(h^0,h^1,\dots,h^t)$. If $v^{m-1-t}\in L$, then let $h^{t+1}=l^t$, $l^{t+1}=l^t+1$, $u^{t+1}=u^t$, and if $v^{m-1-t}\in U$, then let $h^{t+1}=u^t$, $u^{t+1}=u^t-1$, $l^{t+1}=l^t$. Set $t\leftarrow t+1$ and go to Step 1.

Step 3. Stop. Let

$$(z_{10},\dots,z_{1(m-1)})=V^0,$$

$$(z_{20},\dots,z_{2(m-1)})=H^{m-1}.$$

Let $0,1,\dots,q_2,n_2,\dots,n_2-(m-q_2-2)$ be the numbers used to construct $z_{20},z_{21},\dots,z_{2(m-1)}$. Then let $\{z_{jm},z_{j(m+1)},\dots,z_{jn_j}\}=\{q_j+1,q_j+2,\dots,n_j-(m-q_j-1)\}$, $j=1,2$. They can follow each other in any order, because they don't play role in the value of R_I, and on the other hand their order does not change the dual feasible basis structure that we finally obtain.

To construct an upper bound, slight modification is needed in Step 0, while the rest of the algorithm is unchanged. The modified Step 0 is presented below.

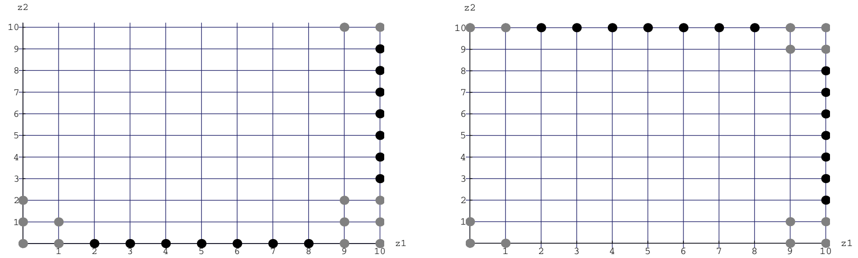
Max Algorithm

Step 0 of algorithm to find $z_{10},\dots,z_{1(m-1)};z_{20},\dots,z_{2(m-1)}$.

Step 0. Initialize $t=0$, $-1\leqslant q_1\leqslant m-1$, $L=(0,1,\dots,q_1)$, $U=(n_1,n_1-1,\dots,n_1-(m-q_1-2))$, $V^0=\{\text{arbitrary merger of the sets } L,U\}=(v^0,v^1,\dots,v^{m-1})$. If $|U|$ is odd, then $h^0=0$, $l^0=1$, $u^0=n_2$, and if $|U|$ is even, then $h^0=n_2$, $l^0=0$, $u^0=n_2-1$. Go to Step 1, etc.

In the general case, where Z_1 is not necessarily $\{0,1,\dots,n_1\}$ and Z_2 is not necessarily $\{0,1,\dots,n_2\}$, we do the following. First we order the elements in both Z_1 and Z_2 increasingly. Then, create pairs out of the elements of Z_1 and $\{0,1,\dots,n_1\}$, where the elements of the sets are assumed to be arranged in increasing order. We do the same to Z_2 and $\{0,1,\dots,n_2\}$. After that, we carry out the Min or Max Algorithm to find a dual feasible basis, using the sets $\{0,1,\dots,n_1\}$, $\{0,1,\dots,n_2\}$, as described in this section. Finally, we create the ordered sets Z_1 and Z_2 by the use of the above mentioned pairings.

Examples of dual feasible bases, obtained by the Min and Max Algorithms, are illustrated in Figure 2.



(a) (b)

Figure 2: (a): Dual feasible basis of the min problem, where $m=4$ and $(z_{10},\ldots,z_{1(m-1)})=(10,9,0,1)$, $(z_{20},\ldots,z_{2(m-1)})=(0,1,2,10)$. (b): Dual feasible basis of the max problem, where $m=4$ and $(z_{10},\ldots,z_{1(m-1)})=(10,9,0,1)$, $(z_{20},\ldots,z_{2(m-1)})=(10,0,1,9)$. The elements of I_0 are gray, those of I_j's are black.

