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CLOSED OPERATOR INEQUALITIES AND OPEN PROBLEMS

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Abstract. In [3, 4], we have given some characterizations of some subclasses of complex Hilbert space normal operators by inequalities. In this note, we shall reformulate these results using a concept of closed inequalities. Also, we shall give some new characterizations. Some open problems concerning closed inequalities are posed at the end of this note.

1. Introduction

We denote by $\mathfrak{B}(H)$ the C^*-algebra of all bounded linear operators on a complex Hilbert space H.

DEFINITION 1. Let $\mathscr{T}(H)$ be a subset of $\mathfrak{B}(H)$, and $\mathfrak{D}(H)$ and $\mathfrak{L}(H)$ be two subsets of $\mathscr{T}(H)$. Let $R(S,X)$ denotes a relation defined on $\mathscr{T}(H)\times\mathfrak{B}(H)$. The proposition

$$\forall S\in\mathfrak{D}(H),\ \forall X\in\mathfrak{B}(H),\ R(S,X)\quad (*)$$

is said to be closed (in $\mathscr{T}(H)\times\mathfrak{B}(H)$) if

$$\mathfrak{D}(H)=\{S\in\mathscr{T}(H):\forall X\in\mathfrak{B}(H),\ R(S,X)\}$$

If the proposition $(*)$ is closed, then it is clear that the proposition

$$\text{“}\forall S\in\mathfrak{L}(H),\ \forall X\in\mathfrak{B}(H),\ R(S,X)''$$

holds iff $\mathfrak{L}(H)\subset\mathfrak{D}(H)$, and by definition, its closure is the closed proposition $(*)$.

NOTATION 1. We denote by

(i) $\mathfrak{I}(H)$, $\mathscr{S}(H)$ and $\mathfrak{N}(H)$, the classes of all invertible operators, of all self-adjoint operators and of all normal operators in $\mathfrak{B}(H)$, respectively,

(ii) $\mathscr{S}_0(H)$ and $\mathfrak{N}_0(H)$, the classes of all invertible elements in $\mathscr{S}(H)$ and $\mathfrak{N}(H)$, respectively,

(iii) $\mathfrak{I}_1(H)$ and $\mathscr{S}_1(H)$, the classes of all unit elements in $\mathfrak{I}(H)$ and $\mathscr{S}_0(H)$, respectively,

(iv) $\mathfrak{U}(H)$ and $\mathfrak{U}_r(H)$, the classes of all unitary operators and of all unitary reflections in $\mathfrak{B}(H)$, respectively.

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NOTATION 2. (1) We denote by $(H)_1$, the unit sphere of H.
(2) Let $x,\ y \in H$. We denote by
(i) $x \otimes y$, the operator defined on H by $(x \otimes y)z = \langle z, y\rangle x,\ z \in H$.
(ii) $\mathscr{F}_1(H) = \{x \otimes y : x,\ y \in (H)_1\}$, the set of all unit rank one operators in $\mathfrak{B}(H)$.

It is easy to see that $A(x \otimes y)B = Ax \otimes B^*y$ and $\|x \otimes y\| = \|x\|\,\|y\|$, for $x, y \in H$ and $A, B \in \mathfrak{B}(H)$.

In $[3,4]$, we have been interested in the following properties:

$$\text{“}\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geqslant 2\|X\|,\ S \in \mathfrak{I}(H)\text{''}$$
$$\text{“}\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geqslant 2\|X\|,\ S \in \mathfrak{I}(H)\text{''}$$
$$\text{“}\forall X \in \mathfrak{B}(H), \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = 2\|X\|,\ S \in \mathfrak{I}(H)\text{''}$$

We have showed that the first property characterizes the class $\mathbb{C}^*\mathscr{S}_0(H)$, the second property characterizes the class $\mathfrak{N}_0(H)$, and the third property characterizes the class $\mathbb{R}^*\mathfrak{U}(H)$. In terms of closed operator inequalities, the following inequalities are closed

$$\forall S \in \mathbb{C}^*\mathscr{S}_0(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geqslant 2\|X\|,$$
$$\forall S \in \mathfrak{N}_0(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geqslant 2\|X\|,$$
$$\forall S \in \mathbb{R}^*\mathfrak{U}(H),\ \forall X \in \mathfrak{B}(H), \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = 2\|X\|.$$

Some others characterizations may be found also in $[4]$. In this note, we shall present these characterizations using the concept of closed inequalities, and we shall present new closed operator inequalities. Some problems concerning closed operator inequalities are posed at the end of this note.