3. Monge property and bounding multivariate probability distribution functions with given marginals and covariances

In this chapter we assume that the function f has the Monge or inverse Monge or some discrete higher order convexity property. First, we need the following

DEFINITION 3.1. An $n_1\times\cdots\times n_s$ s-dimensional array $f=\{f(i_1,\ldots,i_s)\}$ has the *Monge property* or is a *Monge array*, if for all entries $f(i_1,\ldots,i_s)$ and $f(j_1,\ldots,j_s)$, $1\leqslant i_k,j_k\leqslant n_k$, $1\leqslant k\leqslant s$, we have

$$f(l_1,\ldots,l_s)+f(u_1,\ldots,u_s)\leqslant f(i_1,\ldots,i_s)+f(j_1,\ldots,j_s), \tag{3.1}$$

where $l_k=min\{i_k,j_k\}$, $u_k=max\{i_k,j_k\}$, $1\leqslant k\leqslant s$. If the inequality (3.1) holds in reverse order, then it is called the *inverse Monge property* and f is called an *inverse Monge array*.

REMARK 3.1. If $f(z_1,\ldots,z_s), z\in Z=Z_1\times\cdots\times Z_s$ is a (inverse) Monge array on Z, then its second order mixed divided differences are nonpositive (nonnegative). In the two-dimensional case, $f(z_1,z_2), z\in Z=Z_1\times Z_2$ is a (inverse) Monge array on Z if and only if its $(1,1)$ order divided differences are nonpositive (nonnegative).

If we consider problem (1.2) in case of $m=1$, i.e., if only the marginal distributions are known, then it is an s-dimensional transportation problem. In connection with that we have

THEOREM 3.1. (Theorem 2.4 in Hou and Prékopa (2007)) *In the s-dimensional transportation problem any ordered sequence forms a dual feasible basis if and only if f is Monge.*

In case of $m=2$, where the second order moments (covariances) are also known, all the dual feasible bases presented in the mentioned paper can be reproduced by our theorems and the Min and Max Algorithms in a relatively simple way. In the two dimensional case our methodology provides us with additional dual feasible bases as it is shown in the following example.

EXAMPLE 3.1. Consider the minimum problem (1.2), and the equivalent MDMP (1.3) in case of $s=2$ and $m=2$. Suppose that the function $f(z),\ z\in Z$ has nonnegative mixed divided differences of total order 3, i.e., the $(1,2)$-order and $(2,1)$-order divided differences are nonnegative. If we apply the Min Algorithm to the problem, then we obtain the dual feasible bases:

(a) $(z_{10},z_{11}) = V^0 := (0,1) \Longrightarrow (z_{20},z_{21}) = H^{m-1} = (0,1)$

(b) $(z_{10},z_{11}) = V^0 := (0,n_1) \Longrightarrow (z_{20},z_{21}) = H^{m-1} = (n_2,n_2-1)$

(c) $(z_{10},z_{11}) = V^0 := (n_1,0) \Longrightarrow (z_{20},z_{21}) = H^{m-1} = (n_2,0)$

(d) $(z_{10},z_{11}) = V^0 := (n_1,n_1-1) \Longrightarrow (z_{20},z_{21}) = H^{m-1} = (0,n_2)$

The bases are illustrated in Figure 3. Basis (b) is the same as basis B_1 in Figure 4.1 in Hou and Prékopa (2007) (regarding that the order of z_j's there are decreasing) which was the only dual feasible basis there.

n_2	●	○	○	···	○	○	○
n_2-1	●	○	○	···	○	○	○
n_2-2	●	○	○	···	○	○	○
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮
2	●	○	○	···	○	○	○
1	*	*	○	···	○	○	○
0	*	*	●	···	●	●	●
	0	1	2	···	(n_1-2)	(n_1-1)	n_1

(a)

n_2	*	●	●	···	●	●	*
n_2-1	*	○	○	···	○	○	*
n_2-2	●	○	○	···	○	○	○
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮
2	●	○	○	···	○	○	○
1	●	○	○	···	○	○	○
0	●	○	○	···	○	○	○
	0	1	2	···	(n_1-2)	(n_1-1)	n_1

(b)

n_2	*	●	●	···	●	●	*
n_2-1	○	○	○	···	○	○	●
n_2-2	○	○	○	···	○	○	●
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮
2	○	○	○	···	○	○	●
1	○	○	○	···	○	○	●
0	*	○	○	···	○	○	*
	0	1	2	···	(n_1-2)	(n_1-1)	n_1

(c)

n_2	○	○	○	···	○	*	*
n_2-1	○	○	○	···	○	○	●
n_2-2	○	○	○	···	○	○	●
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮
2	○	○	○	···	○	○	●
1	○	○	○	···	○	○	●
0	●	●	●	···	●	*	*
	0	1	2	···	(n_1-2)	(n_1-1)	n_1

(d)

Figure 3: Bases of Example 3.1. The elements of I_0 are denoted by asterisks, those of the I_j's by bullets.