2. On the Corach-Porta-Recht Inequality

In $[1]$, Corach, Porta and Recht showed that the following operator inequality holds

$$\forall S \in \mathscr{S}_0(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geqslant 2\|X\| \tag{1}$$

This inequality is a key factor in their study of differential geometry of self-adjoint operators. They proved this inequality by using the integral representation of a self-adjoint operator with respect to a spectral measure. Note that we may deduce easily this inequality from a previous operator inequality established by McIntosh in $[2]$, that is

$$\forall A, B, X \in \mathfrak{B}(H),\ \|A^*AX + XBB^*\| \geqslant 2\|AXB\| \tag{2}$$

Indeed, if $S \in \mathscr{S}_0(H)$ and $X \in \mathfrak{B}(H)$, then from (2) it follows that

$$\left\|SXS^{-1} + S^{-1}XS\right\| = \left\|SS(S^{-1}XS^{-1}) + (S^{-1}XS^{-1})SS\right\| \geqslant 2\left\|S(S^{-1}XS^{-1})S\right\| = 2\|X\|$$

REMARK 1. (i) The inequality (1) is not closed. Indeed, it is easy to see that the property

$$\text{“}\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geqslant 2\|X\|,\ S \in \mathfrak{I}(H)\text{''}$$

holds also for all operators S in $\mathbb{C}^*\mathscr{S}_0(H)$.

(ii) $\mathbb{C}^*\mathscr{S}_0(H)$ is exactly the class of all invertible normal operators in $\mathfrak{B}(H)$ the spectrum of which is included in straight lines passing through the origin.

(iii) Does the property

$$\text{“}\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}+S^{-1}XS\right\| \geqslant 2\left\|X\right\|,\ S \in \mathfrak{I}(H)\text{''}$$

hold for every invertible normal operator? This is not true. Indeed, in $\mathfrak{B}(\mathbb{C}^2)$ the invertible normal operator $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ and the operator $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ satisfy

$$\left\|SXS^{-1}+S^{-1}XS\right\| = 0 < 2\left\|X\right\| = 2.$$

3. Closed Operator Inequalities and Invertible Self-adjoint Operators

In our paper [3], we have showed that the closure of the inequality (1) is the inequality

$$\forall S \in \mathbb{C}^*\mathscr{S}_0(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}+S^{-1}XS\right\| \geqslant 2\left\|X\right\| \tag{3}$$

Then the class $\mathbb{C}^*\mathscr{S}_0(H)$ is characterized by the property

$$\forall X \in \mathfrak{B}(H), \left\|SXS^{-1}+S^{-1}XS\right\| \geqslant 2\left\|X\right\|,\ S \in \mathfrak{I}(H)$$

On the other hand, using the polar decomposition of an operator and the inequality (1), we easily deduce the following closed inequality

$$\forall S \in \mathfrak{I}(H),\ \forall X \in \mathfrak{B}(H),\ \left\|S^*XS^{-1}+S^{-1}XS^*\right\| \geqslant 2\left\|X\right\| \tag{4}$$

Recently in our paper [4], using the last inequality (4), we have given other two closed inequalities

$$\forall S \in \mathbb{C}^*\mathscr{S}_0(H),\ \forall X \in \mathfrak{B}(H), \left\|SXS^{-1}+S^{-1}XS\right\| = \left\|S^*XS^{-1}+S^{-1}XS^*\right\| \tag{5}$$

$$\forall S \in \mathbb{C}^*\mathscr{S}_0(H),\ \forall X \in \mathfrak{B}(H), \left\|SXS^{-1}+S^{-1}XS\right\| \geqslant \left\|S^*XS^{-1}+S^{-1}XS^*\right\| \tag{6}$$

So from above, the class $\mathbb{C}^*\mathscr{S}_0(H)$ is given by

$$\begin{aligned}
\mathbb{C}^*\mathscr{S}_0(H) &= \left\{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \left\|SXS^{-1}+S^{-1}XS\right\| \geqslant 2\left\|X\right\|\right\} \\
&= \left\{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \left\|SXS^{-1}+S^{-1}XS\right\| = \left\|S^*XS^{-1}+S^{-1}XS^*\right\|\right\} \\
&= \left\{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \left\|SXS^{-1}+S^{-1}XS\right\| \geqslant \left\|S^*XS^{-1}+S^{-1}XS^*\right\|\right\}
\end{aligned}$$

4. Closed Operator Inequalities and Invertible Normal Operators

The following Proposition gives us some preliminary characterizations of normal operators.