4. Bounding the expectations of pseudo-Boolean functions of binary random variables

Let $A_1,\dots,A_s$ be arbitrary events in some probability space, and introduce the notations

$$P(A_{i_1}\cap\cdots\cap A_{i_k})=p_{i_1\dots i_k},\ 1\leqslant i_1<\cdots<i_k\leqslant s. \tag{4.1}$$

We want to give bounds for $P(A_1\cup\cdots\cup A_s)$ assuming, that some of the probabilities of (4.1) are known. The Boolean probability bounding problem is formulated as follows. Define

$$a_{KJ}=\begin{cases}1 \text{ if } K\subset J,\\ 0 \text{ if } K\not\subset J,\end{cases}$$

$$v_J=P\left(\left(\bigcap_{j\in J}A_j\right)\cap\left(\bigcap_{j\notin J}\overline{A_j}\right)\right),\qquad p_K=P\left(\bigcap_{j\in K}A_j\right)$$

for any $K,J\subset\{1,\dots,s\}$. Then we have the equations

$$\sum_{J\subset\{1,\dots s\}}a_{KJ}v_J=p_K,\ K\subset\{1,\dots,s\}.$$

We formulate the following LP:

$$\begin{aligned}&\min(\max)\sum_{\emptyset\neq J\subset\{1,\dots,s\}}x_J\\ &\text{subject to}\\ &\sum_{J\subset\{1,\dots s\}}a_{KJ}x_J=p_K,\ K\subset\{1,\dots,s\}\\ &\text{for } |K|\leqslant m,\\ &x_J\geqslant 0,\ J\subset\{1,\dots,s\}.\end{aligned} \tag{4.2}$$

Problem (4.2) can be reformulated as an MDMP. Consider the event sequence $A_1,\dots,A_s$ and define the random vector $X=(X_1,\dots,X_s)$ such that X_j is the characteristic random variable of the event A_j, $j=1,\dots,s$, i.e.

$$X_j=\begin{cases}1 \text{ if } A_j \text{ occurs},\\ 0 \text{ otherwise.}\end{cases}$$

Define the function $f(z_1,\dots,z_s)$, $(z_1,\dots,z_s)\in Z=\{0,1\}\times\cdots\times\{0,1\}$ as follows:

$$f(z_1,\dots,z_s)=\begin{cases}0 \text{ if } (z_1,\dots,z_s)=(0,\dots,0),\\ 1 \text{ otherwise.}\end{cases} \tag{4.3}$$

If $m+1$ is even (odd) then all divided differences of the function (4.16) of total order $m+1$ are nonpositive (nonnegative).

The equivalent MDMP is the following:

$$\min(\max)\sum_{i_1=0}^{1}\cdots\sum_{i_s=0}^{1}f_{i_1\dots i_s}p_{i_1\dots i_s}$$

subject to

$$\begin{aligned} &\sum_{i_1=0}^{1} \cdots \sum_{i_s=0}^{1} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1\ldots i_s} = \mu_{\alpha_1\ldots\alpha_s} \\ &\text{for } \alpha_j = 0,1;\ j=1,\ldots,s;\ \alpha_1+\cdots+\alpha_s \leqslant m \\ &p_{i_1\ldots i_s} \geqslant 0,\ \text{all } i_1,\ldots,i_s, \end{aligned} \tag{4.4}$$

where $0^0 = 1$, by definition. We can see that the objective function is indeed the probability of the union of the events while the constraints are the same as those in (4.2)

Now, let us consider problem (4.4) with an arbitrary function $f(z_1,\ldots,z_s)$, defined on $(z_1,\ldots,z_s) \in \{0,1\}^s$. Problem (4.2) can be rewritten in a more compact form:

$$\begin{aligned} &\min(\max) \quad f^T p \\ &\text{subject to} \\ &\breve{A}p = \breve{b} \\ &p \geqslant 0. \end{aligned} \tag{4.5}$$

In this section we consider the following subscript set

$$I = \left\{(i_1,\ldots,i_s) \middle|\ 0 \leqslant i_j \leqslant 1,\ \text{integers},\ j=1,\ldots,s,\ i_1+\ldots+i_s \leqslant m\right\}. \tag{4.6}$$

Corresponding to the points Z_I we assign the Lagrange polynomial, given by its Newton's form

$$\begin{aligned} L_I(z_1,\ldots,z_s) = &\sum_{\substack{i_1+\ldots+i_s \leqslant m \\ 0\leqslant i_j \leqslant 1,\ j=1,\ldots,s}} [Z_{1i_1};\cdots;Z_{si_s};f] \prod_{j=1}^{s} \prod_{k=0}^{i_j-1} \left(z_j - z_{jk}\right), \\ &\text{where, by definition, } \prod_{k=0}^{i_j-1}\left(z_j - z_{jk}\right) = 1,\ \text{for } i_j = 0. \end{aligned} \tag{4.7}$$

The residual function is defined as:

$$\begin{aligned} &R_I(z_1,\ldots,z_s) \\ &= \sum_{h=1}^{s} \Bigg(\sum_{\substack{0+i_{h+1}\cdots+i_s = m \\ 0\leqslant i_j\leqslant 1,\ j=(h+1),\ldots,s}} \left[z_1;\cdots;z_{h-1};Z'_{h0};Z_{(h+1)i_{h+1}};\cdots;Z_{si_s};f\right](z_h - z_{h0}) \\ &\qquad \times \prod_{h+1}^{s} \prod_{k=0}^{i_j-1} \left(z_j - z_{jk}\right) \\ &+ \sum_{\substack{1+i_{h+1}+\cdots+i_s\leqslant m \\ 0\leqslant i_j\leqslant 1,\ j=(h+1),\ldots,s}} \left[z_1;\cdots;z_{h-1};Z'_{h1};Z_{(h+1)i_{h+1}};\cdots;Z_{si_s};f\right] \prod_{l=0}^{1} (z_h - z_{hl}) \\ &\qquad \times \prod_{h+1}^{s} \prod_{k=0}^{i_j-1} \left(z_j - z_{jk}\right) \Bigg). \end{aligned} \tag{4.8}$$

THEOREM 4.1. *Consider the Lagrange polynomial* (4.7)*, corresponding to the points* Z_I*, where* I *is defined in* (4.6)*. For any* $z = (z_1, \ldots, z_s)$ *for which the function* f *is defined, we have the equality*

$$L_I(z_1, \ldots, z_s) + R_I(z_1, \ldots, z_s) = f(z_1, \ldots, z_s). \tag{4.9}$$

Proof. The assertion can be proved similarly as the proof of Theorem 4.1 in Prékopa (1998). □

THEOREM 4.2. *Let* $0 = z_{j0} < z_{j1} = 1,\ j = 1, \ldots, s$*. Suppose that the function* $f(z),\ z \in Z$ *has nonnegative mixed divided differences of total order* $m+1$*.*

Under these conditions $L_I(z_1, \ldots, z_s)$*, defined by* (4.7)*, is a unique suitable* H*-type Lagrange polynomial on* Z_I *and satisfies the relations*

$$f(z_1, \ldots, z_s) \geqslant L_I(z_1, \ldots, z_s),\ (z_1, \ldots, z_s) \in Z, \tag{4.10}$$

i.e., the set of columns $\breve{B}$ *of* $\breve{A}$ *in problem* (4.5)*, with the subscript set* I *of* (4.6)*, is a dual feasible basis in the minimization problem* (4.5)*, and*

$$E[f(X_1, \ldots, X_s)] \geqslant E[L_I(X_1, \ldots, X_s)]. \tag{4.11}$$

If $\breve{B}$ *is also a primal feasible basis in problem* (4.5)*, then the inequality* (4.11) *is sharp.*

If all the above mentioned divided differences are nonpositive, then (4.10) *and* (4.11) *hold with reversed inequality signs.*

Proof. Similar to the proof of Theorem 2.2. □

THEOREM 4.3. *Let* $1 = z_{j0} > z_{j1} = 0,\ j = 1, \ldots, s$*. Suppose that the function* $f(z),\ z \in Z$ *has nonnegative mixed divided differences of total order* $m+1$*. Under these conditions we have the following assertions:*

(a) If $m+1$ *is even, then the Lagrange polynomial* $L_I(z_1, \ldots, z_s)$*, defined by* (4.7)*, satisfies*

$$f(z_1, \ldots, z_s) \geqslant L_I(z_1, \ldots, z_s),\ (z_1, \ldots, z_s) \in Z, \tag{4.12}$$

i.e., the set of columns $\breve{B}$ *in* $\breve{A}$*, corresponding to the subscripts* I *of* (4.6)*, is a dual feasible basis in the minimization problem* (4.5)*. We also have the inequality*

$$E[f(X_1, \ldots, X_s)] \geqslant E[L_I(X_1, \ldots, X_s)]. \tag{4.13}$$

If $\breve{B}$ *is also a primal feasible basis in the LP* (4.5)*, then the lower bound* (4.13) *for* $E[f(X_1, \ldots, X_s)]$ *is sharp.*