PROPOSITION 1. *Let $A \in \mathfrak{B}(H)$. The following properties are equivalent:*
(i) A is normal,
(ii) $\forall x \in H,\ \|A^*x\| = \|Ax\|$,
(iii) $\forall X \in \mathfrak{B}(H),\ \|A^*X\| = \|AX\|$,
(iv) $\forall X \in \mathfrak{B}(H),\ \|XA^*\| = \|XA\|$.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ follows immediately from the definition of normal operators.

The equivalences $(ii) \Leftrightarrow (iii)$ and $(iii) \Leftrightarrow (iv)$ are trivial. □

It follows from the above Proposition that the following inequality holds

$$\forall S \in \mathfrak{N}_0(H),\ \forall X \in \mathfrak{B}(H), \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\| \tag{7}$$

Using the inequalities (4) and (7), we deduce that the following inequality holds

$$\forall S \in \mathfrak{N}_0(H),\ \forall X \in \mathfrak{B}(H), \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geqslant 2\|X\| \tag{8}$$

In $[4]$, we have proved that each of the two last inequalities is closed.

From the two last closed inequalities and the inequality (4), we may deduce easily that the following inequality is closed

$$\forall S \in \mathfrak{N}_0(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geqslant \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\| \tag{9}$$

New others characterizations of the class $\mathfrak{N}_0(H)$ are given in the following Proposition.

PROPOSITION 2. *Let $S \in \mathfrak{I}(H)$. The following properties are equivalent*
(i) $S \in \mathfrak{N}_0(H)$,
(ii) $\forall X \in \mathfrak{B}(H), \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \leqslant \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|$,
(iii) $\forall X \in \mathscr{F}_1(H), \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \leqslant \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|$.

Proof. $(i) \Rightarrow (ii)$. This implication follows immediately from (7).
$(ii) \Rightarrow (iii)$. This implication is trivial.
$(iii) \Rightarrow (i)$. From (iii), it follows that the following inequality holds

$$\forall x,\ y \in (H)_1,\ \|Sx\| \left\|(S^*)^{-1}y\right\| + \left\|S^{-1}x\right\| \|S^*y\| \leqslant \|S^*x\| \left\|(S^*)^{-1}y\right\| + \left\|S^{-1}x\right\| \|Sy\|$$

Hence

$$\forall x,\ y \in (H)_1,\ (\|Sx\| - \|S^*x\|) \left\|(S^*)^{-1}y\right\| \leqslant (\|Sy\| - \|S^*y\|) \left\|S^{-1}x\right\| \tag{A}$$

Thus

$$(\forall x \in (H)_1,\ \|Sx\| \geqslant \|S^*x\|) \vee (\forall x \in (H)_1,\ \|Sx\| \leqslant \|S^*x\|)$$

Assume that the inequality $\|Sx\| \geqslant \|S^*x\|$ holds for every $x \in (H)_1$.

Since the relation $\frac{1}{\|T^{-1}\|} \leqslant \|Tx\| \leqslant \|T\|$ holds for every $T \in \mathfrak{I}(H)$ and for every $x \in (H)_1$, then from (A), it follows that

$$\forall x,\ y \in (H)_1,\ \|Sx\| - \|S^*x\| \leqslant k(\|Sy\| - \|S^*y\|)$$

where $k = \|S\| \left\|S^{-1}\right\|$. So we have

$$\forall x,\ y \in (H)_1,\ \|Sx\| + k\|S^*y\| \leqslant \|S^*x\| + k\|Sy\|$$

Hence

$$\forall x \in (H)_1,\ \sup_{\|y\|=1} (\|Sx\| + k\|S^*y\|) \leqslant \sup_{\|y\|=1} (\|S^*x\| + k\|Sy\|)$$

Thus

$$\forall x \in (H)_1,\ \|Sx\| + k\|S\| \leqslant \|S^*x\| + k\|S\|$$

So it follows that the inequality $\|Sx\| \leqslant \|S^*x\|$ holds for every vector x in $(H)_1$. Hence, the equality $\|Sx\| = \|S^*x\|$ holds for every vector x in $(H)_1$. Therefore $S \in \mathfrak{N}_0(H)$.