(b) If $m+1$ *is odd, then the Lagrange polynomial, defined by* (4.7)*, satisfies*

$$f(z_1, \ldots, z_s) \leqslant L_I(z_1, \ldots, z_s),\ (z_1, \ldots, z_s) \in Z, \tag{4.14}$$

i.e., the basis $\breve{B}$ is dual feasible in the maximization problem (4.5). We also have the inequality

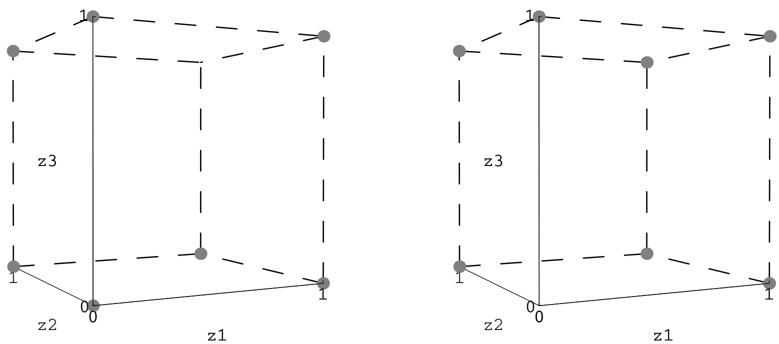
$$E[f(X_1,\ldots,X_s)] \leqslant E[L_I(X_1,\ldots,X_s)]. \tag{4.15}$$

If $\breve{B}$ is also a primal feasible basis in the LP (4.5), then the upper bound (4.15) for $E[f(X_1,\ldots,X_s)]$ is sharp.

If all the above mentioned divided differences are nonpositive, then (4.12), (4.13), (4.14) and (4.15) hold with reversed inequality signs.

Proof. Similar to the proof of Theorem 2.3. □

The dual feasible bases that appear in Theorems 4.2 and 4.3 are illustrated in Figure 4, for the three-dimensional case.



(a) (b)

Figure 4: Dual feasible bases of Theorems 4.2 (figure (a)) and 4.3 (figure (b)), where $m=2$, $s=3$

Prékopa (1998) observed that the probability bounds, corresponding to some special dual feasible bases in MDMP, reproduce the Bonferroni bounds. Since the bounding problem was embedded in an LP, the dual algorithm could be applied, with given dual feasible basis as initial basis, to obtain the best possible bounds. It was also shown on a numerical example that the Bonferroni bound may be meaningless but the obtained algorithmic bound based on the same input data is quite good. Similar phenomenon can be observed here, too.

If $m+1$ is even (odd) then all divided differences of the function (4.16) of total order $m+1$ are nonpositive (nonnegative). It follows from this, that the bases in Theorems 4.2 and 4.3 are dual feasible in problem (4.4), with the objective function (4.3). This also means that we have found dual feasible bases to problem (4.2), which can serve for bounding the probability of the union of events $A_1,\ldots,A_n$. It can be shown that those bounds are the same as the Bonferroni-bounds of order m, that are frequently weak or trivial. However, those dual feasible bases can be used as initial bases carry out the dual method and find the sharp bounds.

If we want to create dual feasible bases for the probability of the intersection, i.e., for $P(A_1 \cap \cdots \cap A_n)$, then we can work with the same constraints that are in the MDMP (4.4) and the new objective function is

$$f(z_1,\ldots,z_s) = \begin{cases} 1 \text{ if } (z_1,\ldots,z_s) = (1,\ldots,1), \\ 0 \text{ otherwise.} \end{cases} \tag{4.16}$$

It is easy to check that all divided differences of any order of the function (4.3) are nonnegative. Using this we can construct dual feasible bases for the new MDMP (4.4), where the objective function is given by (4.16).

EXAMPLE 4.1. The following example is taken from Kuai, Alajaji and Takahara (2000) and Prékopa and Gao (2005). There are 6 events: $A_1,\ldots,A_6$, 15 possible outcomes with probabilities given in the table below.

Outcomes x	$p(x)$	A_1	A_2	A_3	A_4	A_5	A_6
x_0	0.012	×		×		×	
x_1	0.022		×		×		
x_2	0.023	×		×		×	
x_3	0.033		×				
x_4	0.034	×				×	×
x_5	0.044		×	×		×	
x_6	0.045		×			×	×
x_7	0.055			×	×		×
x_8	0.056	×		×			
x_9	0.066				×	×	
x_{10}	0.067		×		×	×	
x_{11}	0.077		×				
x_{12}	0.078	×			×		×
x_{13}	0.088		×				
x_{14}	0.089	×		×		×	×

We give sharp lower and upper bounds for the probability of the union of the events, using the probabilities of some intersections of the A_i's. We solve problem (4.4) with the objective function (4.3). We use the bases of Theorems 4.2 and 4.3 as initial bases of the dual method of CPLEX. The results depending on the parameter m of (4.4) are:

m	Minimum	Iteration	Maximum	Iteration
2	0.789	13	0.955	27
3	0.789	11	0.789	20
4	0.789	0	0.789	12
5	0.789	0	0.789	1

Kuai, Alajaji and Takahara (2000) give 0.7222 as a lower bound, using probabilities of the single events and intersections of pairs of events. Prékopa and Gao (2005) gives 0.73145 as a lower bound and 0.8038333 as an upper bound. They use intersections of up to three events.

Other bounding approaches that are based on the Boolean probability bounding scheme include Bukszár (2003), Bukszár and Prékopa (2001), Bukszár and Szántai (2002), Prékopa and Gao (2005), Prékopa, Vizvári and Regős (1997), Vizvári (2007).

5. Bounding multivariate moment generating functions

The moment generating function of a random variable X is the function M defined by the equation

$$M(t) = E[e^{tX}],\ t \in \mathbb{R}.$$

If $M(t)$ is finite in an open interval J around 0, then M completely determines the distribution of X. Also, M has derivatives of all orders in J and $M^{(n)}(t) = E[X^n e^{tX}]$, $t \in J$. This implies that $M^{(n)}(0) = E[X^n], n = 1, 2, \ldots$.

The joint moment generating function of the random variables $X_1, \ldots, X_s$ is defined as

$$M(t_1, \ldots, t_s) = E[e^{t_1 X_1 + \cdots + t_s X_s}].$$

If it is finite in an open neighborhood around the origin, then M completely determines the distribution of $X = (X_1, \ldots, X_s)$. Other interesting properties are: $M(0, \ldots, 0, t_i, 0, \ldots, 0) = M_i(t_i)$, $i = 1, \ldots, s$ and

$$\frac{\partial^{\alpha_1 + \cdots \alpha_s} M}{\partial {t_1}^{\alpha_1} \cdots \partial {t_s}^{\alpha_s}}(0, \ldots, 0) = \mu_{\alpha_1 \ldots \alpha_s}.$$

More details about (joint) generating function can be found, e.g., in Ross (2002).

If we assume that X has a finite support, then we can use the MDMP for bounding the value of the joint moment generating function for certain values of $(t_1, \ldots, t_s)$ in terms of the (mixed) power moments of X. Recently, Ibrahim and Mugdadi (2005) gave bounds for the values of univariate moment generating functions, based on moments.

For any fixed $(t_1, \ldots, t_s) \geqslant 0$ all divided differences of the function $e^{t_1 z_1 + \cdots + t_s z_s}$ are nonnegative. It is also true that the $m+1$st divided differences are nonnegative (nonpositive) for $(t_1, \ldots, t_s) \leqslant 0$ if $m+1$ is even (odd). In these cases the methods of Section 2 can be applied as it is shown in the following

EXAMPLE 5.1. We use the Min and Max Algorithms of Section 2 to give lower and upper bounds for the bivariate moment generating function $M(t_1, t_2)$, where $t_1 = 0.04$ and $t_2 = 0.05$. The codes for the calculation are written in Wolfram's Mathematica. Assume that the random variables X_1, X_2 have uniform univariate distributions on the supports $Z_1 = Z_2 = \{0, \ldots, 14\}$ and let $Z = Z_1 \times Z_2$. To create the bivariate moments for our example we have used the uniform distribution on Z.

Using the mentioned univariate marginals and the mixed moments of order up to m, we have obtained the following results:

m	Lower	CPU	Upper	CPU
2	1.91194	0.28	1.98564	0.28
3	1.94560	0.58	1.95640	0.56
4	1.95009	1.18	1.95108	1.19
5	1.95051	2.59	1.95060	2.59
6	1.95053	6.11	1.95056	6.13

Note that problem (1.3) as an LP is numerically unstable, CPLEX frequently reported infeasibility even though the problems were feasible by construction. The Min and Max algorithms, however, always provided with us the correct numerical results.