With the second assumption and by the same argument, we find also that $S \in \mathfrak{N}_0(H)$. $\square$

COROLLARY 1. *The following inequality is closed*

$$\forall S \in \mathfrak{N}_0(H),\ \forall X \in \mathfrak{B}(H), \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \leqslant \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\| \quad (10)$$

From above, the class $\mathfrak{N}_0(H)$ is given by

$$\begin{aligned}
\mathfrak{N}_0(H) &= \left\{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geqslant 2\|X\|\right\} \\
&= \left\{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|\right\} \\
&= \left\{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geqslant \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|\right\} \\
&= \left\{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H), \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \leqslant \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|\right\}
\end{aligned}$$

5. Closed Operator Inequalities and Unitary Operators

The following Proposition gives us some preliminary characterizations of the class $\mathbb{R}^*\mathfrak{U}(H)$.

PROPOSITION 3. *Let $S \in \mathfrak{I}(H)$. The following properties are equivalent:*
(i) $S \in \mathbb{R}^*\mathfrak{U}(H)$,
(ii) $\|S\| \left\|S^{-1}\right\| = 1$,
(iii) $\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| = \|X\|$,
(iv) $\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| \leqslant \|X\|$,
(v) $\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| \geqslant \|X\|$.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ is trivial.

The equivalences $(ii) \Leftrightarrow (iii)$, $(ii) \Leftrightarrow (iv)$, $(ii) \Leftrightarrow (v)$ follow from the fact that $\inf\limits_{\|X\|=1} \left\|SXS^{-1}\right\| = \frac{1}{\|S\|\left\|S^{-1}\right\|}$ and $\sup\limits_{\|X\|=1} \left\|SXS^{-1}\right\| = \|S\| \left\|S^{-1}\right\|$. □

From the above Proposition, it follows that the following inequalities hold

$$\forall S \in \mathbb{R}^*\mathfrak{U}(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = 2\|X\| \tag{11}$$

$$\forall S \in \mathbb{R}^*\mathfrak{U}(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \leqslant 2\|X\| \tag{12}$$

$$\forall S \in \mathbb{R}^*\mathfrak{U}(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geqslant 2\|X\| \tag{13}$$

In [4], we have proved that the inequality (11) is closed.

In the next Proposition, we shall show that the inequality (12) is also closed. Note that the inequality (13) is not closed, since its closure is exactly the inequality (8).

PROPOSITION 4. *Let $S \in \mathfrak{I}(H)$. The two following properties are equivalent:*
(i) $S \in \mathbb{R}^*\mathfrak{U}(H)$,
(ii) $\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \leqslant 2\|X\|$.

Proof. $(i) \Rightarrow (ii)$. This implication is clear.

$(ii) \Rightarrow (i)$. Since, the inequality $\left\|SXS^{-1}\right\| \geqslant \frac{\|X\|}{\|S\|\left\|S^{-1}\right\|}$ holds for every $X \in \mathfrak{B}(H)$, then from (ii) it follows that

$$\sup_{\|X\|=1} \left(\frac{\|X\|}{\|S\| \left\|S^{-1}\right\|} + \left\|S^{-1}XS\right\| \right) \leqslant 2$$

Thus $\|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\|\left\|S^{-1}\right\|} \leqslant 2$. Hence, $\|S\| \left\|S^{-1}\right\| = 1$. Therefore, the condition (i) follows immediately from the above Proposition. □

On the other hand, it is easy to see that the following inequality holds

$$\forall S \in \mathbb{R}^*\mathfrak{U}(H),\ \forall X \in \mathfrak{B}(H),\ \left\|S^*XS^{-1} + S^{-1}XS^*\right\| = 2\|X\| \tag{14}$$

In [4], we have proved that this last inequality (14) is closed.