6. Bounding expected utilities

The most general definition of a von Neumann-Morgenstern type utility function $u(z)$, $z \geqslant 0$ only requires that it should be an increasing function, i.e., $u'(z) > 0$. It is called risk averse, if we also have $u''(z) < 0$ which means that the function is also concave.

More generally, we may require:

$$(-1)^{n-1} u^{(n)}(z) > 0, \; n = 1, 2, \ldots. \tag{6.1}$$

Utility functions satisfying (6.1) are called *mixed* by Caballe and Pomansky (1996). For economic justification see Ingersoll (1987). Relation (6.1) means that $u(-z)$ is a *completely monotone function*. Examples of mixed utility functions are:

$$u(z) = a \log\left(1 + \frac{z}{b}\right), \; u(z) = -ae^{-bz},$$

where $a > 0$, $b > 0$.

In multiattribute utility theory (MAU) the well-known multiplicative form of Keeney and Raiffa (1976) is the following:

$$Ku(z_1, \ldots, z_s) + 1 = \prod_{i=1}^{s} (Kk_i u_i(z_i) + 1) \tag{6.2}$$

with $K \neq 0$. (The case $K = 0$ leads to a weighted additive from.)

The risk averse multiattribute utility function may be defined in such a way that $u(z_1, \ldots, z_s)$ is increasing in each variable and concave as an s-variate function.

In addition, we may require

$$(-1)^{i_1 + \cdots + i_s - 1} \frac{\partial^{i_1 + \cdots + i_s} u(z_1, \ldots, z_s)}{\partial z_1^{i_1} \cdots \partial z_s^{i_s}} > 0, \; 1 \leqslant i_1 + \cdots + i_s. \tag{6.3}$$

This is a multivariate counterpart of relations (6.1).

The above properties are usually not true for the functions (6.2). However, it is easy to see that the following is valid for (6.2) in case of $s = 2$:

$$(-1)^{i_1 + i_2} \frac{\partial^{i_1 + i_2} u(z_1, z_2)}{\partial z_1^{i_1} \partial z_2^{i_2}} > 0, \; 1 \leqslant i_1 \text{ and } 1 \leqslant i_2, \tag{6.4}$$

assuming that u_1 and u_2 are mixed utility functions satisfying (6.1).

A class of multiattribute utility functions that fulfill the concavity as well as property (6.3) is defined in Prékopa and Mádi-Nagy (2008).

DEFINITION 6.1. Let $k \geqslant 1$ and D an open convex set. We define the utility function u as:

$$u(z_1,\dots,z_s) := log\left[k(e^{g_1(z_1)}-1)\cdots(e^{g_s(z_s)}-1)-1\right], \tag{6.5}$$

where for every $(z_1,\dots,z_n)\in D$ the following conditions hold:

$$e^{g_j(z_j)} > 2,\ j=1,\dots,s, \tag{6.6}$$

$$\begin{aligned} g_j'(z_j) &> 0 \\ g_j^{(i)}(z_j) &\geqslant 0, \text{ if } i>1 \text{ and is odd} \\ g_j^{(i)}(z_j) &\leqslant 0, \text{ if } i \text{ is even} \\ j &= 1,\dots,s. \end{aligned} \tag{6.7}$$

Let $X=(X_1,\dots,X_s)$ be a random vector where the support of X_j is a known finite set $Z_j=\{z_{j0},\dots,z_{jn_j}\}$. Assume that the marginal distributions and the collection of mixed moments $\mu_{\alpha_1\dots\alpha_s}$, $\alpha_1+\cdots\alpha_s\leqslant m$ are known, and we want to give bounds for the utility

$$E[u(X_1,\dots,X_s)], \tag{6.8}$$

where the utility function u satisfies (6.3) or (6.4). Below we specialize u in three different ways and use the MDMP to create bounds for (6.8).

EXAMPLE 6.1. Let $s=2$. Assume that both random variables have uniform distribution on the sets $Z_1=Z_2=\{0,0.1,0.2,\dots,1\}$ and let $Z=Z_1\times Z_2$. Define the mixed moments of (X_1,X_2) by the use of the uniform distribution on Z. Let

$$u_1(z_1)=\frac{\log(1+z_1)}{\log 2},$$

$$u_2(z_2)=\frac{1-e^{-z_2}}{1-\frac{1}{e}}$$

and define the multiattribute utility function:

$$u(z_1,z_2)=k_1u_1(z_1)+k_2u_2(z_2)+Kk_1k_2u_1(z_1)u_2(z_2). \tag{6.9}$$

We compute bounds for $E[u(X_1,X_2)]$ in case of $k_1=0.3$ and $k_2=0.2$ ($K=1-k_1-k_2$), where the univariate marginal distributions and the mixed moments of order up to m are used. The function $u(z_1,z_2)$ satisfies (6.4), hence lower and upper bounds can be given by the use of the Min and Max Algorithms of Section 2. The calculation uses codes written in Mathematica. The results are:

m	Lower	*CPU*	Upper	*CPU*
2	0.405218	0.08	0.467877	0.08
3	0.432214	0.16	0.445082	0.16
4	0.437435	0.36	0.439693	0.36
5	0.438281	0.86	0.438624	0.86
6	0.438418	2.30	0.438473	2.28