Another closed operator inequality can be found easily from the inequalities (4) and (14), that is

$$\forall S \in \mathbb{R}^*\mathfrak{U}(H),\ \forall X \in \mathfrak{B}(H),\ \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\| = 2\|X\| \tag{15}$$

From above, the class $\mathbb{R}^*\mathfrak{U}(H)$ is given by

$$\begin{aligned}
\mathbb{R}^*\mathfrak{U}(H) &= \left\{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = 2\|X\|\right\} \\
&= \left\{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \leqslant 2\|X\|\right\} \\
&= \left\{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H),\ \left\|S^*XS^{-1} + S^{-1}XS^*\right\| = 2\|X\|\right\} \\
&= \left\{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H),\ \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\| = 2\|X\|\right\}
\end{aligned}$$

If $R(S,X)$ denotes one of the four last relations, then the class $\mathfrak{U}(H)$ is given by

$$\mathfrak{U}(H) = \{S \in \mathfrak{I}_1(H) : \forall X \in \mathfrak{B}(H),\ R(S,X)\}$$

Also, from above it follows that

$$\begin{aligned}
\mathbb{R}^*\mathfrak{U}_r(H) &= \left\{S \in \mathscr{S}_0(H) : \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| = 2\|X\|\right\} \\
&= \left\{S \in \mathscr{S}_0(H) : \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = 2\|X\|\right\} \\
\mathfrak{U}_r(H) &= \left\{S \in \mathscr{S}_1(H) : \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| = 2\|X\|\right\} \\
&= \left\{S \in \mathscr{S}_1(H) : \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = 2\|X\|\right\}
\end{aligned}$$

In [4], we have proved also that the following inequality is closed

$$\forall S \in \mathbb{C}^*\mathfrak{U}_r(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| = 2\|X\| \tag{16}$$

So the class $\mathbb{C}^*\mathfrak{U}_r(H)$ is given by

$$\mathbb{C}^*\mathfrak{U}_r(H) = \left\{S \in \mathfrak{I}(H) : \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| = 2\|X\|\right\}$$

6. Open Problems

It is easy to see that the closed inequality (3) is equivalent to the following inequality

$$\forall S \in \mathbb{C}^*\mathscr{S}_0(H),\ \forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| \geqslant 2\|SXS\|$$

This last is also closed in $\mathfrak{I}(H) \times \mathfrak{B}(H)$. But the relation given in this inequality is defined on $\mathfrak{B}(H) \times \mathfrak{B}(H)$, and using McIntosh inequality (2), it follows immediately that

$$\forall S \in \mathbb{C}\mathscr{S}(H),\ \forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| \geqslant 2\|SXS\| \tag{17}$$

PROBLEM 1. What is (in $\mathfrak{B}(H) \times \mathfrak{B}(H)$) the closure of the inequality (17)?

On the other hand, since $\|S^2X\| = \|S^*SX\|$ and $\|XS^2\| = \|XSS^*\|$, for every $(S,X) \in \mathfrak{N}(H) \times \mathfrak{B}(H)$, it follows from McIntosh inequality that the following inequality holds

$$\forall S \in \mathfrak{N}(H),\ \forall X \in \mathfrak{B}(H),\ \|S^2X\| + \|XS^2\| \geqslant 2\|SXS\| \tag{18}$$

Note that the previous inequality (8) follows immediately from this last inequality.

PROBLEM 2. What is (in $\mathfrak{B}(H) \times \mathfrak{B}(H)$) the closure of the inequality (18)?

PROBLEM 3. It is easy to see that the two following inequalities hold

$$\forall S \in \mathbb{R}^*\mathfrak{U}(H),\ \forall X \in \mathfrak{B}(H),\ \|SXS^{-1} + S^{-1}XS\| \leqslant 2\|X\| \tag{19}$$

$$\forall S \in \mathbb{C}^*\mathscr{S}_0(H) \cup \mathbb{R}^*\mathfrak{U}(H),\ \|SXS^{-1} + S^{-1}XS\| \leqslant \|S^*XS^{-1} + S^{-1}XS^*\| \tag{20}$$

What is the closure of each of these two inequalities?

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