EXAMPLE 6.2. Let $s = 2$, $Z_1 = Z_2 = \{0, \ldots, 19\}$. Consider the independent random variables X, Y_1, Y_2 that have Poisson distributions with parameters $3, 4, 5$, respectively. Define the random vector:

$$(X_1, X_2) := (min(X + Y_1, 19), min(X + Y_2, 19)).$$

Take the bivariate utility function

$$u(z_1, z_2) = \log[(e^{\alpha z_1 + a} - 1)(e^{\beta z_2 + b} - 1) - 1], \tag{6.10}$$

defined for z_1, z_2, satisfying

$$e^{\alpha z_1 + a} > 2,\ e^{\beta z_2 + b} > 2.$$

The function (6.10) is a special case of the function in (6.5), hence it satisfies the relations (6.3). This means that in order to give bounds for the expected utility we can apply the Min and Max Algorithms of Section 2. Let $\alpha = 0.75$, $\beta = 1.25$, $a = 2$, $b = 3$. We have obtained the following results:

m	Lower	*CPU*	Upper	*CPU*
2	20.2456980	0.88	20.2458790	0.81
3	20.2456982	1.75	20.2458790	1.70
4	20.2456989	3.45	20.2458790	3.45
5	20.2457012	2.63	20.2458790	7.20
6	20.2457066	15.44	20.2458790	15.47

EXAMPLE 6.3. Let $s = 3$. We calculate the expected value of $u(X_1, X_2, X_3)$ where u is the following utility function

$$\begin{gathered} u(z_1, z_2, z_3) = \log\left[(e^{\alpha_1 z_1 + a_1} - 1)(e^{\alpha_2 z_2 + a_2} - 1)(e^{\alpha_3 z_3 + a_3} - 1) - 1\right] \\ (z_1, z_2, z_3) \in Z, \end{gathered} \tag{6.11}$$

and

$$Z = (0,1,2,3,4,5,6,7,8,9) \times (0,1,2,3,4,5,6,7,8,9) \times (0,1,2,3,4,5,6,7,8,9).$$

We choose $\alpha_1 = \alpha_2 = \alpha_3 = a_1 = a_2 = a_3 = 1$. The function (6.11) is again a special case of the function (6.5).

Assume that X_1, X_2, X_3 are independent and each has uniform distribution on $\{0,1,2,3,4,5,6,7,8,9\}$. We have used the joint uniform distribution to generate mixed moments. The results are summarized below.

m	Minimum	Iteration	Maximum	Iteration
2	16.272644221	375	16.294708615	65
3	16.279932313	428	16.294702240	515
4	16.288384112	779	16.294688894	391
5	16.290690088	1121	16.294643748	1326
6	16.292421158	1605	16.294587574	1198

The application of the dual method of CPLEX frequently reported infeasibility even though the problems are feasible by construction. However, using one of our dual feasible bases as initial basis in the dual algorithm, we were able to carry out the algorithm and obtain the correct result.

7. Conclusions

We have presented examples to show that the MDMP technique can efficiently be used for bounding expectations of functions of random variables. In the examples the univariate marginals and mixed moments of total order up to m are assumed to be known. We have obtained results not mentioned in Hou and Prékopa (2007), for bounding expectations of Monge arrays (as functions) of random variables. We have presented an efficient method for bounding pseudo-Boolean functions of binary random variables, where the functions have special monotonicity property. We have also shown that the technique is useful for bounding the values of multivariate moment generating functions and the expectation of some multiattribute utility functions. Sometimes the bounds are given in terms of formulas, sometimes in terms of algorithms. In the latter case the dual algorithm of linear programming is adapted for the problems at hand. The application of standard LP packages, e.g., CPLEX, frequently failed to provide us with the correct numerical answers. However, using one of our dual feasible bases, as initial basis in the dual algorithm or use the Min and Max algorithms in the bivariate case, the correct numerical results could be obtained.

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